MATH/COSC 303

Assignment 4

Due: Mar 11 in LAB, assignments are due at the END of lab.

Hand Written Questions:

1. (Big oh Calculus) Suppose $F(h) = O(h^k)$ (at $h = 0$) and $G(h) = O(h^k)$ (at $h = 0$) where k is a positive integer. Show that

   a) $(F + G)(h) = O(h^k)$,

   b) $\alpha F(h) = O(h^k)$ for any $\alpha \in \mathbf{R}$, and

   c) $\frac{F(h)}{h} = O(h^{k-1})$.

2. Let
$$CD^{(1)}_{h^4}(h) = \frac{-f(\bar{x} + 2h) + 8f(\bar{x} + h) - 8f(\bar{x} - h) + f(\bar{x} - 2h)}{12h}.$$
   Prove that, if $f \in \mathcal{C}^5$, then $f'(\bar{x}) = CD^{(1)}_{h^4}(h) + O(h^4)$.
   (You may use the results of question 1 to do the 'simple proof'.)

3. Let
$$CD^{(3)}_{h^2}(h) = \frac{f(\bar{x} + 2h) - 2f(\bar{x} + h) + 2f(\bar{x} - h) - f(\bar{x} - 2h)}{2h^3}.$$
   Prove that, if $f \in \mathcal{C}^5$, then $f'''(\bar{x}) = CD^{(3)}_{h^2}(h) + O(h^2)$.
   (You may use the results of question 1 to do the 'simple proof'.)

4. Let $f(x) = x^3 + 4x^2 - 1$.

   a) Use Lagrange Interpolation to construct interpolating polynomials of degree 2, 3, and 4 on the interval $[0, 3]$.

   b) Use Newton Interpolation to construct interpolating polynomials of degree 2, 3, and 4 on the interval $[0, 3]$.

   c) What do you notice about the Lagrange versus the Newton polynomial of higher degrees?

5. Prove that
$$\frac{f[x_2, x_0] - f[x_1, x_0]}{x_2 - x_1} = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}.$$

Computer Assisted Questions:

6. Three formulae for approximating f' are
$$\begin{aligned} DD(h) &= \tfrac{f(\bar{x}+h)-f(\bar{x})}{h} \\ CD_{h^2}(h) &= \tfrac{f(\bar{x}+h)-f(\bar{x}-h)}{2h} \\ CD_{h^4}(h) &= \tfrac{-f(\bar{x}+2h)+8f(\bar{x}+h)-8f(\bar{x}-h)+f(\bar{x}-2h)}{12h}. \end{aligned}$$
   Consider $f(x) = \frac{1}{2}e^{2x}$ at $\bar{x} = 1$.

   a) Use each formula and $h = \{1, 10^{-1}, 10^{-2}, \ldots, 10^{-16}\}$ to approximate $f'(\bar{x})$.

b) Compute the relative error for each formula and value of h.

c) Which formula approximates $f'(\bar{x})$ the 'fastest'?

d) What value of h provides the highest accuracy for each formula?

7. Two formulae for approximating f'' are

$$\begin{aligned} CD^{(2)}_{h^2}(h) &= \tfrac{f(\bar{x}+h)-2f(\bar{x})+f(\bar{x}-h)}{h^2} \\ CD^{(2)}_{h^4}(h) &= \tfrac{-f(\bar{x}+2h)+16f(\bar{x}+h)-30f(\bar{x})+16f(\bar{x}-h)-f(\bar{x}-2h)}{12h^2}. \end{aligned}$$

Consider $f(x) = \frac{1}{2}e^{2x}$ at $\bar{x} = 1$.

a) Use each formula and $h = \{1, 10^{-1}, 10^{-2}, \ldots, 10^{-16}\}$ to approximate $f''(\bar{x})$.

b) Compute the relative error for each formula and value of h.

c) Which formula approximates $f''(\bar{x})$ the 'fastest'?

d) What value of h provides the highest accuracy for each formula?

8. Let $f(x) = \cos(x)$.

a) Write a MATLAB function that uses Lagrange **or** Newton Interpolation to create a polynomial interpolation $p(x)$ of degree d for $f(x) = \cos(x)$ over the interval $[0, 1]$ (d should be an input to the function).

c) Use (a) to approximate $f(0.5)$ by using polynomials of degree $d = \{2, 3, 4, ...10\}$. Compute the relative error for each estimation.