The Commutator and the Uncertainty Principle

Podcast Learn & Fun *

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Introduction

We will explore the concept of the *commutator* in quantum mechanics and understand its significance in the context of measurement theory. The commutator helps us understand the relationship between two operators, and it has profound implications on whether certain physical properties can be measured simultaneously.

We will define the commutator, discuss its physical significance, and look at some important examples, particularly the position and momentum operators. Along the way, we will derive some important results and explore how these results shape the way we understand quantum measurements.

1 What is a Commutator?

In quantum mechanics, observables like position, momentum, and energy are represented by *operators*. The commutator of two operators $\hat{A}$ and $\hat{B}$ is defined as:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

Here, $\hat{A}$ and $\hat{B}$ are two operators, and $[\hat{A}, \hat{B}]$ is the commutator, which is also an operator. The key feature of the commutator is that it quantifies the difference between applying the operators in different orders.

If two operators commute, i.e., $[\hat{A}, \hat{B}] = 0$, this means that the operators are compatible. In other words, the corresponding physical observables can be measured *simultaneously* with precise values. Mathematically, this means that the operators have a *common set of eigenstates*, allowing both quantities to have well-defined values in the same state.

However, if the commutator is non-zero, i.e., $[\hat{A}, \hat{B}] \neq 0$, the operators do not commute, and it is *not possible* to measure both properties with exact precision at the same time. The act of measuring one observable will disturb the measurement of the other.

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Example: Position and Momentum

Let's now consider one of the most famous examples in quantum mechanics: the *position operator* $\hat{x}$ and the *momentum operator* $\hat{p}$.

The commutator between position and momentum is given by:

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$$

By using the canonical commutation relations, we know the result of this commutator:

$$[\hat{x}, \hat{p}] = i\hbar$$

where i is the imaginary unit and $\hbar$ is the reduced Planck constant.

This result is deeply significant because it tells us that position and momentum *do not commute*. This non-commutativity implies that position and momentum cannot be simultaneously measured with arbitrary precision. The underlying reason for this limitation stems from the fact that these operators do not share a common set of eigenstates.

2 The Heisenberg Uncertainty Principle

The commutator $[\hat{x}, \hat{p}] = i\hbar$ leads directly to one of the most famous results in quantum mechanics: the *Heisenberg uncertainty principle*.

The uncertainty principle states that there is a fundamental limit to how precisely we can know both the position and momentum of a particle at the same time. This limit is given by:

$$\sigma_x \cdot \sigma_p \geq \frac{\hbar}{2}$$

where $\sigma_x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$ is the standard deviation in position, and $\sigma_p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2}$ is the standard deviation in momentum.

This is a fundamental result of quantum mechanics, showing that the product of the uncertainties in position and momentum cannot be smaller than a fixed value, $\hbar/2$.

In general, for any two operators $\hat{A}$ and $\hat{B}$, we have

$$\sigma_A \sigma_B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right| .$$

3 Angular Momentum Operators

Another important set of operators in quantum mechanics are the *angular momentum operators*. The components of angular momentum $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ do not commute in general. The commutation relations between the components of angular momentum are:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

These commutation relations imply that we cannot simultaneously measure the components of angular momentum along different axes (e.g., $\hat{L}_x$ and $\hat{L}_y$) with arbitrary precision.

4 Why Can't We Measure Non-Commuting Quantities Simultaneously?

The inability to simultaneously measure non-commuting quantities with arbitrary precision arises because of the fundamental structure of quantum mechanics. When you try to measure one observable, the measurement process disturbs the system in a way that makes the measurement of the other observable uncertain. This disturbance comes from the *non-commuting nature* of the corresponding operators.

For example, in the case of position and momentum, the measurement of one property (say position) disturbs the momentum of the particle. Since the commutator of position and momentum is non-zero, this disturbance is unavoidable.

5 Key Takeaways

- **Commutator definition:** The commutator of two operators $\hat{A}$ and $\hat{B}$ is defined as $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. If the commutator is zero, the operators commute and can be measured simultaneously.

- **Non-zero commutator:** If $[\hat{A}, \hat{B}] \neq 0$, the operators do not commute, and their corresponding observables cannot be measured with arbitrary precision at the same time.

- **Heisenberg uncertainty principle:** The commutation relation between position and momentum leads to the uncertainty principle: $\sigma_x \cdot \sigma_p \geq \hbar/2$.

- **Physical interpretation:** The commutator provides insight into the limitations of simultaneous measurements in quantum mechanics, reflecting the inherent disturbances introduced by measurement.

6 Conclusion

The commutator is a powerful tool in quantum mechanics that helps us understand the fundamental limitations of simultaneous measurements. The position-momentum commutation relation and the resulting uncertainty principle are just one of the many ways in which non-commuting operators reveal the strange and non-intuitive nature of the quantum world. The idea that certain pairs of observables cannot be measured simultaneously with arbitrary precision is one of the key distinctions between classical and quantum systems.