Angular Momentum in Quantum Mechanics

Podcast Learn & Fun *

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Classical Angular Momentum and Transition to Quantum Mechanics

In classical mechanics, the angular momentum $\mathbf{L}$ of a point particle is defined as the cross product of its position vector $\mathbf{r}$ and its linear momentum $\mathbf{p}$:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

This vector quantity describes the rotational motion of the particle about the origin. In quantum mechanics, however, we do not deal with classical vectors but instead use operators to represent physical observables.

The quantum mechanical counterpart of angular momentum is represented by an operator $\hat{\mathbf{L}}$. In this framework, quantum states, denoted by kets $|\psi\rangle$, are the fundamental objects on which operators act. Angular momentum is described by the operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ for the components of angular momentum along the x-, y-, and z-axes, respectively. The total angular momentum is represented by the operator $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$.

Commutation Relations for Angular Momentum Operators

The angular momentum operators obey the following fundamental commutation relations:

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{jk\ell}\hat{L}_k$$

where $j, k, \ell \in \{x, y, z\}$, and $\epsilon_{jk\ell}$ is the Levi-Civita symbol. These commutation relations express the non-commutative nature of the angular momentum components, meaning that it is impossible to simultaneously measure the three components of angular momentum with arbitrary precision. This reflects the underlying uncertainty principle in quantum mechanics.

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Importantly, the operators $\hat{L}^2$ and $\hat{L}_z$ commute with each other:

$$[\hat{L}^2, \hat{L}_z] = 0$$

Thus, they share a common set of eigenstates, and we can diagonalize both operators simultaneously.

Eigenvalue Equations for Angular Momentum Operators

The eigenvalue equations for the angular momentum operators are central to understanding their quantization. Consider a quantum state $|\psi\rangle$ that is an eigenstate of both $\hat{L}^2$ and $\hat{L}_z$:

$$\hat{L}^2|\psi\rangle = \hbar^2 l(l+1)|\psi\rangle$$
$$\hat{L}_z|\psi\rangle = \hbar m|\psi\rangle$$

Here, l is the orbital angular momentum quantum number, which can take integer or half-integer values, and m is the magnetic quantum number, associated with the projection of angular momentum along the z-axis. The quantum number m can take values ranging from $-l$ to l in integer steps. These quantized values for angular momentum arise from the non-commuting nature of the angular momentum components and the requirement that the system's wavefunction be single-valued.

The quantum number l represents the total angular momentum, and m represents the component of angular momentum along the z-axis. Thus, the eigenvalues of $\hat{L}^2$ are $\hbar^2 l(l+1)$, and the eigenvalues of $\hat{L}_z$ are $\hbar m$.

Spherical Harmonics and Angular Part of the Wavefunction

The eigenfunctions of the angular momentum operators $\hat{L}^2$ and $\hat{L}_z$ are the spherical harmonics, $Y_l^m(\theta, \phi)$, which form a complete orthonormal set for describing the angular part of the wavefunction. These functions are solutions to the angular part of the Schrödinger equation when expressed in spherical coordinates (r, θ, ϕ).

The spherical harmonics $Y_l^m(\theta, \phi)$ are defined as:

$$Y_l^m(\theta, \phi) = N_{l,m} P_l^m(\cos\theta) e^{im\phi}$$

where $P_l^m(\cos\theta)$ are the associated Legendre polynomials, and $N_{l,m}$ is a normalization constant. The spherical harmonics satisfy the following orthonormality condition:

$$\int_0^\pi \int_0^{2\pi} Y_l^m(\theta, \phi) Y_{l'}^{m'}(\theta, \phi)^* \sin\theta \, d\theta \, d\phi = \delta_{l,l'} \delta_{m,m'}$$

This orthonormality is important for expressing wavefunctions as linear combinations of angular momentum eigenstates. The spherical harmonics describe the angular dependence of the wavefunction, and the quantum numbers l and m determine the shape and orientation of the corresponding angular wavefunction.

Addition of Angular Momenta

When multiple particles with individual angular momenta interact, the total angular momentum of the system is the vector sum of the individual angular momenta. Consider two angular momenta $\hat{L}_1$ and $\hat{L}_2$, corresponding to two particles, with quantum numbers l_1 and l_2. The total angular momentum quantum number L of the system can take values in the range:

$$L = |l_1 - l_2|, |l_1 - l_2| + 1, \ldots, l_1 + l_2$$

This is the result of the addition of angular momenta, and the total magnetic quantum number M for the system can range from $-L$ to L in integer steps.

The process of adding angular momenta is mathematically formalized using Clebsch-Gordan coefficients, which describe how the product of two angular momentum states can be decomposed into a sum of total angular momentum states. If the system is composed of two particles with angular momenta l_1 and l_2, the total state $|L, M\rangle$ can be written as a linear combination of the product states $|l_1, m_1\rangle|l_2, m_2\rangle$, where m_1 and m_2 are the magnetic quantum numbers for each particle.

Spin Angular Momentum

In addition to orbital angular momentum, quantum particles also possess intrinsic angular momentum called spin. Spin is characterized by the spin quantum number s, which can take integer or half-integer values. For a particle with spin s, the eigenvalues of the spin angular momentum operators $\hat{S}^2$ and $\hat{S}_z$ are:

$$\hat{S}^2|s, m_s\rangle = \hbar^2 s(s+1)|s, m_s\rangle$$

$$\hat{S}_z|s, m_s\rangle = \hbar m_s|s, m_s\rangle$$

where m_s is the spin quantum number and can take values from $-s$ to s in integer steps. For example, for an electron, $s = \frac{1}{2}$, so $m_s = \pm\frac{1}{2}$.

The total spin angular momentum and its component along any axis are treated similarly to orbital angular momentum, and spin plays a crucial role in determining the quantum statistics of particles. Fermions, with half-integer spin, obey the Pauli exclusion principle, while bosons, with integer spin, do not.

Conclusion

Angular momentum in quantum mechanics is a deeply fundamental concept that governs the rotational properties of quantum systems. Through the use of operators $\hat{L}^2$, $\hat{L}_z$, and the associated spherical harmonics, quantum states with well-defined angular momentum are mathematically described. The quantization of angular momentum, the addition of angular momenta, and the inclusion of spin angular momentum provide a rigorous framework for understanding the dynamics of both single-particle and multi-particle quantum systems. The conservation of angular momentum, stemming from rotational symmetry, underpins many physical phenomena, from atomic spectra to particle interactions in quantum field theory.