Quantum Coherence and Decoherence

Podcast Learn & Fun *

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Quantum coherence refers to the property of a quantum system where its wavefunction exhibits well-defined phase relationships between its components. It lies at the heart of quantum mechanics and plays a crucial role in phenomena such as quantum computing, quantum optics, and even quantum biology.

At the heart of quantum mechanics is the idea of superposition: a system can exist in multiple states simultaneously, with each state associated with a probability amplitude. The *superposition principle* allows quantum systems to exhibit *interference effects*, much like how waves can interfere constructively or destructively.

1 Mathematical Framework of Coherence

When we talk about coherence, we usually refer to the ability of different components of a quantum system to interfere with each other. In other words, coherence ensures that a system's quantum states are *in sync* and can interact in a predictable manner.

To mathematically describe quantum coherence, we need to work within the formalism of quantum mechanics. The state of a quantum system is described by a *state vector* (or a *wavefunction*) in a Hilbert space. The wavefunction can be written as a linear superposition of basis states:

$$|\psi\rangle = \sum_i c_i |u_i\rangle$$

where $\{|u_i\rangle\}$ are basis states (typically, energy eigenstates or position or momentum eigenstates), and c_i are complex coefficients (called probability amplitudes). The square of the modulus of c_i, $|c_i|^2$, gives the probability of finding the system in state $|u_i\rangle$.

A coherent superposition implies that these coefficients have fixed relative phases. Mathematically, the coherence between two states $|u_i\rangle$ and $|u_j\rangle$ is captured by the *off-diagonal elements* of the density matrix, ρ, which is a more general way of describing a quantum state than just the wavefunction. The density matrix is defined as:

$$\rho = |\psi\rangle\langle\psi|$$

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For the pure state $|\psi\rangle = \sum_i c_i|u_i\rangle$, we can express the density matrix as:

$$\rho = \sum_{i,j} c_i c_j^* |u_i\rangle\langle u_j|$$

The off-diagonal elements $\rho_{ij} = c_i c_j^*$ $(i \neq j)$ represent the *coherences* between the states $|u_i\rangle$ and $|u_j\rangle$. When these off-diagonal terms are large, the system is highly coherent.

For a system to maintain coherence, these off-diagonal elements need to remain non-zero and stable over time. In pure states, coherence is fully preserved, as the off-diagonal elements of the density matrix are large and well-defined. In mixed states, however, the off-diagonal elements may be diminished or even vanish entirely, indicating that the system has lost coherence. Any loss of coherence can significantly affect the quantum properties of the system.

2 Coherence for a Single Particle

Consider a particle in a superposition of two position eigenstates $|x_1\rangle$ and $|x_2\rangle$, with the state given by:

$$|\psi(t)\rangle = \alpha|x_1\rangle + \beta|x_2\rangle$$

where α and β are complex coefficients, and we can express them in polar form as:

$$\alpha = |\alpha|e^{i\theta_1}, \quad \beta = |\beta|e^{i\theta_2}$$

The relative phase between the two states is: $\Delta\theta = \theta_1 - \theta_2$.

The density matrix for the pure state $|\psi(t)\rangle$ is:

$$\rho = |\psi(t)\rangle\langle\psi(t)|$$

Substituting the superposition $|\psi(t)\rangle = \alpha|x_1\rangle + \beta|x_2\rangle$ into this expression:

$$\rho = (\alpha|x_1\rangle + \beta|x_2\rangle)(\alpha^*\langle x_1| + \beta^*\langle x_2|)$$

Expanding this, we get:

$$\rho = |\alpha|^2|x_1\rangle\langle x_1| + \alpha\beta^*|x_1\rangle\langle x_2| + \beta\alpha^*|x_2\rangle\langle x_1| + |\beta|^2|x_2\rangle\langle x_2|$$

Thus, the density matrix in the $\{|x_1\rangle, |x_2\rangle\}$ basis is:

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix}$$

This matrix has two important components: (1) Diagonal elements: $|\alpha|^2$ and $|\beta|^2$, which represent the probabilities of the particle being found in the position states $|x_1\rangle$ and $|x_2\rangle$, respectively. (2) Off-diagonal elements: $\alpha\beta^*$ and $\beta\alpha^*$, which encode the *interference* information. These terms reflect the coherence between the two states $|x_1\rangle$ and $|x_2\rangle$.

The off-diagonal elements of the density matrix are directly responsible for quantum coherence. These elements involve the product of the coefficients α and β, which include the phase information. To better understand their significance, we rewrite the off-diagonal terms explicitly:

$$\rho_{12} = \alpha\beta^*, \quad \rho_{21} = \beta\alpha^*$$

Now, using the polar form of α and β:

$$\rho_{12} = |\alpha||\beta|e^{i(\theta_1 - \theta_2)} = |\alpha||\beta|e^{i\Delta\theta}$$

Thus, the off-diagonal elements contain the relative phase $\Delta\theta = \theta_1 - \theta_2$. This phase governs the interference between the two position states.

Coherence and Interference

When the off-diagonal elements ρ_{12} (or ρ_{21}) are non-zero, the quantum system exhibits coherence, and interference phenomena can arise. Specifically, the relative phase $\Delta\theta$ determines the nature of the interference between the states $|x_1\rangle$ and $|x_2\rangle$. The *interference* between these two states is a hallmark of quantum coherence.

For example, in the double-slit experiment, if the particle is in a superposition of two paths (represented by $|x_1\rangle$ and $|x_2\rangle$), the off-diagonal elements ρ_{12} contain the phase information that leads to an interference pattern in the probability distribution.

The time evolution of a quantum state is governed by the Schrödinger equation. For a single particle, the evolution of the wavefunction $\psi(x,t)$ is given by:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \hat{H}\psi(x,t)$$

where $\hat{H}$ is the Hamiltonian of the system. The solution to this equation will describe the evolution of the particle's wavefunction, and as long as there are no external interactions or measurements, the state will maintain coherence.

If the off-diagonal elements ρ_{12} become zero, the quantum system has lost coherence. This happens if the relative phase $\Delta\theta$ becomes undefined or randomized due to interactions with an environment (such as in *decoherence*). When the phase information is lost, the system behaves as if it is in a statistical mixture of $|x_1\rangle$ and $|x_2\rangle$ rather than a superposition of the two.

For example, consider a situation where the particle interacts with an environment in such a way that the coherence between $|x_1\rangle$ and $|x_2\rangle$ is destroyed. The density matrix would then take the form of a *mixed state*, with no off-diagonal elements:

$$\rho = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$$

In this case, the system behaves classically, and the interference pattern disappears. The system is no longer in a coherent superposition, and we can only describe it as being in one of the two states $|x_1\rangle$ or $|x_2\rangle$ with probabilities $|\alpha|^2$ and $|\beta|^2$, respectively.

Decoherence and its Implications

Now, let's talk about *decoherence*—the process through which a quantum system loses its coherence with the environment. Decoherence is one of the key challenges in quantum technologies, especially in quantum computing.

Decoherence occurs when a quantum system interacts with its environment, causing the system's wavefunction to become entangled with the environment. This interaction effectively leads to a *loss of the off-diagonal terms* in the system's density matrix. Mathematically, decoherence can be described by the Lindblad master equation, which governs the time evolution of the density matrix:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \mathcal{L}(\hat{\rho})$$

where $\hat{H}$ is the Hamiltonian of the system, and $\mathcal{L}(\hat{\rho})$ represents the dissipative processes that account for decoherence, such as the interaction with an external environment.

The critical point about decoherence is that while it destroys *quantum coherence*, it does not necessarily destroy the total probability of the system (since the diagonal elements of the density matrix can remain non-zero). However, once coherence is lost, quantum interference effects cannot be observed.

Decoherence explains why classical behavior emerges from quantum systems when they interact with their environment—this loss of coherence leads to classical probabilities rather than quantum superpositions.

Example: Loss of Coherence in the Double-Slit Experiment

Imagine a particle passing through a double slit, with its state described as a superposition of position eigenstates $|x_1\rangle$ and $|x_2\rangle$, corresponding to the two slits. The wavefunction is:

$$|\psi\rangle = \alpha|x_1\rangle + \beta|x_2\rangle$$

In the absence of decoherence, the density matrix contains off-diagonal terms that encode the relative phase between the two slits. The probability distribution $P(x)$ for finding the particle at a particular position x is:

$$P(x) = |\langle x|\psi\rangle|^2 = |\alpha|^2|\langle x|x_1\rangle|^2 + |\beta|^2|\langle x|x_2\rangle|^2 + 2\Re\left(\alpha\beta^*\langle x|x_1\rangle\langle x_2|x\rangle\right)$$

The cross term $\langle x|x_1\rangle\langle x_2|x\rangle$ is an interference term that depends on the relative phase between $|x_1\rangle$ and $|x_2\rangle$.

However, if the particle interacts with its environment in such a way that it becomes entangled with the environment (e.g., due to which-path information), the coherence between $|x_1\rangle$ and $|x_2\rangle$ is lost. The off-diagonal elements of the density matrix decay, and the interference term vanishes. The result is a disappearance of the interference pattern, and the system behaves classically as if the particle had passed through one slit or the other, rather than both.

3 Coherence in Multibody Quantum Systems

In quantum mechanics, coherence refers to the phase relationships between different components of a quantum state, which are critical for phenomena like interference and entanglement. When dealing with quantum systems composed of multiple particles, coherence can exhibit intricate behaviors due to the interactions between the individual components of the system. We will provide a mathematical framework to understand and quantify quantum coherence in such multibody systems, using the language of density matrices, quantum states, and operators.

We begin by discussing the general form of quantum states and the significance of coherence in a system of multiple particles. A quantum state in a system of multiple particles can be described by a *wavefunction* (in pure states) or by a *density matrix* (in mixed states). For N particles, the state of the system is described in a combined Hilbert space, denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, where each $\mathcal{H}_i$ corresponds to the Hilbert space of the i-th particle.

In the case of a pure state, the system's quantum state $|\psi\rangle$ is a vector in the joint Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$. The general form of a pure state is:

$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_N} c_{i_1, i_2, \ldots, i_N} |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle,$$

where $|i_j\rangle$ are the basis states for the j-th particle's Hilbert space, and $c_{i_1, i_2, \ldots, i_N}$ are complex coefficients. The coefficients satisfy the normalization condition:

$$\sum_{i_1, i_2, \ldots, i_N} |c_{i_1, i_2, \ldots, i_N}|^2 = 1.$$

A more general description of quantum states, especially mixed states, is provided by the *density matrix* ρ. For example, the density matrix for two particles can be written as:

$$\rho = \sum_{i_1, i_2, j_1, j_2} \rho_{i_1, i_2, j_1, j_2} |i_1\rangle\langle j_1| \otimes |i_2\rangle\langle j_2| = \sum_{i_1, i_2, j_1, j_2} c_{i_1, i_2} c^*_{j_1, j_2} |i_1\rangle\langle j_1| \otimes |i_2\rangle\langle j_2|.$$

The off-diagonal elements $\rho_{i_1, i_2, j_1, j_2}$ correspond to the coherences between the basis states of the two particles. These off-diagonal elements are crucial for phenomena like *quantum entanglement* and *interference*. For instance, when the particles are entangled, the off-diagonal elements in the appropriate basis (such as the total spin or position basis) do not vanish.

In general, the density matrix for a pure state $|\psi\rangle$ of N particles is given by:

$$\rho = |\psi\rangle\langle\psi| = \sum_{\substack{i_1, i_2, \ldots, i_N \\ j_1, j_2, \ldots, j_N}} c_{i_1, i_2, \ldots, i_N} c^*_{j_1, j_2, \ldots, j_N} |i_1\rangle\langle j_1| \otimes |i_2\rangle\langle j_2| \otimes \cdots \otimes |i_N\rangle\langle j_N|.$$

For mixed states, the density matrix is a statistical mixture of pure states:

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|,$$

where p_k are the probabilities of the system being in the pure states $|\psi_k\rangle$.

Classical vs Quantum Correlations

To distinguish between classical and quantum correlations in a multibody system, one often uses the *quantum mutual information* $I(\rho)$. For a bipartite system, the mutual information is defined as:

$$I(\rho) = S(\rho_A) + S(\rho_B) - S(\rho),$$

where $S(\rho)$ is the von Neumann entropy, $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$, and ρ_A and ρ_B are the reduced density matrices of subsystems A and B, respectively. The mutual information quantifies the total correlations between subsystems A and B, while the von Neumann entropy measures the uncertainty (or disorder) of a quantum state.

When the mutual information is primarily due to *quantum entanglement* (rather than classical correlations), coherence plays a crucial role in the interdependence between the particles' states.

Unitary Evolution

The time evolution of a closed quantum system is governed by the *Schrödinger equation*, and the density matrix evolves according to the unitary operator $\hat{U}(t)$:

$$\rho(t) = \hat{U}(t)\rho(0)\hat{U}^\dagger(t).$$

For a system of multiple particles, this evolution takes into account the Hamiltonian of the entire system, including both the individual particles and their interactions.

If the system evolves under a Hamiltonian $\hat{H} = \sum_{i=1}^N \hat{H}_i + \sum_{i<j} \hat{V}_{ij}$, where $\hat{H}_i$ is the Hamiltonian of the i-th particle and $\hat{V}_{ij}$ describes the interaction between particles i and j, the evolution is:

$$\rho(t) = e^{-i\hat{H}t/\hbar}\rho(0)e^{i\hat{H}t/\hbar}.$$

The unitary operator $e^{-i\hat{H}t/\hbar}$ preserves the coherence of the system, meaning that if the initial state has off-diagonal elements, these will evolve over time in a manner consistent with the Hamiltonian dynamics.

Measurement and Collapse of Coherence

When we perform a measurement on a system, the quantum state collapses to one of the eigenstates of the observable being measured. If the system was initially in a coherent superposition, the measurement will typically destroy the coherence and collapse the state into one of the eigenstates. The post-measurement state is:

$$\rho_{\text{post}} = \frac{P_m \rho P_m}{\mathrm{Tr}(P_m \rho)},$$

where P_m is the projection operator onto the measurement outcome m, and $\mathrm{Tr}(P_m\rho)$ is the probability of observing outcome m.

4 Applications in Quantum Technologies

Quantum coherence is fundamental to numerous emerging technologies, particularly quantum computing. Quantum computers rely on the ability of quantum bits (qubits) to exist in superpositions of states, which requires coherence between the qubit's states. The ability to manipulate and maintain coherence is the key challenge in building scalable quantum computers.

For instance, *quantum gates* in quantum computing operate by manipulating the quantum coherence of qubits. A quantum gate applies a unitary transformation to a qubit's state, preserving the coherence between the components of the superposition. If coherence is lost, the gate may not perform as expected, resulting in errors.

Other fields, such as *quantum optics*, rely heavily on coherence. The behavior of photons in lasers, for example, depends on the coherent superposition of quantum states. In *quantum cryptography*, maintaining coherence allows for secure communication via quantum key distribution, where the coherence of the quantum states ensures the security of the transmitted information.