Observables in Quantum Physics

Podcast Learn & Fun *

December 25, 2024

Introduction

Observables in quantum mechanics is a crucial topic in understanding the measurement process, the structure of quantum theories, and how we make predictions about the physical world at the quantum level. Today, we will explore the mathematical structure of observables, their connection to physical measurements, and how different representations affect our interpretation of quantum systems.

1 The Concept of Observables

In classical mechanics, we are accustomed to the idea of observable quantities such as position, momentum, energy, and angular momentum. These quantities have well-defined values at any given time. However, quantum mechanics operates quite differently, and observables in quantum theory represent measurable quantities, but their values are not always definite until a measurement is made.

In quantum mechanics, observables are represented by **operators** acting on the quantum state of the system. These operators are not just abstract mathematical objects but play a crucial role in predicting the results of measurements.

1.1 Definition of Observables

An **observable** in quantum mechanics is associated with a Hermitian operator $\hat{O}$, where the operator is said to be Hermitian because the expectation value of an observable must be a real number, as that corresponds to the outcome of a measurement.

Mathematically, we can express this by saying that for a quantum state $|\psi\rangle$, the expectation value of an observable $\hat{O}$ is given by:

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle \, .$$

This expectation value represents the average of many measurements of the observable $\hat{O}$ on identically prepared systems in the state $|\psi\rangle$.

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2 The Role of Eigenstates and Eigenvalues

For an observable to have a well-defined value when measured, the quantum state of the system must be an **eigenstate** of the corresponding operator. This brings us to two crucial concepts: **eigenvalues** and **eigenstates**.

Let's suppose $\hat{O}$ is the operator corresponding to some observable. If $|\phi\rangle$ is an eigenstate of $\hat{O}$, then:

$$\hat{O}|\phi\rangle = o|\phi\rangle \, ,$$

where o is the **eigenvalue** corresponding to the observable $\hat{O}$. The eigenvalue o is the specific value that would be measured if the system is in the eigenstate $|\phi\rangle$.

The set of eigenstates $\{|\phi_n\rangle\}$ of the observable operator $\hat{O}$ forms a **basis** for the Hilbert space, and any quantum state $|\psi\rangle$ can be written as a linear combination (or superposition) of these eigenstates:

$$|\psi\rangle = \sum_n c_n |\phi_n\rangle \, ,$$

where c_n are complex coefficients. The probability of measuring a particular eigenvalue o_n is given by the modulus squared of the coefficient c_n:

$$P(o_n) = |c_n|^2 \, .$$

This probabilistic nature is a hallmark of quantum mechanics and distinguishes it from classical mechanics.

3 Representations of Observables

Quantum mechanics can be described in various representations, depending on the choice of basis. The most common representations are the **position** and **momentum** representations, but we can also work in other bases, such as the energy basis. Let's explore these representations.

3.1 Position and Momentum Representations

In the **position representation**, the quantum state is described as a wavefunction $\psi(x)$, which is the projection of the quantum state onto the position eigenstates $|x\rangle$. In this case, the position operator $\hat{x}$ and momentum operator $\hat{p}$ act as differential operators:

$$\hat{x}\psi(x) = x\psi(x) \, ,$$

$$\hat{p}\psi(x) = -i\hbar \frac{\partial \psi(x)}{\partial x} \, .$$

Here, the eigenstates of $\hat{x}$ are the Dirac delta functions $\delta(x - x_0)$, and the eigenstates of $\hat{p}$ are plane waves of the form e^{ikx}.

In the **momentum representation**, the wavefunction is denoted as $\tilde{\psi}(p)$, and the operators take the form:

$$\hat{x}\tilde{\psi}(p) = i\hbar \frac{\partial \tilde{\psi}(p)}{\partial p} ,$$

$$\hat{p}\tilde{\psi}(p) = p\tilde{\psi}(p) .$$

The choice between position or momentum representation often depends on the specific problem at hand. Some problems, such as those dealing with particles in potentials, are more easily solved in the position representation, while others, like free particles, are more naturally described in the momentum representation.

3.2 Energy Representation

In quantum mechanics, energy is often one of the most important observables. In many systems, especially those with time-independent Hamiltonians $\hat{H}$, the energy eigenstates $|E_n\rangle$ form a basis for the system's Hilbert space.

For a time-independent Hamiltonian, the Schrödinger equation:

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

describes how the energy eigenstates $|E_n\rangle$ evolve. The energy eigenvalues E_n are the possible measured energy values when the system is in a corresponding eigenstate.

3.3 Spin Representations

For systems involving spin, such as electrons, observables like the **spin operator** $\hat{S}_z$ (for spin in the z-direction) have eigenstates $|+\rangle$ and $|-\rangle$, with eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ respectively. These states form a basis for spin-1/2 systems, and the general state of the spin can be expressed as a superposition of these eigenstates.

4 The Measurement Process

A central feature of quantum mechanics is that the act of measurement causes a collapse of the quantum state. When an observable $\hat{O}$ is measured, the state $|\psi\rangle$ collapses to one of the eigenstates of $\hat{O}$, and the corresponding eigenvalue is obtained as the measurement outcome.

If the system is initially in a superposition of eigenstates of $\hat{O}$, the measurement process is probabilistic. The probability of measuring a particular eigenvalue o_n is given by:

$$P(o_n) = |\langle \phi_n|\psi\rangle|^2 ,$$

where $|\phi_n\rangle$ is the eigenstate of $\hat{O}$ corresponding to the eigenvalue o_n.

After the measurement, the state of the system is no longer the initial superposition $|\psi\rangle$, but rather the eigenstate corresponding to the measured eigenvalue.

5 Summary

To summarize, in quantum mechanics:

- Observables correspond to Hermitian operators, whose eigenvalues represent the possible measurement outcomes.
- The quantum state can be expressed as a superposition of eigenstates of an observable, with the coefficients determining the probability of obtaining each eigenvalue upon measurement.
- Different representations, such as position, momentum, and energy, offer different mathematical descriptions of observables, and the choice of representation depends on the problem at hand.
- The act of measurement causes the collapse of the quantum state, which is a key feature distinguishing quantum mechanics from classical physics.