Applying Operators on Tensor Products

Podcast Learn & Fun *

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In quantum mechanics, linear operators are often applied to states that are elements of tensor product spaces, such as when dealing with composite systems. The concept of tensor products is central in quantum theory, and understanding how to apply operators to these product spaces is crucial for the correct description of multi-particle systems, entanglement, and measurement.

1 Tensor Product of Vector Spaces

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces (or vector spaces in a simpler context). The *tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a new vector space constructed from the elements of both $\mathcal{H}_1$ and $\mathcal{H}_2$. A general element in the tensor product space can be written as a linear combination of tensor products of elements from the individual spaces.

For instance, if $\psi_1 \in \mathcal{H}_1$ and $\psi_2 \in \mathcal{H}_2$, then their tensor product is written as $\psi_1 \otimes \psi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$. This element represents a state of a composite system formed by the subsystems described by $\mathcal{H}_1$ and $\mathcal{H}_2$.

2 Operators on Tensor Products

Operators on Individual Spaces

Suppose $\hat{A}$ is a linear operator acting on $\mathcal{H}_1$, and $\hat{B}$ is a linear operator acting on $\mathcal{H}_2$. These operators can be extended to the tensor product space in a natural way.

- $\hat{A}$ acts on the first factor, i.e., $\hat{A}(\psi_1 \otimes \psi_2) = (\hat{A}\psi_1) \otimes \psi_2$.
- $\hat{B}$ acts on the second factor, i.e., $\hat{B}(\psi_1 \otimes \psi_2) = \psi_1 \otimes (\hat{B}\psi_2)$.

Operators on Tensor Products: General Case

If we have two operators $\hat{A}$ and $\hat{B}$, acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then the operator $\hat{A} \otimes \hat{B}$ acts on the tensor product space as follows:

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$$(\hat{A} \otimes \hat{B})(\psi_1 \otimes \psi_2) = (\hat{A}\psi_1) \otimes (\hat{B}\psi_2)$$

This rule is very important because it generalizes the action of operators on composite systems. The operator $\hat{A} \otimes \hat{B}$ does not mix the subsystems but instead acts independently on each subsystem.

Generalization to More Than Two Spaces

For more than two spaces, say $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$, the tensor product space is $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. If we have operators $\hat{A}_i$ acting on $\mathcal{H}_i$, then the total operator acting on the entire system is:

$$\hat{A}_1 \otimes \hat{A}_2 \otimes \cdots \otimes \hat{A}_n$$

This operator acts on a tensor product state $\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n$ as:

$$(\hat{A}_1 \otimes \hat{A}_2 \otimes \cdots \otimes \hat{A}_n)(\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n) = (\hat{A}_1\psi_1) \otimes (\hat{A}_2\psi_2) \otimes \cdots \otimes (\hat{A}_n\psi_n)$$

3 Applying Operators to Tensor Product States

Example 1: A Two-Qubit System

Consider a two-qubit system, where the Hilbert spaces of the individual qubits are $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$. Let the operators $\hat{X}_1$ and $\hat{X}_2$ act on the first and second qubit, respectively. If we have the state:

$$\psi = \psi_1 \otimes \psi_2$$

and the operators act as follows:

$$\hat{X}_1(\psi_1 \otimes \psi_2) = (\hat{X}_1\psi_1) \otimes \psi_2, \quad \hat{X}_2(\psi_1 \otimes \psi_2) = \psi_1 \otimes (\hat{X}_2\psi_2)$$

The operator $\hat{X}_1 \otimes \hat{X}_2$ will act as:

$$(\hat{X}_1 \otimes \hat{X}_2)(\psi_1 \otimes \psi_2) = (\hat{X}_1\psi_1) \otimes (\hat{X}_2\psi_2)$$

This shows how each operator acts only on its respective subsystem, leaving the other subsystem unchanged.

Example 2: Entangled States

Consider a state $\psi = \frac{1}{\sqrt{2}}(\psi_1 \otimes \psi_2 + \psi_3 \otimes \psi_4)$ where $\psi_1, \psi_2, \psi_3, \psi_4$ are states in $\mathcal{H}_1$ and $\mathcal{H}_2$. An operator $\hat{A}_1 \otimes \hat{B}_2$ acting on this state will act as:

$$(\hat{A}_1 \otimes \hat{B}_2)\left(\frac{1}{\sqrt{2}}(\psi_1 \otimes \psi_2 + \psi_3 \otimes \psi_4)\right)$$

$$= \frac{1}{\sqrt{2}} \Big((\hat{A}_1 \psi_1) \otimes (\hat{B}_2 \psi_2) + (\hat{A}_1 \psi_3) \otimes (\hat{B}_2 \psi_4) \Big)$$

This demonstrates how operators can modify entangled states by applying the respective operator to each subsystem.

4 Summary

- Operators act independently on each factor of the tensor product space.
- The operator $\hat{A} \otimes \hat{B}$ acts as $\hat{A}$ on the first subsystem and $\hat{B}$ on the second subsystem.
- Tensor products of operators extend to more than two spaces.
- The action of operators on entangled states is similarly independent on each subsystem, but the overall effect may still change the correlations between subsystems.

Understanding how operators act on tensor product spaces is essential for a deeper comprehension of composite systems in quantum mechanics and other fields involving multi-component systems.