Time Dependent Expectation Values and Constants of Motion

Introduction

We will focus on time-dependent expectation values, constants of motion, and the time evolution of operators.

To get started, let's remind ourselves of some key ideas in quantum mechanics. In quantum theory, physical quantities, such as energy, momentum, or position, are represented by operators. When we want to make a prediction about a physical quantity, we calculate the expectation value of the corresponding operator.

Let's break it down, starting with the basic notion of expectation values.

1 Expectation Values

In quantum mechanics, the expectation value of an operator $\hat{A}$ in a given state $|\psi\rangle$ is a number that represents the **average value** you would expect to obtain if you measured the observable corresponding to $\hat{A}$ many times in the state $|\psi\rangle$.

Mathematical Definition

The expectation value of an operator $\hat{A}$ in the state $|\psi\rangle$ is given by:

$$\langle A\rangle_\psi = \langle\psi|\hat{A}|\psi\rangle$$

This is the inner product of the state $|\psi\rangle$ with the operator $\hat{A}$ acting on it. Physically, it's the average value of the observable A when the system is in the state $|\psi\rangle$.

We can think of this as the **probability distribution** over all possible outcomes of the measurement of $\hat{A}$ in the state $|\psi\rangle$. When we write $|\psi\rangle$ in the basis of eigenstates of $\hat{A}$, the expectation value reflects the **weighted average** of the possible outcomes, weighted by the probabilities of each eigenstate.

2 Time-Dependent Expectation Values

In many quantum systems, the state of the system changes over time. Thus, the expectation value of an operator $\hat{A}$ can also change with time.

The time-dependent expectation value is written as:

$$\langle A \rangle(t) = \langle \psi(t) | \hat{A}(t) | \psi(t) \rangle$$

where $|\psi(t)\rangle$ is the state of the system at time t and $\hat{A}(t)$ is the operator, which may also evolve with time.

To understand how this changes, we need to examine how the quantum state evolves over time. The time evolution of the quantum state is governed by the **Schrödinger equation**:

$$\frac{\partial}{\partial t} |\psi(t)\rangle = \frac{1}{i\hbar} \hat{H}(t) |\psi(t)\rangle$$

where $\hat{H}(t)$ is the Hamiltonian operator of the system, and $\hbar$ is the reduced Planck's constant. The Schrödinger equation tells us how the quantum state changes over time. Additionally, the conjugate equation for the bra vector is:

$$\frac{\partial}{\partial t} \langle \psi(t) | = -\frac{1}{i\hbar} \langle \psi(t) | \hat{H}(t)$$

To compute the time derivative of the expectation value $\langle A \rangle(t)$, we apply the product rule:

$$\frac{d}{dt} \langle A \rangle(t) = \frac{d}{dt} \langle \psi(t) | \hat{A}(t) | \psi(t) \rangle$$

Expanding this:

$$\frac{d}{dt} \langle A \rangle(t) = \left(\frac{\partial}{\partial t} \langle \psi(t) | \right) \hat{A}(t) | \psi(t) \rangle + \langle \psi(t) | \frac{\partial}{\partial t} \hat{A}(t) | \psi(t) \rangle + \langle \psi(t) | \hat{A}(t) \frac{\partial}{\partial t} | \psi(t) \rangle$$

Using the Schrödinger equation and its conjugate, we get:

$$\frac{d}{dt} \langle A \rangle(t) = -\frac{1}{i\hbar} \langle \psi(t) | \hat{H}(t) \hat{A}(t) | \psi(t) \rangle + \frac{1}{i\hbar} \langle \psi(t) | \hat{A}(t) \hat{H}(t) | \psi(t) \rangle + \langle \psi(t) | \frac{\partial}{\partial t} \hat{A}(t) | \psi(t) \rangle$$

Simplifying:

$$\frac{d}{dt} \langle A \rangle(t) = \frac{1}{i\hbar} \langle \psi(t) | [\hat{A}(t), \hat{H}(t)] | \psi(t) \rangle + \langle \psi(t) | \frac{\partial}{\partial t} \hat{A}(t) | \psi(t) \rangle$$

Here, $[\hat{A}(t), \hat{H}(t)]$ is the **commutator** of $\hat{A}(t)$ and $\hat{H}(t)$, and $\partial \hat{A}(t) / \partial t$ is the explicit time dependence of the operator $\hat{A}(t)$.

3 Constants of Motion

In certain situations, the expectation value of an operator does not change over time. These operators are called **constants of motion**. For an operator $\hat{A}$ to be a constant of motion, the following conditions must be satisfied:

- The operator does not explicitly depend on time: $\frac{\partial \hat{A}}{\partial t} = 0$,
- The operator must commute with the Hamiltonian: $[\hat{A}, \hat{H}] = 0$.

When these conditions hold, the time derivative of the expectation value is zero:

$$\frac{d}{dt}\langle A \rangle = 0$$

This implies that $\langle A \rangle(t)$ is constant in time, and this is a hallmark of a conserved quantity in quantum mechanics.

4 Stationary States and Time Evolution

Now, let's discuss **stationary states**. A stationary state is one where the quantum state has a well-defined energy E_n, and the state does not change in time except for a phase factor. Mathematically, for a stationary state $|n, p\rangle$, we have:

$$\hat{H}|n, p\rangle = E_n|n, p\rangle$$

In this case, the time evolution of the state is given by:

$$|\psi(t)\rangle = e^{-iE_n(t-t_0)/\hbar}|n, p\rangle$$

This phase factor $e^{-iE_n(t-t_0)/\hbar}$ does not affect the probabilities of measurement outcomes, but it does affect the time evolution of the state in the complex plane.

When the system is in a stationary state, measurements of the observable $\hat{A}$ will always yield the same result a_p, which is the eigenvalue corresponding to the eigenstate $|n, p\rangle$. Since the state is stationary, the **probability distribution** for the measurement of $\hat{A}$ does not change with time. If the system is initially in the state $|\psi(t_0)\rangle = \sum_{n,p} c_{n,p}(t_0)|n, p\rangle$, then the probability of measuring a_p at time t_0 is:

$$P(a_p, t_0) = |c_{n,p}(t_0)|^2$$

The time evolution of the coefficients $c_{n,p}(t)$ is given by:

$$c_{n,p}(t) = e^{-iE_n(t-t_0)/\hbar}c_{n,p}(t_0)$$

Thus, the probability of measuring a_p at time t remains:

$$P(a_p, t) = |c_{n,p}(t)|^2 = |c_{n,p}(t_0)|^2 = P(a_p, t_0)$$

This shows that for a stationary state, the probability of measuring any particular outcome does not change with time.

5 Conclusion

In summary, we've discussed:

- **Expectation values**, which represent the average of a measurement in a given state.
- **Time-dependent expectation values**, which describe how expectation values evolve as the state changes over time.
- **Constants of motion**, where operators that commute with the Hamiltonian have time-independent expectation values.
- **Stationary states**, where the system's time evolution is governed by phase factors, and measurement probabilities remain constant.

These concepts are fundamental to understanding quantum mechanics and its predictive power. By mastering them, you can gain a deeper insight into how quantum systems behave and how to make predictions about the outcomes of measurements.