Legendre Transforms

Podcast Learn & Fun *

February 4, 2025

The *Legendre transform* is a fundamental mathematical operation that connects a function defined in terms of one set of variables to a function defined in terms of another set of conjugate variables. This transformation is crucial in various fields, including thermodynamics and classical mechanics, as it provides a way to switch between different representations of physical systems. In addition to its utility in physics, the Legendre transform has deep connections to convex analysis and differential geometry, offering a powerful method for studying duality in mathematical problems.

1 Definition of Legendre Transform

Mathematically, the *Legendre transform* of a smooth, differentiable function $f(x)$, where x is a vector of variables, is defined by the expression

$$g(p) = \sup_x \left(p \cdot x - f(x) \right),$$

where p is the conjugate variable defined as:

$$p = \frac{df}{dx}.$$

The supremum (maximum) is taken over all x. The Legendre transform $g(p)$ is then constructed by maximizing the linear function $p \cdot x - f(x)$ with respect to x. If the function $f(x)$ is convex, this transformation is well-defined, and the Legendre transform provides a natural dual representation of the original function.

2 Geometrical Interpretation of the Legendre Transform

Geometrically, the Legendre transform can be viewed as a way to shift from a description of a system in terms of the variable x to a description in terms

*YouTube Channel: `https://www.youtube.com/@Podcast_Learn_Fun`

of its conjugate variable p. The original function $f(x)$ is typically plotted as a curve or surface in the x-space. For each point on this curve, one can find a tangent line (or hyperplane in higher dimensions) that best approximates the function locally. The slope of this tangent line is given by $p = df/dx$, which is the conjugate variable to x.

The Legendre transform takes the value of p as the independent variable and computes the area between the original curve and the tangent line. This area is maximized in the sense that we are looking for the best linear approximation to the function $f(x)$ at each point. The resulting function $g(p)$ represents the dual description of the system, which is often more convenient for certain physical or mathematical applications, such as moving from a position-based formulation to a momentum-based one.

3 Legendre Transform in One Dimension

Let us consider the case of a function $f(x)$ of a single variable x. The Legendre transform of $f(x)$ is constructed in a few steps. First, we define the conjugate variable p as the derivative of $f(x)$:

$$p = \frac{df}{dx}.$$

This implies that for every value of p, there is a corresponding value of x. If the function $p = df/dx$ is invertible, we can solve for x as a function of p, which we denote by $x(p)$. Next, the Legendre transform $g(p)$ is given by the expression

$$g(p) = p \cdot x(p) - f(x(p)),$$

where $x(p)$ is the inverse function of $p(x)$. This expression represents the Legendre transform as a function of the conjugate variable p.

Example: Let us consider the function $f(x) = \frac{1}{2}ax^2$.

First, we calculate the derivative of $f(x)$:

$$p = \frac{df}{dx} = ax.$$

Solving for x in terms of p, we get:

$$x = \frac{p}{a}.$$

Now, we substitute this expression for x into the formula for the Legendre transform:

$$g(p) = p \cdot \frac{p}{a} - \frac{1}{2}a\left(\frac{p}{a}\right)^2.$$

Simplifying this expression gives:

$$g(p) = \frac{p^2}{a} - \frac{p^2}{2a} = \frac{p^2}{2a},$$

which is the Legendre transform of $f(x)$.

4 Properties of the Legendre Transform

One important property of the Legendre transform is that it is *involutive*. This means that if we apply the Legendre transform twice, we recover the original function. More formally, if $g(p)$ is the Legendre transform of $f(x)$, then applying the Legendre transform to $g(p)$ will yield $f(x)$. In mathematical terms, if

$$p = \frac{df}{dx}, \quad x = \frac{dg}{dp},$$

then

$$f(x) = g(p) + p \cdot x.$$

This property allows us to switch back and forth between the two representations, making the Legendre transform a reversible operation. This reversibility is crucial in applications like thermodynamics, where one might switch between different thermodynamic potentials depending on the variables of interest.

Another important property is that the Legendre transform preserves the convexity or concavity of the function. If the original function $f(x)$ is convex, the Legendre transform $g(p)$ will also be convex. Similarly, if $f(x)$ is concave, then $g(p)$ will be concave. This duality is a key feature in convex analysis and has important implications for optimization theory, as the Legendre transform helps relate the primal and dual problems.

5 Legendre Transform in the Multivariable Case

In the multivariable case, the Legendre transform extends naturally from the one-dimensional case to functions that depend on multiple variables. Suppose we have a function $f(x_1, x_2, \ldots, x_n)$, where the function depends on n variables $x_1, x_2, \ldots, x_n$. The goal is to perform a Legendre transform with respect to each of these variables.

For each of the variables x_i, we define the conjugate variable p_i as the partial derivative of f with respect to x_i:

$$p_i = \frac{\partial f}{\partial x_i}.$$

This relationship indicates that the conjugate variables p_i are the derivatives of the function $f(x_1, x_2, \ldots, x_n)$ with respect to each of the n independent variables x_i. Intuitively, these conjugate variables represent the "slopes" or rates of change of the function f in each of the x_i-directions.

The Legendre transform in the multivariable case is defined by transitioning from the original variables $x_1, x_2, \ldots, x_n$ to the conjugate variables $p_1, p_2, \ldots, p_n$. The transformation is expressed as follows:

$$g(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} p_i x_i - f(x_1, x_2, \ldots, x_n),$$

where $g(p_1, p_2, \ldots, p_n)$ is the Legendre transform of $f(x_1, x_2, \ldots, x_n)$. In this expression, $x_1, x_2, \ldots, x_n$ are implicitly functions of $p_1, p_2, \ldots, p_n$, obtained by inverting the relations $p_i = \partial f / \partial x_i$. This inversion is possible provided that f is suitably convex, ensuring that the partial derivatives are invertible.

The term $\sum_{i=1}^{n} p_i x_i$ can be interpreted as the "total contribution" of all the conjugate variables p_i multiplied by their corresponding x_i-values, while $f(x_1, x_2, \ldots, x_n)$ represents the original function. The Legendre transform $g(p_1, p_2, \ldots, p_n)$ represents the dual function expressed in terms of the conjugate variables.

Geometrically, the Legendre transform in the multivariable case can be interpreted as follows: The function $f(x_1, x_2, \ldots, x_n)$ is a hypersurface in an n-dimensional space. For each point on this hypersurface, we can construct a tangent hyperplane whose coordinates are determined by the conjugate variables $p_1, p_2, \ldots, p_n$. The Legendre transform $g(p_1, p_2, \ldots, p_n)$ gives the value of the function at the point where this tangent hyperplane best approximates the surface. In this sense, the Legendre transform generalizes the idea of switching from a curve to its tangent line in the one-dimensional case to higher-dimensional spaces.

Example: Suppose we have a function $f(x_1, x_2) = \frac{1}{2} a x_1^2 + \frac{1}{2} b x_2^2$.

The conjugate variables are computed by taking the partial derivatives with respect to x_1 and x_2:

$$p_1 = \frac{\partial f}{\partial x_1} = a x_1, \quad p_2 = \frac{\partial f}{\partial x_2} = b x_2.$$

To perform the Legendre transform, we first solve for x_1 and x_2 in terms of p_1 and p_2:

$$x_1 = \frac{p_1}{a}, \quad x_2 = \frac{p_2}{b}.$$

Now, we can substitute these expressions into the Legendre transform formula:

$$g(p_1, p_2) = p_1 x_1 + p_2 x_2 - f(x_1, x_2) = p_1 \cdot \frac{p_1}{a} + p_2 \cdot \frac{p_2}{b} - \left(\frac{1}{2} a \left(\frac{p_1}{a} \right)^2 + \frac{1}{2} b \left(\frac{p_2}{b} \right)^2 \right).$$

Simplifying:

$$g(p_1, p_2) = \frac{p_1^2}{a} + \frac{p_2^2}{b} - \frac{p_1^2}{2a} - \frac{p_2^2}{2b} = \frac{p_1^2}{2a} + \frac{p_2^2}{2b}.$$

which is the Legendre transform of $f(x_1, x_2)$.

6 Applications of Legendre Transform

6.1 Thermodynamics

In thermodynamics, the Legendre transform is used to convert between different thermodynamic potentials, which are functions of different sets of natural variables. For example, the *internal energy* $U(S, V, N)$ is a function of entropy S, volume V, and particle number N. The *Helmholtz free energy* $F(T, V, N)$ is obtained by performing a Legendre transform with respect to S, where the conjugate variable to S is the temperature T. The Legendre transform is given by:

$$F(T, V, N) = U(S, V, N) - T \cdot S.$$

This expression switches the description from entropy S to temperature T, providing a more convenient potential for certain processes, such as those at constant temperature.

Similarly, the *Gibbs free energy* $G(T, P, N)$ is derived by performing a Legendre transform with respect to V, where the conjugate variable to V is the pressure P. The Gibbs free energy is given by:

$$G(T, P, N) = U(S, V, N) - T \cdot S - P \cdot V.$$

These thermodynamic potentials are valuable because they are formulated in terms of different natural variables, such as temperature, pressure, or volume, depending on the process under consideration.

6.2 Classical Mechanics

In classical mechanics, the Legendre transform is essential in transitioning from the *Lagrangian* formulation to the *Hamiltonian* formulation. The Lagrangian $L(q, \dot{q}, t)$ is a function of generalized coordinates q, their time derivatives (velocities) $\dot{q}$, and time t. The conjugate momenta p are defined as:

$$p = \frac{\partial L}{\partial \dot{q}}.$$

The *Hamiltonian* $H(p, q, t)$ is related to the Lagrangian by the Legendre transform:

$$H(p, q, t) = p \cdot \dot{q} - L(q, \dot{q}, t).$$

This transformation changes the description of the system from velocities $\dot{q}$ to momenta p, and the Hamiltonian formulation is particularly useful for describing systems in phase space, where both positions and momenta are treated symmetrically.

7 Conclusion

The Legendre transform is a powerful and versatile mathematical tool that allows for the conversion between different representations of a system, whether in classical mechanics, thermodynamics, or optimization theory. Geometrically, it provides a way to switch from a function expressed in terms of one set of variables to a function expressed in terms of conjugate variables. It also plays a central role in convex analysis, duality, and optimization. Understanding the Legendre transform and its properties is essential for various areas of both theoretical and applied mathematics and physics.