Integration by Parts

Podcast Learn & Fun *

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Integration by parts is a technique used to evaluate integrals where direct integration is difficult. It is derived from the product rule of differentiation, and it allows us to transform a challenging integral into simpler ones, often leveraging known formulas or easier integration methods.

The Fundamental Formula

The core idea behind integration by parts comes from the product rule of differentiation, which states that:

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$

Rearranging this, we get:

$$u'(x)v(x) = \frac{d}{dx}[u(x)v(x)] - u(x)v'(x)$$

Integrating both sides with respect to x, we obtain:

$$\int u'(x)v(x)\,dx = \int \frac{d}{dx}[u(x)v(x)]\,dx - \int u(x)v'(x)\,dx$$

Since the integral of a derivative is simply the function itself (ignoring the constant of integration), we arrive at the final formula for integration by parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

This formula expresses the integral of the product of two functions in terms of the original product and a new integral that might be easier to solve.

Strategy for Applying Integration by Parts

The key to effectively using integration by parts is selecting the appropriate functions to assign to $u(x)$ and $v'(x)$. In general, the function $u(x)$ should be

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chosen so that it is easier to differentiate, while $v'(x)$ should be chosen so that its integral is simpler.

A useful mnemonic for determining which function to pick as $u(x)$ is the acronym **LIATE**, which stands for Logarithmic, Inverse Trigonometric, Algebraic, Trigonometric, and Exponential functions. According to this rule, the preference for $u(x)$ should follow the order: Logarithmic $\rightarrow$ Inverse Trigonometric $\rightarrow$ Algebraic $\rightarrow$ Trigonometric $\rightarrow$ Exponential. This order is designed to ensure that differentiating $u(x)$ makes the integral simpler, while the integral of $v'(x)$ remains manageable.

Example 1: $\int xe^x \, dx$

Let's apply integration by parts to solve the integral $\int xe^x \, dx$.

First, using the LIATE rule, we choose:

$$u(x) = x \quad \text{and} \quad v'(x) = e^x$$

Next, we differentiate $u(x)$ and integrate $v'(x)$:

$$u'(x) = 1 \quad \text{and} \quad v(x) = e^x$$

Now, applying the formula for integration by parts:

$$\int xe^x \, dx = xe^x - \int e^x \, dx$$

The remaining integral, $\int e^x \, dx$, is straightforward:

$$\int e^x \, dx = e^x$$

Thus, the solution is:

$$\int xe^x \, dx = xe^x - e^x + C = e^x(x-1) + C$$

Example 2: $\int \ln(x) \, dx$

Now, let's solve $\int \ln(x) \, dx$ using integration by parts.

First, we choose:

$$u(x) = \ln(x) \quad \text{and} \quad v'(x) = 1$$

Differentiating $u(x)$ and integrating $v'(x)$:

$$u'(x) = \frac{1}{x} \quad \text{and} \quad v(x) = x$$

Applying the integration by parts formula:

$$\int \ln(x)\,dx = x\ln(x) - \int x \cdot \frac{1}{x}\,dx$$

The remaining integral is:

$$\int 1\,dx = x$$

Thus, the solution is:

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

Special Cases of Integration by Parts

There are certain situations in which integration by parts is applied multiple times, or the integral can be simplified further. For instance, when integrating products of polynomials and exponential or trigonometric functions, it may be necessary to apply integration by parts repeatedly. A classic example of this is the integral $\int x^n e^x\,dx$, where each application of integration by parts reduces the degree of the polynomial by one.

In addition, for certain types of integrals involving polynomials and simple functions like exponentials or trigonometric functions, the method of **tabular integration** (also known as the DI method) can be more efficient. This method involves creating a table with the derivatives of the polynomial and the integrals of the simpler function, alternating signs to simplify the process.

Applications of Integration by Parts

Integration by parts is widely used in various applications. One common use is in evaluating integrals that involve logarithmic and trigonometric functions, such as:

$$\int \ln(x)\sin(x)\,dx$$

It is also used for integrals involving powers of x and exponential functions, like:

$$\int x^n e^x\,dx$$

In more advanced applications, integration by parts plays a key role in deriving reduction formulas for integrals of trigonometric functions, as well as in the analysis of Fourier transforms and Laplace transforms.

Further Considerations

In some cases, integration by parts must be applied multiple times to reduce an integral to a solvable form. When applying the method iteratively, it is important to carefully track all terms and simplify the integrals at each step. Additionally, when dealing with definite integrals, it is crucial to properly evaluate the boundary terms—specifically, the term $u(x)v(x)$ at the limits of integration.

Conclusion

Integration by parts is an invaluable technique in calculus for solving integrals that involve products of functions. By appropriately choosing $u(x)$ and $v'(x)$, and sometimes applying the method iteratively, we can break down complex integrals into simpler ones. The method's versatility makes it essential for a wide range of integrals, from basic polynomial-exponential integrals to more complex applications in Fourier and Laplace transforms.

Practice Problems

To further solidify your understanding of integration by parts, try solving the following integrals:

1. $\int x \cos(x) \, dx$
2. $\int x^2 \ln(x) \, dx$
3. $\int e^x \sin(x) \, dx$
4. $\int \ln(x) \, dx$
5. $\int x \ln(x^2) \, dx$