Displacement Operator in Quantum Mechanics

Podcast Learn & Fun *

January 17, 2025

The displacement operator is a unitary operator that shifts or "displaces" the state of a system in phase space, and it has applications in the description of coherent states, squeezing operators, and quantum measurement theory.

The displacement operator, often denoted $\hat{D}(\alpha)$, is defined as:

$$\hat{D}(\alpha) = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right)$$

where α is a complex parameter, α^* is the complex conjugate of α, $\hat{a}$ is the annihilation (lowering) operator, and $\hat{a}^\dagger$ is the creation (raising) operator. The commutation relation for $\hat{a}$ and $\hat{a}^\dagger$ is:

$$[\hat{a}, \hat{a}^\dagger] = 1$$

The parameter α gives insight into the displacement of a quantum state in phase space, where position and momentum are conjugate variables.

Properties of the Displacement Operator

Relationship with Adjoints

One of the key consequences of the unitarity of $\hat{D}(\alpha)$ is the relationship between the displacement operator and its adjoint. Specifically:

$$\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha)$$

This means that the Hermitian conjugate (or adjoint) of the displacement operator is simply the displacement operator evaluated at the negative of the original argument.

Proof: Since $\hat{D}(\alpha)$ involves the exponential of the combination $\alpha \hat{a}^\dagger - \alpha^* \hat{a}$, we can compute its adjoint by taking the Hermitian conjugate of each component. The Hermitian conjugate of $\hat{a}^\dagger$ is $\hat{a}$, and the Hermitian conjugate of

*YouTube Channel: https://www.youtube.com/@Podcast_Learn_Fun

$\hat{a}$ is $\hat{a}^\dagger$. Therefore, the adjoint of $\alpha\hat{a}^\dagger - \alpha^*\hat{a}$ is $-\alpha\hat{a} + \alpha^*\hat{a}^\dagger$, which is precisely $\hat{D}(-\alpha)$. ■

This is a crucial property because it implies that applying the displacement operator with the parameter α is the inverse of applying it with $-\alpha$, in the sense of undoing the displacement. This property also highlights that $\hat{D}(\alpha)$ does not introduce any non-physical or non-unitary behavior, as its adjoint (inverse operation) is simply the displacement by the negative of the same amount.

Unitarity of $\hat{D}(\alpha)$

To say that $\hat{D}(\alpha)$ is *unitary* means that it preserves the inner product of quantum states, and therefore it preserves the normalization of quantum states. Mathematically, this means:

$$\hat{D}^\dagger(\alpha)\hat{D}(\alpha) = \hat{D}(\alpha)\hat{D}^\dagger(\alpha) = \mathbb{I}$$

where $\hat{D}^\dagger(\alpha)$ is the adjoint (or Hermitian conjugate) of $\hat{D}(\alpha)$, and $\mathbb{I}$ is the identity operator.

Proof: We need to calculate the product $\hat{D}^\dagger(\alpha)\hat{D}(\alpha)$. Using the result from the previous property, we have:

$$\hat{D}^\dagger(\alpha)\hat{D}(\alpha) = \hat{D}(-\alpha)\hat{D}(\alpha)$$

We now compute this product.

$$\hat{D}(-\alpha)\hat{D}(\alpha) = \exp\left(-\alpha\hat{a}^\dagger + \alpha^*\hat{a}\right)\exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)$$

To simplify this product, we use the Baker-Campbell-Hausdorff formula for the product of two exponential operators. For operators $\hat{A}$ and $\hat{B}$, the Baker-Campbell-Hausdorff formula gives:

$$\exp(\hat{A})\exp(\hat{B}) = \exp\left(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] + \cdots\right)$$

In our case, $\hat{A} = -\alpha\hat{a}^\dagger + \alpha^*\hat{a}$ and $\hat{B} = \alpha\hat{a}^\dagger - \alpha^*\hat{a}$. We compute the commutator:

$$[\hat{A}, \hat{B}] = \left[-\alpha\hat{a}^\dagger + \alpha^*\hat{a}, \alpha\hat{a}^\dagger - \alpha^*\hat{a}\right]$$

Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, we obtain:

$$[\hat{A}, \hat{B}] = \alpha\alpha^*[\hat{a}^\dagger, \hat{a}] + \alpha^*\alpha[\hat{a}, \hat{a}^\dagger] = \alpha\alpha^*(-1) + \alpha^*\alpha(1) = 0$$

Since $[\hat{A}, \hat{B}] = 0$, the higher order terms in the Baker-Campbell-Hausdorff formula also vanish. Thus, the commutator is zero, and we conclude that:

$$\hat{D}(-\alpha)\hat{D}(\alpha) = \exp(0) = \mathbb{I}$$

Hence, we have shown that:

$$\hat{D}^\dagger(\alpha)\hat{D}(\alpha) = \mathbb{I}$$

Similarly, we can compute $\hat{D}(\alpha)\hat{D}^\dagger(\alpha)$ using the same method. Since $\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha)$, we find:

$$\hat{D}(\alpha)\hat{D}^\dagger(\alpha) = \hat{D}(\alpha)\hat{D}(-\alpha) = \mathbb{I}\,. \quad \blacksquare$$

This property ensures that applying the displacement operator to any quantum state does not change its norm, i.e., if a state $|\psi\rangle$ is normalized, then $\hat{D}(\alpha)|\psi\rangle$ remains normalized:

$$\langle\psi|\hat{D}^\dagger(\alpha)\hat{D}(\alpha)|\psi\rangle = \langle\psi|\psi\rangle = 1$$

Thus, $\hat{D}(\alpha)$ does not distort the physical probabilities of measurement outcomes, making it a fundamental operator for manipulating quantum states in a physically meaningful way.

Action on Raising and Lowering Operators

The displacement operator $\hat{D}(\alpha)$ acts non-trivially on the creation and annihilation operators. We now derive the action of the displacement operator on these operators using the Baker-Campbell-Hausdorff formula.

Action on $\hat{a}$ We begin by computing the action of $\hat{D}(\alpha)$ on the annihilation operator $\hat{a}$. We are interested in the operator $\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)$. Using the Baker-Campbell-Hausdorff formula, which states that for two operators $\hat{A}$ and $\hat{B}$, we have (see the Appendix):

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots$$

We apply this formula to $\hat{A} = \alpha^*\hat{a} - \alpha\hat{a}^\dagger$ and $\hat{B} = \hat{a}$. The commutator between $\hat{A}$ and $\hat{a}$ is:

$$[\hat{A}, \hat{a}] = [\alpha^*\hat{a} - \alpha\hat{a}^\dagger, \hat{a}] = \alpha[\hat{a}, \hat{a}^\dagger] = \alpha$$

This gives the first term of the Baker-Campbell-Hausdorff expansion as α. Higher commutators vanish because the commutator $[\hat{a}, \hat{a}^\dagger] = 1$ and further commutators will involve only identity operators. Therefore, the action of the displacement operator on $\hat{a}$ is:

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha$$

Similarly, we can show:

$$\hat{D}(\alpha)\hat{a}\hat{D}^\dagger(\alpha) = \hat{a} - \alpha \quad \blacksquare$$

Action on $\hat{a}^\dagger$ Next, we compute the action of $\hat{D}(\alpha)$ on the creation operator $\hat{a}^\dagger$. We are interested in $\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha)$. Again, using the Baker-Campbell-Hausdorff formula with $\hat{A} = \alpha^*\hat{a} - \alpha\hat{a}^\dagger$ and $\hat{B} = \hat{a}^\dagger$, we find the commutator:

$$[\hat{A}, \hat{a}^\dagger] = [\alpha\hat{a}^\dagger - \alpha^*\hat{a}, \hat{a}^\dagger] = \alpha^*[\hat{a}, \hat{a}^\dagger] = \alpha^*$$

Thus, the action of $\hat{D}(\alpha)$ on $\hat{a}^\dagger$ is:

$$\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*$$

Similarly, we can show:

$$\hat{D}(\alpha)\hat{a}^\dagger\hat{D}^\dagger(\alpha) = \hat{a}^\dagger - \alpha^* \quad \blacksquare$$

These relations show that the displacement operator displaces the annihilation operator by α and the creation operator by α^*, which reflects its action in phase space.

Action on Vacuum State

A key property of the displacement operator is its action on the vacuum state $|0\rangle$. By definition, the vacuum state is annihilated by the annihilation operator $\hat{a}$, i.e.,

$$\hat{a}|0\rangle = 0.$$

Now, we want to compute the result of applying the displacement operator $\hat{D}(\alpha)$ to the vacuum state:

$$\hat{D}(\alpha)|0\rangle = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)|0\rangle.$$

Since $\hat{a}|0\rangle = 0$, the operator $\hat{a}$ does not contribute when acting on the vacuum state. Therefore, we have:

$$\hat{D}(\alpha)|0\rangle = \exp\left(\alpha\hat{a}^\dagger\right)|0\rangle.$$

We can expand the exponential operator $\exp\left(\alpha\hat{a}^\dagger\right)$ as a power series:

$$\exp\left(\alpha\hat{a}^\dagger\right) = 1 + \alpha\hat{a}^\dagger + \frac{\alpha^2}{2!}\hat{a}^{\dagger 2} + \frac{\alpha^3}{3!}\hat{a}^{\dagger 3} + \cdots .$$

Therefore, acting with the series on $|0\rangle$ gives:

$$\exp\left(\alpha\hat{a}^\dagger\right)|0\rangle = \left(1 + \alpha\hat{a}^\dagger + \frac{\alpha^2}{2!}\hat{a}^{\dagger 2} + \frac{\alpha^3}{3!}\hat{a}^{\dagger 3} + \cdots\right)|0\rangle.$$

The vacuum state $|0\rangle$ is the state with no particles, and any application of $\hat{a}^\dagger$ creates a particle in the corresponding mode: $\hat{a}^\dagger|0\rangle$ creates the single-particle state $|\alpha_1\rangle$, $\hat{a}^{\dagger 2}|0\rangle$ creates the two-particle state, and so on, we recognize that this series is simply the expansion of a coherent state. Thus, the result is a coherent state, which we denote as $|\alpha\rangle$:

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle\,. \quad \blacksquare$$

By applying the displacement operator $\hat{D}(\alpha)$ to the vacuum state, we obtain a coherent state $|\alpha\rangle$, which is a minimum-uncertainty state. This process effectively “displaces” the vacuum state in phase space, generating a state with a well-defined amplitude and phase.

Time Evolution of Coherent States

The time evolution of a coherent state is governed by the time-evolution operator $\hat{U}(t) = \exp\left(-i\hat{H}t/\hbar\right)$, where $\hat{H}$ is the Hamiltonian of the system. The state of the system at time t is:

$$|\alpha(t)\rangle = \hat{U}(t)|\alpha(0)\rangle = \hat{U}(t)\hat{D}(\alpha_0)|0\rangle.$$

Since $\hat{D}(\alpha_0)|0\rangle = |\alpha_0\rangle$, we have:

$$|\alpha(t)\rangle = \exp\left(-i\hat{H}t/\hbar\right)|\alpha_0\rangle.$$

Applications of the Displacement Operator

The displacement operator $\hat{D}(\alpha)$ plays a significant role in quantum mechanics, particularly in the study of coherent states and quantum optics. Below, we provide an overview of some of the main applications of $\hat{D}(\alpha)$.

Coherent States in Quantum Optics

One of the most important applications of the displacement operator is in the construction of coherent states. As previously discussed, the coherent state $|\alpha\rangle$ is defined as the state obtained by applying the displacement operator to the vacuum state:

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle,$$

where α is a complex number. The coherent state is a minimum-uncertainty state, meaning that it minimizes the Heisenberg uncertainty principle. This makes the coherent state an idealized description of a classical harmonic oscillator, and it is widely used in quantum optics to model light fields that behave classically, such as laser light.

The displacement operator allows us to shift the vacuum state in phase space, creating a coherent state characterized by well-defined amplitude and phase. Coherent states $|\alpha\rangle$ are eigenstates of the annihilation operator $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$

where α is the eigenvalue corresponding to the coherent state. This property is vital for the description of coherent light sources and provides an essential link between quantum mechanics and classical physics.

Quantum State Engineering

The displacement operator is also used in quantum state engineering, particularly in the creation and manipulation of squeezed states and other non-classical states. By applying the displacement operator to squeezed states, we can "displace" the state in phase space, moving it away from the origin while maintaining its squeezed nature.

A squeezed state $|z\rangle$ is defined as the state generated by applying the squeezing operator $\hat{S}(r)$ to the vacuum state:

$$|z\rangle = \hat{S}(r)|0\rangle,$$

where r is the squeezing parameter. Applying the displacement operator to this squeezed state results in a displaced squeezed state, $|\alpha, r\rangle = \hat{D}(\alpha)|z\rangle$, which has properties of both squeezed states (minimizing uncertainty in one quadrature) and coherent states (described by a specific displacement α).

These displaced squeezed states are crucial in the development of quantum information protocols, where entangled or squeezed states of light are used for tasks such as quantum key distribution and quantum computing.

Photon Number Distribution

The displacement operator is also useful in calculating the photon number distribution of a quantum state. For a coherent state $|\alpha\rangle$, the photon number distribution is given by the Poisson distribution, which can be derived by calculating the expectation value of the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ in the coherent state:

$$\langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2.$$

This result reflects the fact that the average number of photons in a coherent state is $|\alpha|^2$, and the variance in the number of photons is also $|\alpha|^2$, which is characteristic of the Poisson distribution. The displacement operator allows us to easily compute such quantities and provides a powerful tool for understanding the statistics of quantum optical systems.

Displacement Operator and Quantum Measurement

In the context of quantum measurements, the displacement operator plays a crucial role in the description of quantum states prepared by optical devices. For example, by applying the displacement operator to the vacuum state, we can describe the output of an optical system (such as a laser or a coherent light source) in terms of the coherent states that the system produces. Additionally, the displacement operator allows us to analyze the effects of different optical components, such as beamsplitters, on the phase space representation of quantum states.

Summary

The displacement operator $\hat{D}(\alpha)$ is a unitary operator that shifts the quantum state in phase space. It is primarily used to generate coherent states, which have properties similar to classical harmonic oscillators, and they saturate the Heisenberg uncertainty principle. The displacement operator plays a central role in quantum optics, quantum information, and harmonic oscillator systems, providing a tool for creating and manipulating quantum states with classical-like behavior.

Appendix

The Baker-Campbell-Hausdorff Formula on $e^{\hat{A}}\hat{B}e^{-\hat{A}}$

We aim to derive the Baker-Campbell-Hausdorff formula:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots .$$

Step 1: Define a time-dependent operator We introduce a time-dependent operator $\hat{B}(t)$ that evolves with respect to the parameter t. Specifically, we define:

$$\hat{B}(t) = e^{t\hat{A}}\hat{B}e^{-t\hat{A}},$$

where t is a parameter, and $\hat{A}$ and $\hat{B}$ are fixed operators. Note that when $t = 0$, we have:

$$\hat{B}(0) = \hat{B}.$$

Step 2: Compute the time derivative of $\hat{B}(t)$ To understand how $\hat{B}(t)$ evolves with respect to t, we compute its time derivative:

$$\frac{d}{dt}\hat{B}(t) = \frac{d}{dt}\left(e^{t\hat{A}}\hat{B}e^{-t\hat{A}}\right).$$

Using the product rule of differentiation, we get:

$$\frac{d}{dt}\hat{B}(t) = \hat{A}e^{t\hat{A}}\hat{B}e^{-t\hat{A}} - e^{t\hat{A}}\hat{B}e^{-t\hat{A}}\hat{A}.$$

Now, factoring out the exponential terms, we get:

$$\frac{d}{dt}\hat{B}(t) = e^{t\hat{A}}\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)e^{-t\hat{A}}.$$

Thus, we have:

$$\frac{d}{dt}\hat{B}(t) = e^{t\hat{A}}[\hat{A}, \hat{B}]e^{-t\hat{A}},$$

where $[\hat{A}, \hat{B}]$ is the commutator of $\hat{A}$ and $\hat{B}$.

Step 3: Solve the differential equation The equation we just derived tells us that the time derivative of $\hat{B}(t)$ is proportional to the commutator $[\hat{A}, \hat{B}]$. We can solve this differential equation by expanding $\hat{B}(t)$ in a Taylor series around $t = 0$:

$$\hat{B}(t) = \hat{B}(0) + t\frac{d}{dt}\hat{B}(0) + \frac{t^2}{2!}\frac{d^2}{dt^2}\hat{B}(0) + \cdots .$$

First, we know that $\hat{B}(0) = \hat{B}$, and the first derivative is:

$$\frac{d}{dt}\hat{B}(t)\bigg|_{t=0} = \frac{d}{dt}\hat{B}(0) = [\hat{A}, \hat{B}].$$

Next, we compute the second derivative:

$$\frac{d^2}{dt^2}\hat{B}(t)\bigg|_{t=0} = \frac{d}{dt}\left(e^{t\hat{A}}[\hat{A}, \hat{B}]e^{-t\hat{A}}\right)\bigg|_{t=0}.$$

This second derivative gives us:

$$\frac{d^2}{dt^2}\hat{B}(t)\bigg|_{t=0} = [\hat{A}, [\hat{A}, \hat{B}]].$$

We can continue this process to compute higher derivatives. In general, the n-th derivative is given by:

$$\frac{d^n}{dt^n}\hat{B}(t)\bigg|_{t=0} = [\hat{A}, [\hat{A}, \cdots, [\hat{A}, \hat{B}]]] \quad \text{(nested } n \text{ times)}.$$

Step 4: Write the full series solution Now we can write the full solution for $\hat{B}(t)$ by summing the terms of the Taylor series:

$$\hat{B}(t) = \hat{B} + t[\hat{A}, \hat{B}] + \frac{t^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots .$$

Thus, when $t = 1$, we get:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots .$$

This is the desired Baker-Campbell-Hausdorff formula. ■