Quantum Tunneling

Podcast Learn & Fun *

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1 Introduction

We will be discussing an important phenomenon in quantum mechanics: **quantum tunneling**. It is a process where a particle passes through a potential barrier that it classically could not overcome, based on its energy. This phenomenon is in stark contrast to classical physics, where objects with insufficient energy to surmount a barrier are simply confined to one side of the barrier. The key concepts includes: (1) **Wave-particle duality:** Quantum mechanics tells us that particles like electrons have both particle-like and wave-like characteristics. The wave function (described by Schrödinger's equation) represents a probability distribution of where a particle might be found. (2) **Potential Barrier:** Imagine a particle that is moving towards a barrier. In classical mechanics, if the particle's energy is less than the potential energy of the barrier, it cannot cross the barrier. However, in quantum mechanics, the situation is different due to the wave-like nature of particles. (3) **Tunneling:** Quantum tunneling occurs when a particle has a nonzero probability of passing through the barrier, even if its energy is less than the height of the barrier. The idea of tunneling is essential in several areas of physics, including nuclear reactions, semiconductor devices (like tunnel diodes), and even biological systems (such as enzyme catalysis).

2 The Schrödinger Equation and Quantum Tunneling

To understand quantum tunneling mathematically, we need to solve the Schrödinger equation for a particle approaching a potential barrier.

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Time-Independent Schrödinger Equation:

The time-independent Schrödinger equation describes the behavior of a particle in a stationary state, without time dependence:

$$\hat{H}\psi(x) = E\psi(x)$$

where $\hat{H}$ is the Hamiltonian operator, which includes the kinetic and potential energy, $\psi(x)$ is the wave function of the particle, and E is the total energy of the particle..

The Hamiltonian for a particle moving in a one-dimensional potential is given by:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

where $\hbar$ is the reduced Planck's constant, m is the mass of the particle, and $V(x)$ is the potential energy function, which is the energy profile the particle experiences as it moves. For simplicity, we will look at a **one-dimensional potential barrier**.

Potential Barrier and Tunneling Scenario

Consider a potential energy function $V(x)$ that represents a rectangular barrier of height V_0 and width a, with the particle approaching from the left. The potential looks like this:

- $V(x) = 0$ for $x < 0$ (before the barrier),
- $V(x) = V_0$ for $0 \leq x \leq a$ (inside the barrier),
- $V(x) = 0$ for $x > a$ (beyond the barrier).

For simplicity, let the energy of the particle be E, and assume $E < V_0$, which means the particle does not have enough energy to classically go over the barrier. Yet, quantum mechanics allows us to investigate the probability of the particle "tunneling" through the barrier. Let's solve the Schrödinger equation for the three regions defined by the potential.

Region 1: $x < 0$ (Before the Barrier)

In this region, the potential $V(x) = 0$, so the Schrödinger equation becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

This simplifies to:

$$\frac{d^2\psi(x)}{dx^2} + k_1^2\psi(x) = 0$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

The general solution to this differential equation is:

$$\psi(x) = Ae^{ik_1 x} + Be^{-ik_1 x}$$

where A and B are constants to be determined by boundary conditions, and the exponential terms represent waves moving in opposite directions. If we follow a $e^{-iEt/\hbar}$ convention for time modulation, the first term, $Ae^{ik_1 x}$, represents a wave moving to the right (towards the barrier), and the second term, $Be^{-ik_1 x}$, represents a wave moving to the left (reflected wave).

Region 2: $0 \leq x \leq a$ (Inside the Barrier)

In the region where the potential is $V(x) = V_0$, the Schrödinger equation becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V_0\psi(x) = E\psi(x)$$

or

$$\frac{d^2\psi(x)}{dx^2} - k_2^2\psi(x) = 0$$

where

$$k_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

Since $E < V_0$, k_2 is real, which implies that the solutions inside the barrier will be exponentially decaying or growing. The general solution is:

$$\psi(x) = Ce^{-k_2 x} + De^{k_2 x}$$

Region 3: $x > a$ (Beyond the Barrier)

In the region beyond the barrier, the potential $V(x) = 0$, so the Schrödinger equation is similar to the equation in Region 1:

$$\frac{d^2\psi(x)}{dx^2} + k_1^2\psi(x) = 0$$

The general solution is:

$$\psi(x) = Fe^{ik_1 x}$$

where F is a constant to be determined. In this region, we only consider a solution that moves away from the barrier since the wave propagates from the left to the right.

Boundary Conditions

To solve for the constants A, B, C, D, and F, we apply boundary conditions. These conditions arise from the requirement that the wave function and its derivative must be continuous at the boundaries of the potential barrier ($x = 0$ and $x = a$).

Boundary at $x = 0$:

At $x = 0$, the wave functions from Regions 1 and 2 must match. This gives two conditions:

$$\psi_1(0) = \psi_2(0) \quad \text{and} \quad \psi_1'(0) = \psi_2'(0)$$

Substituting the expressions for the wave functions:

$$Ae^{ik_1(0)} + Be^{-ik_1(0)} = Ce^{-k_2(0)} + De^{k_2(0)}$$

Simplifying:

$$A + B = C + D$$

Now, for the derivatives:

$$ik_1 Ae^{ik_1(0)} - ik_1 Be^{-ik_1(0)} = -k_2 Ce^{-k_2(0)} + k_2 De^{k_2(0)}$$

Simplifying:

$$ik_1(A - B) = -k_2(C - D)$$

The combination of these two conditions gives

$$2A = C\left(1 + \frac{ik_2}{k_1}\right) + D\left(1 - \frac{ik_2}{k_1}\right)$$

Boundary at $x = a$:

At $x = a$, the wave functions from Regions 2 and 3 must also match:

$$\psi_2(a) = \psi_3(a) \quad \text{and} \quad \psi_2'(a) = \psi_3'(a)$$

Substituting the expressions for the wave functions:

$$Ce^{-k_2 a} + De^{k_2 a} = Fe^{ik_1 a}$$

For the derivatives:

$$-k_2 Ce^{-k_2 a} + k_2 De^{k_2 a} = ik_1 Fe^{ik_1 a}$$

The combination of these two conditions gives

$$2Ce^{-k_2 a} = \left(1 - \frac{ik_1}{k_2}\right) Fe^{ik_1 a}\,, \quad 2De^{k_2 a} = \left(1 + \frac{ik_1}{k_2}\right) Fe^{ik_1 a}$$

Solving for the Transmission Coefficient

Now, we combine the boundary conditions and solve for the constants. The critical relation for the transmission coefficient T comes from the ratio of the transmitted amplitude F to the incident amplitude A.

Based on the above boundary conditions, we can get a relation between A and F:

$$Ae^{-ik_1 a} = \left[\cosh(k_2 a) + \frac{i}{2}\left(\frac{k_1}{k_2} - \frac{k_2}{k_1}\right)\sinh(k_2 a)\right] F$$

Therefore, the transmission coefficient is

$$T = \left|\frac{F}{A}\right|^2 = \frac{1}{\cosh^2(k_2 a) + \frac{1}{4}\left(\frac{k_1}{k_2} - \frac{k_2}{k_1}\right)^2 \sinh^2(k_2 a)}$$

Interpretation of the Transmission Coefficient

The final result for the transmission coefficient reveals the following: The probability of tunneling decreases with the width a of the potential barrier. The particle is less likely to tunnel through a wider barrier. The energy E of the particle affects k_2. If E is close to V_0, k_2 is smaller, and the particle has a higher probability of tunneling. If E is much less than V_0, the tunneling probability becomes much smaller.

In the limit of high potential energy: $V_0 \gg E$ and wide energy barrier: $k_2 a \gg 1$, we can approximate the transmission coefficient as:

$$T \approx 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right) e^{-2k_2 a} \propto e^{-2k_2 a}$$

3 Examples of Quantum Tunneling

Quantum tunneling is not just an abstract theory; it happens in the real world. Here are a couple of notable examples:

Alpha Decay (Nuclear Tunneling)

In radioactive materials, unstable atomic nuclei can undergo a process called alpha decay. In this process, an alpha particle (which consists of two protons and two neutrons) is trapped inside the nucleus by a potential barrier. Classically, the alpha particle doesn't have enough energy to escape. However, quantum tunneling allows the alpha particle to escape the nucleus, causing radioactive decay.

Semiconductors and Tunnel Diodes

Quantum tunneling plays a key role in modern electronics, particularly in semiconductor devices. In devices like tunnel diodes, electrons tunnel through barriers in a way that would be impossible in classical physics. This is one of the reasons why semiconductor technology works, even at scales smaller than the classical barrier would allow.

Scanning Tunneling Microscope (STM)

The STM is one of the most famous applications of quantum tunneling. It allows us to image surfaces at the atomic level by measuring the tunneling current between a sharp tip and the surface. As the tip approaches the surface, electrons tunnel through the gap, and this current is used to map the surface's atomic structure with incredible precision.

Fusion in Stars

In stars, nuclear fusion happens at temperatures and pressures so high that, classically, the nuclei should not have enough energy to overcome their electrostatic repulsion (since they are both positively charged). Quantum tunneling allows these nuclei to fuse together, powering stars and providing energy for life on Earth.

4 Conclusion

Quantum tunneling is a fascinating and counterintuitive phenomenon where particles can pass through barriers they classically could not overcome. This effect arises from the wave-like properties of particles and is governed by the Schrödinger equation, with the wave function decaying exponentially within the barrier. Tunneling plays a crucial role in phenomena such as nuclear fusion in stars, the operation of tunnel diodes, and certain chemical reactions.

The solution to the Schrödinger equation in the presence of a potential barrier reveals the underlying mathematics behind quantum tunneling and shows that there is always a nonzero probability for a particle to tunnel through the barrier, depending on the barrier's height and width as well as the particle's energy.