Solving Partial Differential Equations via Separation of Variables

Podcast Learn & Fun *

January 25, 2025

Partial Differential Equations (PDEs) describe a wide range of physical phenomena, such as heat conduction, wave propagation, and fluid dynamics. One common and powerful technique for solving certain types of PDEs is *Separation of Variables*. This method works well when the PDE can be written in such a way that the solution can be factored into products of functions, each depending on only one of the independent variables.

1 The Heat Equation Example

Consider the heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2},$$

with the boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad t > 0,$$

and initial condition:

$$u(x,0) = f(x) \quad \text{for} \quad 0 \leq x \leq L.$$

We assume the solution can be separated as:

$$u(x,t) = X(x)T(t).$$

Substituting this form into the heat equation:

$$\frac{\partial}{\partial t}\left(X(x)T(t)\right) = \alpha \frac{\partial^2}{\partial x^2}\left(X(x)T(t)\right).$$

Since $X(x)$ depends only on x and $T(t)$ depends only on t, the partial derivatives give:

$$X(x)\frac{dT(t)}{dt} = \alpha T(t)\frac{d^2X(x)}{dx^2}.$$

*YouTube Channel: `https://www.youtube.com/@Podcast_Learn_Fun`

Now, divide both sides of the equation by $X(x)T(t)$:

$$\frac{1}{\alpha T(t)}\frac{dT(t)}{dt} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2}.$$

The left-hand side depends only on t and the right-hand side depends only on x. Since the equation must hold for all x and t, both sides must equal a constant, say $-\lambda$. Thus, we obtain two ordinary differential equations. For $T(t)$:

$$\frac{dT(t)}{dt} = -\lambda T(t),$$

which has the solution:

$$T(t) = Ae^{-\lambda t}.$$

For $X(x)$:

$$\frac{d^2X(x)}{dx^2} = -\lambda X(x).$$

This is a second-order linear differential equation. The general solution to the equationis:

$$X(x) = B\sin(\sqrt{\lambda}x) + C\cos(\sqrt{\lambda}x).$$

We now apply the boundary conditions. Recall that $u(0,t) = 0$ and $u(L,t) = 0$, which imply:

$$X(0) = 0, \quad X(L) = 0.$$

The boundary condition $X(0) = 0$ implies $C = 0$, so:

$$X(x) = B\sin(\sqrt{\lambda}x).$$

The boundary condition $X(L) = 0$ gives:

$$B\sin(\sqrt{\lambda}L) = 0.$$

For non-trivial solutions ($B \neq 0$), we must have:

$$\sqrt{\lambda}L = n\pi \quad \text{where} \quad n = 1, 2, 3, \ldots$$

Thus:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Now that we know the eigenvalues λ_n, the general solution for $X(x)$ is:

$$X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right).$$

The corresponding solution for $T(t)$ is:

$$T_n(t) = A_n e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}.$$

Thus, the solution to the heat equation is:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}.$$

The initial condition is $u(x,0) = f(x)$, so:

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = f(x).$$

This is a Fourier sine series expansion of $f(x)$. The coefficients B_n can be determined using the standard Fourier series formula:

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$

Thus, the solution to the heat equation is:

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx\right) \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 \alpha t}.$$

2 Higher-Dimensional Problems

In the case of higher-dimensional partial differential equations (PDEs), separation of variables can still be applied effectively, though the procedure is slightly more involved due to the increased number of spatial dimensions. A classic example is the *heat equation* in two dimensions, which models the distribution of temperature over a two-dimensional plate.

Consider the heat equation in two spatial dimensions:

$$\frac{\partial u(x,y,t)}{\partial t} = \alpha \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right),$$

where $u(x,y,t)$ is the temperature at position (x,y) and time t, and α is the thermal diffusivity constant.

For this equation, we typically have boundary conditions that describe how the temperature behaves at the edges of the plate (say, at $x = 0, L_x$ and $y = 0, L_y$) and an initial condition that gives the temperature distribution at $t = 0$.

$$u(x,y,0) = f(x,y) \quad \text{for} \quad 0 \le x \le L_x, \quad 0 \le y \le L_y,$$

with boundary conditions such as:

$$u(0,y,t) = u(L_x,y,t) = 0, \quad u(x,0,t) = u(x,L_y,t) = 0 \quad \text{for} \quad t > 0.$$

These boundary conditions imply that the edges of the plate are kept at zero temperature (for simplicity, though other conditions are possible).

We use the same idea of separation of variables and assume that the solution $u(x, y, t)$ can be factored into the product of three functions, each depending on only one of the variables:

$$u(x, y, t) = X(x)Y(y)T(t) .$$

Substituting this form into the two-dimensional heat equation:

$$\frac{\partial}{\partial t}\left(X(x)Y(y)T(t)\right) = \alpha \left(\frac{\partial^2}{\partial x^2}\left(X(x)Y(y)T(t)\right) + \frac{\partial^2}{\partial y^2}\left(X(x)Y(y)T(t)\right) \right) .$$

On the left-hand side, the partial derivative with respect to t only acts on $T(t)$. On the right-hand side, the second derivatives with respect to x and y act on $X(x)$ and $Y(y)$. Thus, the equation becomes:

$$X(x)Y(y)\frac{dT(t)}{dt} = \alpha \left(Y(y)\frac{d^2X(x)}{dx^2}T(t) + X(x)\frac{d^2Y(y)}{dy^2}T(t) \right) .$$

Now, divide through by $X(x)Y(y)T(t)$:

$$\frac{1}{\alpha T(t)}\frac{dT(t)}{dt} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} .$$

At this point, we notice that the left-hand side depends only on t, and the right-hand side depends on x and y. For the equation to hold for all values of x, y, and t, both sides must equal a constant, say $-\lambda$. This leads to two spatial eigenvalue problems and a temporal equation. For $T(t)$:

$$\frac{dT(t)}{dt} = -\lambda T(t),$$

which has the solution:

$$T(t) = Ae^{-\lambda t}.$$

For $X(x)$:

$$\frac{d^2X(x)}{dx^2} = -\mu X(x),$$

where μ is a new constant. The boundary conditions $X(0) = 0$ and $X(L_x) = 0$ imply that:

$$X(x) = B \sin\left(\sqrt{\mu}x\right) ,$$

$$\sqrt{\mu}L_x = n\pi, \quad \mu_n = \left(\frac{n\pi}{L_x}\right)^2, \quad n = 1, 2, 3, \ldots$$

For $Y(y)$:

$$\frac{d^2Y(y)}{dy^2} = -\nu Y(y),$$

where ν is a second constant, and $\mu+\nu = \lambda$. The boundary conditions $Y(0) = 0$ and $Y(L_y) = 0$ yields the solution:

$$Y(y) = C \sin\left(\sqrt{\nu}y\right) ,$$

$$\sqrt{\nu} L_y = m\pi, \quad \nu_m = \left(\frac{m\pi}{L_y}\right)^2, \quad m = 1, 2, 3, \ldots$$

The general solution is then the product of the solutions for $X(x)$, $Y(y)$, and $T(t)$:

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \sin\left(\frac{n\pi}{L_x} x\right) \sin\left(\frac{m\pi}{L_y} y\right) e^{-\alpha\left(\frac{n^2\pi^2}{L_x^2} + \frac{m^2\pi^2}{L_y^2}\right)t}.$$

The initial condition is $u(x, y, 0) = f(x, y)$, so:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \sin\left(\frac{n\pi}{L_x} x\right) \sin\left(\frac{m\pi}{L_y} y\right) = f(x, y).$$

This is a double Fourier sine series expansion of $f(x, y)$. The coefficients B_{nm} can be computed using the Fourier series formulas:

$$B_{nm} = \frac{4}{L_x L_y} \int_0^{L_x} \int_0^{L_y} f(x, y) \sin\left(\frac{n\pi}{L_x} x\right) \sin\left(\frac{m\pi}{L_y} y\right) \, dx \, dy.$$

3 Summary and Key Points

- The separation of variables method is useful for solving linear PDEs with appropriate boundary conditions.
- The method involves assuming a solution in the form of a product of functions, each depending on only one variable.
- After substituting this form into the PDE, we obtain separate ODEs for each variable.
- Eigenvalue problems emerge from boundary conditions, leading to solutions that can often be expressed as series expansions.
- For the heat equation, this method gives solutions as sums of decaying exponential functions weighted by Fourier sine series.