Quantum Phases and Their Significance

Podcast Learn & Fun [*]

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In quantum mechanics, the concept of phase plays a central role in the behavior of quantum systems. The phase, often referred to as the quantum phase, manifests not only in the mathematical form of wavefunctions but also in the physical phenomena observed in experiments. Understanding phase is critical for grasping the full spectrum of quantum behavior, including interference, superposition, and quantization.

1 Global and Relative Phases

The phase in quantum mechanics has profound consequences on observable quantities. First, recall that the probability density of finding a particle at a position x is given by the squared modulus of the wavefunction:

$$P(x,t) = |\psi(x,t)|^2$$

This expression shows that a *global phase* factor $e^{i\alpha}$, where α is a constant, does not affect any physical measurement. This is because the probability density, which is related to the square of the wavefunction's modulus, is independent of the global phase. For example, for a single state $\psi' = e^{i\alpha}\psi$, we have:

$$P(x,t) = |\psi'(x,t)|^2 = |e^{i\alpha}\psi(x,t)|^2 = |\psi(x,t)|^2$$

Hence, global phase changes leave all physical observables unchanged.

However, *relative phases* between different components of a superposition state are physically significant. Consider a superposition of two quantum states:

$$\psi_{\text{total}} = \frac{1}{\sqrt{2}}\left(\psi_1 + e^{i\theta}\psi_2\right)$$

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The relative phase θ between the two components will affect the probability distributions, as seen in interference experiments. For example, in the famous double-slit experiment, the phase difference between paths leads to constructive or destructive interference, thus determining the overall probability distribution on the detection screen.

The total probability at a point on the screen is given by:

$$P = |\psi_{\text{total}}|^2 = |\psi_1 + e^{i\theta}\psi_2|^2$$

Expanding this expression:

$$P = |\psi_1|^2 + |\psi_2|^2 + 2\,\text{Re}(\psi_1^* e^{i\theta}\psi_2)$$

Here, the interference term $\text{Re}(\psi_1^* e^{i\theta}\psi_2)$ depends explicitly on the relative phase θ, influencing the intensity pattern observed. If $\theta = 0$, constructive interference occurs, while for $\theta = \pi$, destructive interference happens.

2 Quantum Entanglement

In quantum mechanics, the concept of *entanglement* plays a central role in many quantum phenomena, such as quantum teleportation and quantum cryptography. Entangled states represent a situation where the properties of two or more quantum particles (e.g., photons, electrons, etc.) are strongly correlated, regardless of the distance between them.

Consider the two-particle entangled quantum state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{i\theta}|11\rangle)$$

This is an example of a *superposition* of two basis states $|00\rangle$ and $|11\rangle$, where the qubits are in the computational basis states $|0\rangle$ and $|1\rangle$. The probability of measuring the first qubit in state $|0\rangle$ is determined by the amplitude $1/\sqrt{2}$ associated with the $|00\rangle$ term in the superposition. Similarly, the probability of measuring the first qubit in state $|1\rangle$ is determined by the amplitude $e^{i\theta}/\sqrt{2}$ associated with the $|11\rangle$ term.

Suppose we measure the first particle in the computational basis $|0\rangle$ or $|1\rangle$. The measurement of the first particle will collapse the state of the second particle, leading to the following outcomes: (1) If the first particle is measured in state $|0\rangle$: The entangled state will collapse to $|00\rangle$, and the second particle will also be in state $|0\rangle$. (2) If the first particle is measured in state $|1\rangle$: The entangled state will collapse to $|11\rangle$, and the second particle will also be in state $|1\rangle$.

3 The Role of Phase in the Hadamard Basis

Consider the two-particle entangled state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{i\theta}|11\rangle)$$

This state is a superposition of two terms: $|00\rangle$ and $|11\rangle$, with a relative phase $e^{i\theta}$ between them. We are interested in how this phase affects the measurement outcomes when we transform the basis in which we measure the qubits. In many quantum algorithms, we often want to measure qubits in a different basis, such as the *Hadamard basis*.

The *Hadamard basis* is defined by the states $|+\rangle$ and $|-\rangle$, which are superpositions of the computational basis states:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

These states are obtained by applying the Hadamard operator $\hat{H}$ on the computational basis states as follows:

$$\hat{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad \hat{H}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Thus, if we apply a Hadamard transform to a qubit, it maps $|0\rangle$ to $|+\rangle$ and $|1\rangle$ to $|-\rangle$.

Now, consider the two-particle entangled state $\psi = (|00\rangle + e^{i\theta}|11\rangle)/\sqrt{2}$. If we apply a Hadamard transform to both qubits, we are mapping the computational basis states to the Hadamard basis states.

Let's see what happens when we apply the Hadamard transform H on this state:

$$\hat{H} \otimes \hat{H} \left(\frac{1}{\sqrt{2}} \left(|00\rangle + e^{i\theta}|11\rangle \right) \right) = \frac{1}{\sqrt{2}} \left(\hat{H}|0\rangle \otimes \hat{H}|0\rangle + e^{i\theta}\hat{H}|1\rangle \otimes \hat{H}|1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + e^{i\theta}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right)$$

$$= \frac{1}{2} \Big((|00\rangle + |01\rangle + |10\rangle + |11\rangle) + e^{i\theta}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \Big)$$

$$= \frac{1}{2} \Big((1 + e^{i\theta})|00\rangle + (1 - e^{i\theta})|01\rangle + (1 - e^{i\theta})|10\rangle + (1 + e^{i\theta})|11\rangle \Big)$$

At this stage, the state is a superposition of $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, but with coefficients that depend on the phase θ. To measure the second qubit in the Hadamard basis, we need to express the states in terms of $|+\rangle$ and $|-\rangle$: Start with $|00\rangle$. Applying the definitions of $|+\rangle$ and $|-\rangle$, we can rewrite $|00\rangle$ as:

$$
\begin{aligned}
|00\rangle &= \left(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right) \\
&= \frac{1}{2}(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle)
\end{aligned}
$$

Similarly, for $|11\rangle$, we can express it as:

$$
\begin{aligned}
|11\rangle &= \left(\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\right) \\
&= \frac{1}{2}(|++\rangle - |+-\rangle - |-+\rangle + |--\rangle)
\end{aligned}
$$

Now, let's consider the state $|\psi\rangle$. Substitute the expressions for $|00\rangle$ and $|11\rangle$ in terms of the Hadamard basis:

$$
|\psi\rangle = \frac{1}{2}\left((1+e^{i\theta})|++\rangle + (1-e^{i\theta})|+-\rangle + (e^{i\theta}-1)|-+\rangle + (1+e^{i\theta})|--\rangle\right)
$$

When we measure the second qubit in the Hadamard basis, we are measuring in the $|+\rangle$, $|-\rangle$ basis. The state ψ that we derived will have certain coefficients in front of the $|+\rangle$ and $|-\rangle$ terms. The probabilities of measuring the second qubit in those states will depend on these coefficients, and hence on the phase θ.

Quantum Information Applications: Teleportation and Cryptography

The *relative phase* θ is central to quantum information protocols, such as *quantum teleportation* and *quantum cryptography.* These protocols rely on the *non-local correlations* between entangled particles. In quantum teleportation, for instance, two parties (Alice and Bob) share an entangled pair of qubits. Alice measures her qubit, and based on the outcome, she sends classical information to Bob. Bob then uses this classical information to perform a unitary operation on his qubit, effectively "teleporting" the quantum state.

The *phase of the entangled state* governs how the measurement outcomes at Alice's side correlate with Bob's operations. Changing θ changes these correlations, which can affect the success probability of teleportation, and similarly for cryptographic protocols such as *quantum key distribution* (QKD), where phase plays a role in the security of the transmitted information.

4 The Aharonov-Bohm Effect

The Aharonov-Bohm effect is a striking example of the physical importance of phase. In this phenomenon, a charged particle moving in a region with a magnetic field will experience a phase shift even if the field does not directly interact with the particle. The phase shift can be observed by measuring interference patterns, even in regions where the magnetic field is absent, indicating that the vector potential (and thus the phase) is an important physical quantity in quantum mechanics.

5 Summary and Conclusion

To summarize, the phase in quantum mechanics plays a crucial role in determining the outcomes of experiments and is embedded within the mathematical description of quantum states. While global phases are unobservable, relative phases lead to measurable interference effects and influence physical phenomena such as entanglement, interference, and the Aharonov-Bohm effect. Understanding the significance of phase is essential for advancing both theoretical and applied quantum mechanics, especially in fields like quantum computing, quantum cryptography, and quantum field theory.