Reduced Mass

Podcast Learn & Fun *

January 13, 2025

The concept of reduced mass emerges in classical mechanics when analyzing systems with two interacting objects. For many practical problems, such as orbital motion or molecular vibrations, dealing with the interaction between two bodies directly can be cumbersome. Instead, we often simplify the problem by introducing an effective mass that allows us to reduce a two-body system to an equivalent single-body problem. This effective mass is known as the reduced mass.

1 Derivation of the Reduced Mass

Consider two particles of masses m_1 and m_2 interacting via a central force (such as the Coulomb force between an electron and a hole in an exciton). Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the positions of the two particles. The relative position between the two particles is given by:

$$\mathbf{r}_{\text{rel}} = \mathbf{r}_1 - \mathbf{r}_2$$

The dynamics of the two particles can be described by their individual motions, but it's often more convenient to express the system in terms of the *relative motion* of the particles rather than tracking both particles individually.

In classical mechanics, for a system of two particles under a central force, we write Newton's second law for each particle: For particle 1:

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_1$$

For particle 2:

$$m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_2$$

Since the force on one particle comes from the other, we have $\mathbf{F}_1 = -\mathbf{F}_2$, meaning that the forces are equal in magnitude but opposite in direction.

A useful concept in the two-body problem is the *center of mass* (COM) of the system. The center of mass is defined by:

$$\mathbf{R}_{\text{COM}} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

*YouTube Channel: https://www.youtube.com/@Podcast_Learn_Fun

The relative position $\mathbf{r}_{\mathrm{rel}} = \mathbf{r}_1 - \mathbf{r}_2$ and the motion of the center of mass allow us to describe the system more simply.

The total kinetic energy T of the system can be written as:

$$T = \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2$$

We can rewrite this total kinetic energy in terms of the *relative velocity* $\mathbf{v}_{\mathrm{rel}} = \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2$ and the velocity of the center of mass $\mathbf{V}_{\mathrm{COM}} = (m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2)/(m_1 + m_2)$:

$$T = \frac{1}{2}(m_1 + m_2)\mathbf{V}_{\mathrm{COM}}^2 + \frac{1}{2}\mu \mathbf{v}_{\mathrm{rel}}^2$$

where μ is the *reduced mass* defined by:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

The reduced mass μ effectively represents the *effective inertia* of the two-particle system, allowing us to treat the relative motion of the system as if it were a single particle of mass μ. The advantage of this simplification is that it turns the problem into a *one-body problem*, which is much easier to solve. The *total kinetic energy* is now expressed as the sum of the kinetic energy of the center of mass (which is independent of the interaction between the particles) and the kinetic energy of the relative motion (which depends on the interaction force). The dynamics of the two particles can be reduced to the dynamics of a single particle of mass μ, interacting with the other particle through the central force.

2 Applications

2.1 The Two-Body Problem in Classical Mechanics

In classical mechanics, when two objects interact through a central force (like gravity, electrostatic force, etc.), each object feels a force due to the other. The force can be described by Newton's laws of motion, but if we try to describe the system by tracking both objects individually, it can get complicated.

However, there is a simplification: we can switch to a *center of mass* frame of reference. In this frame, instead of considering the two objects individually, we can treat the relative motion of the objects as if one object has mass equal to the reduced mass μ, moving in the same potential created by both objects. This allows the problem to be reduced to a one-body problem.

For example, when analyzing planetary orbits, we can use the reduced mass to calculate the relative motion of two bodies (like the Earth and the Moon) as if the motion of a single body with reduced mass occurs under the influence of the combined gravitational forces.

2.2 Orbital Mechanics: Simplification in Celestial Motion

Consider a system like the Earth orbiting the Sun. The two bodies (Earth and Sun) exert gravitational forces on each other, which causes them to both move. However, instead of tracking the motion of both objects, we can analyze the system by focusing on the center of mass, using the reduced mass to describe the relative motion between the two bodies.

For a simplified two-body orbit problem, we can treat the Sun-Earth system as though the Earth is orbiting the Sun at a fixed distance, while the Sun also moves slightly due to the Earth's gravitational pull. The reduced mass helps simplify this by allowing us to treat the problem as if one of the objects has an "effective mass" in the center-of-mass system.

The reduced mass is particularly useful when applying Kepler's laws or calculating the orbital period of planets or moons. By considering the reduced mass of the system, we can simplify the orbital mechanics equations, ensuring that the solutions are mathematically tractable.

2.3 Vibrational Motion in Molecules (Diatomic Molecules)

When atoms in a diatomic molecule vibrate, the motion is usually described as the relative motion between the two atoms. In this case, the reduced mass plays a central role in determining the frequency of the vibration.

The vibrational frequency ν of a diatomic molecule is given by:

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

where k is the force constant (which depends on the chemical bond between the atoms) and μ is the reduced mass.

This relationship tells us that the vibrational frequency depends on the reduced mass of the molecule. A larger reduced mass results in a lower vibrational frequency, while a smaller reduced mass leads to a higher frequency.

For example, in a hydrogen molecule (H_2), the atoms are very light, and the reduced mass is small, resulting in a high-frequency vibration. On the other hand, a heavier molecule like oxygen (O_2) has a larger reduced mass, so its vibrational frequency will be lower.

2.4 Quantum Mechanics and the Schrödinger Equation

In quantum mechanics, the reduced mass is a key factor in solving the Schrödinger equation for two-particle systems. When analyzing the interaction between two particles, such as an electron and a nucleus (like in the hydrogen atom or hydrogen-like ions), we don't treat the electron and nucleus as independent particles moving through space. Instead, we focus on their relative motion.

The relative motion is governed by a potential that depends on the separation between the two particles, and the system can be described by a single particle moving with the reduced mass μ in this effective potential.

For example, in the hydrogen atom, the electron orbits the nucleus. The electron and proton both move in response to each other's gravitational or electrostatic forces. By using the reduced mass, we can simplify the system and solve for energy levels and wavefunctions.

2.5 Elastic Collisions and Reduced Mass

When analyzing *elastic collisions* (such as between gas molecules), the reduced mass simplifies calculations involving momentum transfer. In a head-on elastic collision, the total momentum and kinetic energy are conserved, but the velocities of the particles after collision depend on their masses.

The relative velocity of the particles before and after the collision, and the transfer of kinetic energy, can be more easily described using the reduced mass. In a two-body collision, the formula for calculating the relative velocities and energy exchange is based on the reduced mass, making it easier to predict the outcome of the collision.

2.6 Atomic and Molecular Spectroscopy

Spectroscopic techniques are based on the interaction of electromagnetic radiation with the discrete energy levels of atoms and molecules. The energy levels of molecules are influenced by various factors, including their reduced mass, which plays a significant role in determining the spacing between energy levels. This section highlights how the reduced mass affects both rotational and vibrational spectra of molecules.

2.6.1 Rotational Spectra

The rotational spectra of molecules depend on their moment of inertia, which is related to the reduced mass. The moment of inertia I for a diatomic molecule is given by:

$$I = \mu r^2$$

where μ is the reduced mass of the molecule and r is the distance between the two nuclei.

The rotational constant B, which determines the spacing between rotational energy levels, is inversely proportional to the moment of inertia I, and thus to the reduced mass μ. This relationship can be expressed as:

$$B = \frac{h}{8\pi^2 I}$$

where h is Planck's constant. As a result, the rotational spectra of molecules are sensitive to the reduced mass, with heavier molecules having smaller rotational constants and thus a lower frequency for rotational transitions.

2.6.2 Vibrational Spectra

The vibrational spectra of molecules, which correspond to transitions in the infrared region, are also influenced by the reduced mass. The frequency of vibration ν for a diatomic molecule can be described by the following equation:

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

where k is the force constant, which depends on the strength of the chemical bond, and μ is the reduced mass.

From this relationship, we can see that the vibrational frequency is inversely proportional to the square root of the reduced mass. Therefore, a larger reduced mass results in a lower vibrational frequency, and vice versa. This effect plays a crucial role in molecular spectroscopy, as the vibrational frequencies determine the absorption or emission of infrared radiation, which is characteristic of different molecules.

The reduced mass thus influences both the rotational and vibrational spectra, making it a key factor in the study of molecular energy levels and the identification of molecules through spectroscopic techniques.

3 Conclusion

The concept of reduced mass is not just a mathematical abstraction, but an important tool for simplifying a wide variety of problems in physics and chemistry. By reducing the complexity of two-body problems, the reduced mass allows us to model and understand systems such as planetary motion, molecular vibrations, quantum mechanical interactions, and collisions in an efficient manner. Whether you're working with classical mechanics, quantum mechanics, or spectroscopy, reduced mass provides an essential simplification that makes many real-world phenomena easier to analyze and predict.