Noether's Theorem: Symmetries and Conservation Laws

Podcast Learn & Fun *

February 2, 2025

Noether's Theorem, developed by the mathematician Emmy Noether in 1915 and published in 1918 [1], is a fundamental result in both theoretical physics and mathematics. It highlights a profound relationship between symmetries of physical systems and conserved quantities. Specifically, the theorem states that for every continuous symmetry of the action in a physical system, there exists a corresponding conserved quantity. This theorem is essential in classical mechanics, quantum mechanics, and field theory, as it offers deep insights into the underlying structure of physical laws.

At its core, Noether's theorem provides a systematic way to understand why certain quantities in nature—such as energy, momentum, and angular momentum—are conserved. The power of this result lies in its generality and applicability across a wide variety of physical systems.

Action

To properly understand Noether's theorem, we must first introduce a few key concepts related to Lagrangian mechanics. The *action* S of a system is defined as the integral of the Lagrangian L over time:

$$S = \int L \, dt$$

The Lagrangian itself is a function of the generalized coordinates q_i, their time derivatives $\dot{q}_i$, and potentially time t. The action describes the dynamics of the system, and the principle of least action tells us that the actual

*YouTube Channel: `https://www.youtube.com/@Podcast_Learn_Fun`

[1] Noether, E. (1918). *Invariante Variationsprobleme*. Nachr. Ges. Wiss. Göttingen.

path of a system will make the action S stationary (typically a minimum or saddle point).

A *symmetry* in a physical system refers to the invariance of the system's behavior under certain transformations. For example, time translation symmetry means that the system's behavior is unchanged if it is shifted in time. Similarly, space translation symmetry means the system's behavior is unchanged under spatial shifts, and rotational symmetry means the system remains invariant under rotations in space. Symmetries are crucial because they reflect the fundamental properties of nature, such as the laws of physics being the same at all times or in all locations.

Noether's Theorem: Statement and Significance

Noether's theorem asserts that *every continuous symmetry of the action of a system corresponds to a conserved quantity.* If a system's action is invariant under a continuous transformation, this invariance implies the existence of a conserved quantity.

Noether's theorem provides a unified mathematical framework that explains why many physical quantities are conserved. It generalizes classical conservation laws and shows that they are not independent, but rather a consequence of deeper symmetries in the system.

Time Translation Symmetry If the Lagrangian does not explicitly depend on time, the system is invariant under time translations. According to Noether's theorem, this symmetry corresponds to the conservation of the total energy of the system. This is a direct generalization of the first law of thermodynamics.

Space Translation Symmetry If the Lagrangian is invariant under spatial translations, meaning the system behaves the same way no matter where it is located in space, Noether's theorem implies the conservation of linear momentum. This is a manifestation of Newton's first law of motion, which states that a body at rest or in uniform motion will remain in that state unless acted upon by an external force.

Rotational Symmetry If the system's Lagrangian is invariant under rotations in space, meaning the system's behavior does not change under a rotation, Noether's theorem implies the conservation of angular momentum. This is a direct consequence of the rotational invariance of the physical laws governing the system, such as in the case of central force motion (e.g., planetary motion).

Noether's theorem illuminates the structure of physical laws by linking symmetries with conserved quantities. Conservation laws, which were historically discovered through experimentation, are seen to emerge naturally from the invariance properties of the system's action. This insight deepens our understanding of the fundamental principles governing physical systems.

Proof of Noether's Theorem

The proof of Noether's theorem is grounded in the idea that symmetries of the action imply certain conserved quantities.

Consider a system described by a set of generalized coordinates $q_i(t)$ and velocities $\dot{q}_i(t)$, where $i = 1, 2, \ldots, n$. The dynamics of the system are encoded in the action S, which is given by the integral of the Lagrangian L over time:

$$S = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t)\, dt$$

Suppose now that the system undergoes a continuous symmetry transformation, which we will denote by a transformation $\delta q_i(t)$, where the coordinates $q_i(t)$ change infinitesimally. This transformation could, for example, represent a translation in time, space, or a rotation. The essential point is that this symmetry transformation leaves the form of the action S invariant, meaning that the total change in the action δS is zero:

$$\delta S = 0$$

In other words, the transformation $q_i(t) \to q_i(t) + \delta q_i(t)$ does not affect the physics described by the action.

When the generalized coordinates change infinitesimally, we can expand the action to first order in $\delta q_i(t)$. The change in the action δS can be written as:

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt$$

Using the fact that $\delta \dot{q}_i = d(\delta q_i)/dt$, we can rewrite the second term:

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}(\delta q_i) \right) dt$$

Next, apply integration by parts to the second term, assuming that $\delta q_i(t)$ vanishes at the endpoints t_1 and t_2:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}(\delta q_i) dt = -\int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt$$

Thus, the change in the action becomes:

$$\delta S = \int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i \right] dt$$

For the action to be invariant under the transformation, the integrand must be zero for arbitrary $\delta q_i(t)$. This condition leads to the Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0$$

These are the usual equations of motion that describe the system's dynamics.

Now, let's focus on the fact that the transformation $\delta q_i(t)$ corresponds to a symmetry of the system. By assumption, the action remains invariant under the symmetry transformation. This means that, while the generalized coordinates change, the Lagrangian L itself may change in a particular way, typically by a total time derivative of a function $F(q_i, \dot{q}_i, t)$. That is, the change in the Lagrangian under the symmetry is given by:

$$\delta L = \frac{d}{dt} F(q_i, \dot{q}_i, t)$$

This expression indicates that the Lagrangian changes by a total time derivative, which is a crucial feature of Noether's theorem.

To find the conserved quantity associated with the symmetry, we need to examine how the action changes under the symmetry transformation. Recall the expression for δS:

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt$$

Using the fact that $\delta L = \frac{d}{dt} F(q_i, \dot{q}_i, t)$, the change in the action becomes:

$$\delta S = \int_{t_1}^{t_2} \left(\frac{d}{dt} F(q_i, \dot{q}_i, t) \right) dt$$

This can be rewritten as:

$$\delta S = \left[F(q_i, \dot{q}_i, t) \right]_{t_1}^{t_2}$$

Since the variation $\delta q_i(t)$ vanishes at the boundaries t_1 and t_2, this boundary term must be zero. Thus, the total change in the action is zero, which implies that the system's equations of motion remain unchanged under the symmetry transformation.

The key point here is that the term $F(q_i, \dot{q}_i, t)$, which is the total time derivative of some function of the coordinates and velocities, corresponds to a conserved quantity. This conserved quantity is the quantity associated with the symmetry of the system.

From the above derivation, we conclude that the symmetry of the system implies the existence of a conserved quantity. Specifically, the total time derivative of the function $F(q_i, \dot{q}_i, t)$ corresponds to a conserved quantity that remains constant over time. This is the essence of Noether's theorem: the invariance of the action under a continuous symmetry transformation leads to a conserved quantity, and the conservation law is directly linked to the specific symmetry transformation of the system.

Examples of Noether's Theorem

Time Translation Symmetry and Energy Conservation

Time translation symmetry implies that the system's behavior does not depend on the absolute time. If the Lagrangian $L(q_i, \dot{q}_i, t)$ does not explicitly depend on time t, then the system is invariant under shifts in time, and Noether's theorem guarantees the conservation of energy. However, if the Lagrangian explicitly depends on time, the system no longer has time translation symmetry, and energy is not conserved.

Example: Consider a system where the Lagrangian explicitly depends on time, such as a driven harmonic oscillator:

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 - \frac{1}{2}Atq^2$$

Here, the term $\frac{1}{2}Atq^2$ explicitly depends on time. This time-dependence breaks time translation symmetry and, consequently, the total energy of the system is no longer conserved. The energy is influenced by the external time-varying driving force, which supplies or extracts energy from the system.

In this case, energy conservation fails because the time-dependent term in the Lagrangian introduces an external driving force that alters the energy of the system.

Space Translation Symmetry and Momentum Conservation

If a system's Lagrangian is invariant under spatial translations, meaning the system's physics does not depend on where it is located in space, the conserved quantity is linear momentum. This is guaranteed by Noether's theorem. If the Lagrangian explicitly depends on the spatial position, then space translation symmetry is broken, and momentum is no longer conserved.

Example: Consider a particle moving in a potential $V(x) = kx^2$, where k is a constant. The Lagrangian for such a system is:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

In this case, the Lagrangian depends on x, and the system is not invariant under spatial translation. In this system, momentum is no longer conserved because the symmetry of the system has been broken by the position-dependent potential. The force arising from the potential is not constant in space, meaning the system experiences a non-conservative force that causes the momentum to change over time.

Rotational Symmetry and Angular Momentum Conservation

If the system is invariant under rotations (i.e., the system behaves the same regardless of its orientation in space), then angular momentum is conserved according to Noether's theorem. If the system experiences an external torque or if the potential is not rotationally symmetric, the rotational symmetry is broken, and angular momentum is no longer conserved.

Example: Consider a particle moving in a central potential $V(r)$, such as the gravitational potential:

$$V(r) = -\frac{GMm}{r}$$

This system is invariant under rotations because the potential depends only on the radial distance r and not on the angle.

However, if the particle is subject to an external torque, such as a magnetic field acting on a charged particle or a non-central force, the rotational symmetry is broken. For example, if the force acting on the particle depends on the angle or if an external torque is applied, the Lagrangian of the system no longer exhibits rotational symmetry.

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r) - \tau\theta$$

where τ represents an external torque.

In this case, the system no longer exhibits rotational symmetry, and angular momentum is no longer conserved due to the external torque (or asymmetric force) acting on the system.

These examples illustrate how the breakdown of symmetries, whether in time, space, or other aspects, leads to the failure of conservation laws. Symmetry violations play a key role in understanding many real-world phenomena, such as time-dependent forces, external fields, and interactions that introduce non-conservative effects.

Applications in Modern Physics

Noether's theorem extends far beyond classical mechanics and has crucial applications in modern physics. In field theory, for example, the theorem applies to various physical fields, such as the electromagnetic field, where symmetries like gauge invariance lead to the conservation of charge. In quantum mechanics, Noether's theorem is essential for understanding the symmetries of quantum fields and particle interactions, helping to explain conservation laws in the context of particle physics.

In quantum field theory, Noether's theorem plays a central role in understanding symmetries such as Lorentz invariance, which leads to the conservation of energy, momentum, and angular momentum in relativistic theories. It is a cornerstone in the study of the Standard Model of particle physics, where symmetries dictate fundamental interactions and conservation laws governing elementary particles.

Conclusion

Noether's theorem is one of the most elegant and powerful results in theoretical physics. By connecting symmetries in physical systems to conserved quantities, it provides a framework for understanding why certain quantities remain constant in nature. The theorem not only illuminates classical conservation laws, such as energy, momentum, and angular momentum, but also has profound implications in modern physics, particularly in the context of quantum mechanics and field theory. Understanding Noether's theorem is essential for anyone wishing to delve deeper into the structure of physical laws and the interplay between symmetry and conservation.