# Bloch Representation

Podcast Learn & Fun *

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The *Bloch representation* is a powerful mathematical framework used to describe quantum states in the context of quantum mechanics, particularly when dealing with qubits and two-level systems. The representation provides a clear and intuitive geometrical picture of quantum states, and it is especially useful in the analysis of spin systems, qubits, and other two-level quantum systems. In this lecture, we will derive the essential features of the Bloch representation, including its connection to quantum mechanics, and we will explore its properties, operations, and applications.

## The State Space of a Two-Level Quantum System

A *two-level quantum system* can be described by a quantum state $|\psi\rangle$ in a two-dimensional Hilbert space. The most general state of such a system is a linear combination of the two orthonormal basis states $|0\rangle$ and $|1\rangle$, which are the computational basis states.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha$ and $\beta$ are complex numbers, and the normalization condition requires that:

$$|\alpha|^2 + |\beta|^2 = 1$$

We can write the coefficients $\alpha$ and $\beta$ in polar form:

$$\alpha = re^{i\theta_1}, \quad \beta = se^{i\theta_2}$$

where $r, s$ are real non-negative numbers, and $\theta_1, \theta_2$ are phase angles. Since the overall global phase $e^{i\gamma}$ of the state does not affect its physical properties, we can always choose to remove it by adjusting $\alpha$ and $\beta$ appropriately. This leaves us with a single complex number degree of freedom, ensuring that the general form of the state can be reduced to a simpler expression.

Alternatively, we can use a more geometrical approach by noting that any state of the two-level system can be written in terms of a *Bloch vector*.

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## The Bloch Sphere

The *Bloch sphere* is a geometric representation of quantum states in a two-level system. It provides an elegant and intuitive way to visualize pure quantum states and their evolution. Any pure state $|\psi\rangle$ of a two-level system can be parametrized by two angles, $\theta$ and $\phi$, in spherical coordinates. This allows us to express the state as a point on the unit sphere.

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

where $\theta \in [0, \pi]$ is the polar angle (also called the "latitude"), $\phi \in [0, 2\pi)$ is the azimuthal angle (also called the "longitude").

This parametrization reflects the fact that the state vector lies on the surface of a unit sphere, with the polar angle $\theta$ specifying the "height" of the state on the sphere, and the azimuthal angle $\phi$ determining its position in the plane.

The state $|\psi\rangle$ on the Bloch sphere corresponds to a point with spherical coordinates $(\theta, \phi)$. The vector can be associated with the vector $\vec{r}$ in three-dimensional space:

$$\vec{r} = \sin(\theta)\cos(\phi)\hat{\imath} + \sin(\theta)\sin(\phi)\hat{\jmath} + \cos(\theta)\hat{k}$$

This is known as the Bloch vector.

## Density Matrices

The Bloch sphere representation can be extended to describe mixed states. In this case, a quantum state is represented by a *density matrix* $\rho$, which is a more general description than the pure state vector $|\psi\rangle$.

**Pure States** The density matrix of a pure state is given by:

$$\rho = |\psi\rangle\langle\psi|$$

The density matrix corresponding to this pure two-level system state is:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

Using the trigonometric identities:

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}, \quad \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}, \quad \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\sin\theta}{2},$$

we can rewrite the density matrix as:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix}$$

In the Bloch representation, the density matrix $\rho$ of a two-level system can be written as:

$$\rho = \frac{1}{2}\left(\mathbf{I} + \vec{r}\cdot\vec{\sigma}\right) = \frac{1}{2}\left(\mathbf{I} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right)$$

where $\vec{r} = (r_x, r_y, r_z)$ is the Bloch vector (a 3D vector), which represents the state in the Bloch sphere, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices, and $\mathbf{I}$ is the identity matrix.

The Pauli matrices are essential tools in the Bloch representation, as they correspond to the quantum mechanical operators acting on the state of the system. The Pauli matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices satisfy the following properties:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I, \quad \sigma_x\sigma_y = i\sigma_z, \quad \sigma_y\sigma_z = i\sigma_x, \quad \sigma_z\sigma_x = i\sigma_y$$

Using the Pauli matrices, any quantum state can be described by the density matrix expression mentioned earlier.

**Mixed States** A mixed state is a probabilistic mixture of pure states. The density matrix for a mixed state is written as:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

where $p_i$ are the probabilities associated with the pure states $|\psi_i\rangle$, and $\sum_i p_i = 1$. For a general mixed state, the density matrix is not a projector, and it can have rank greater than 1.

Consider a mixed state as a probabilistic mixture of two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ with probabilities $p_1$ and $p_2$, where $p_1 + p_2 = 1$. The density matrix for each pure state is:

$$\rho_i = |\psi_i\rangle\langle\psi_i| = \frac{1}{2}\left(\mathbf{I} + \vec{r}_i\cdot\vec{\sigma}\right)$$

where $\vec{r}_i = (r_{i,x}, r_{i,y}, r_{i,z})$ is the Bloch vector for the state $|\psi_i\rangle$. The total density matrix is:

$$\rho = p_1\rho_1 + p_2\rho_2 = p_1\frac{1}{2}\left(\mathbf{I} + \vec{r}_1\cdot\vec{\sigma}\right) + p_2\frac{1}{2}\left(\mathbf{I} + \vec{r}_2\cdot\vec{\sigma}\right)$$

Simplifying, we get:

$$\rho = \frac{1}{2}\left((p_1 + p_2)\mathbf{I} + p_1\vec{r}_1\cdot\vec{\sigma} + p_2\vec{r}_2\cdot\vec{\sigma}\right)$$

Since $p_1 + p_2 = 1$, this becomes:

$$\rho = \frac{1}{2}\left(\mathbf{I} + (p_1\vec{r}_1 + p_2\vec{r}_2)\cdot\vec{\sigma}\right)$$

We define the effective Bloch vector for the mixed state as:

$$\vec{r} = p_1\vec{r}_1 + p_2\vec{r}_2$$

Thus, for the mixed state, the density matrix is:

$$\rho = \frac{1}{2}\left(\mathbf{I} + \vec{r}\cdot\vec{\sigma}\right)$$

where $\vec{r}$ is the weighted sum of the Bloch vectors of the pure states, and $|\vec{r}|$ is less than 1 for a mixed state.

**Conclusion** The Bloch representation

$$\rho = \frac{1}{2}\left(\mathbf{I} + \vec{r}\cdot\vec{\sigma}\right)$$

is valid for both pure and mixed states. For pure states, $|\vec{r}| = 1$, and for mixed states, $|\vec{r}| < 1$. The maximally mixed state corresponds to $\vec{r} = 0$, where $\rho = (1/2)\mathbf{I}$. This corresponds to the case where the qubit is in the states $|0\rangle$ or $|1\rangle$ with equal probability.

The trace of the density matrix must be equal to 1. Using the properties of the Pauli matrices, where $\mathrm{Tr}(\sigma_i) = 0$ for $i = x, y, z$ and $\mathrm{Tr}(\mathbf{I}) = 2$, we get:

$$\mathrm{Tr}(\rho) = \frac{1}{2}\left(\mathrm{Tr}(\mathbf{I}) + r_x\mathrm{Tr}(\sigma_x) + r_y\mathrm{Tr}(\sigma_y) + r_z\mathrm{Tr}(\sigma_z)\right) = \frac{1}{2}(2) = 1$$

This confirms that the normalization condition is satisfied.

## Expectation Values of the Pauli Matrices

The components of the Bloch vector $\vec{r}$ can be related to the expectation values of the Pauli matrices:

$$r_x = \langle\sigma_x\rangle, \quad r_y = \langle\sigma_y\rangle, \quad r_z = \langle\sigma_z\rangle$$

These quantities represent the expectation values of the corresponding Pauli matrices in the state $\rho$.
**Proof:** The expectation value of a Pauli matrix $\sigma_i$ in the quantum state $\rho$ is:

$$\langle\sigma_i\rangle = \mathrm{Tr}(\rho\sigma_i)$$

Using the expression for $\rho$, we can compute the expectation value of each Pauli matrix:

$$\langle\sigma_x\rangle = \mathrm{Tr}\left(\rho\sigma_x\right) = \mathrm{Tr}\left(\frac{1}{2}(\mathbf{I} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)\sigma_x\right)$$

Since the Pauli matrices are traceless, and using the property $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$, we get:

$$\begin{aligned}\langle\sigma_x\rangle &= \frac{1}{2}\left(\mathrm{Tr}(\mathbf{I}\sigma_x) + r_x\mathrm{Tr}(\sigma_x^2) + r_y\mathrm{Tr}(\sigma_y\sigma_x) + r_z\mathrm{Tr}(\sigma_z\sigma_x)\right)\\ &= \frac{1}{2}\left(0 + r_x\cdot 2 + 0 + 0\right) = r_x\end{aligned}$$

Similarly, for $\sigma_y$:

$$
\begin{aligned}
\langle \sigma_y \rangle = \mathrm{Tr}(\rho\sigma_y) &= \mathrm{Tr}\left(\frac{1}{2}(\mathbf{I} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)\sigma_y\right) \\
&= \frac{1}{2}\left(\mathrm{Tr}(\mathbf{I}\sigma_y) + r_x\mathrm{Tr}(\sigma_x\sigma_y) + r_y\mathrm{Tr}(\sigma_y^2) + r_z\mathrm{Tr}(\sigma_z\sigma_y)\right) \\
&= \frac{1}{2}\left(0 + 0 + r_y \cdot 2 + 0\right) = r_y
\end{aligned}
$$

For $\sigma_z$:

$$
\begin{aligned}
\langle \sigma_z \rangle = \mathrm{Tr}(\rho\sigma_z) &= \mathrm{Tr}\left(\frac{1}{2}(\mathbf{I} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)\sigma_z\right) \\
&= \frac{1}{2}\left(\mathrm{Tr}(\mathbf{I}\sigma_z) + r_x\mathrm{Tr}(\sigma_x\sigma_z) + r_y\mathrm{Tr}(\sigma_y\sigma_z) + r_z\mathrm{Tr}(\sigma_z^2)\right) \\
&= \frac{1}{2}\left(0 + 0 + 0 + r_z \cdot 2\right) = r_z
\end{aligned}
$$

## Measurement and Probabilities

The Bloch representation also provides a clear picture of measurement in quantum mechanics. For a measurement in the computational basis $\{|0\rangle, |1\rangle\}$, the probability of observing the system in state $|0\rangle$ is:

$$
P(0) = |\langle 0|\psi\rangle|^2 = \left|\cos\left(\frac{\theta}{2}\right)\right|^2 = \frac{1}{2}(1 + r_z)
$$

Similarly, the probability of observing the system in state $|1\rangle$ is:

$$
P(1) = |\langle 1|\psi\rangle|^2 = \left|\sin\left(\frac{\theta}{2}\right)\right|^2 = \frac{1}{2}(1 - r_z)
$$

Thus, the measurement outcomes are directly related to the component $r_z$ of the Bloch vector, with the probabilities given by the projections onto the $|0\rangle$ and $|1\rangle$ states.

## Conclusion

The Bloch representation offers a simple and geometrically intuitive framework for understanding quantum states, especially in two-level systems like qubits. By mapping the state space to a unit sphere, it provides a powerful way to visualize quantum states, their evolution, and measurements. This approach plays a central role in quantum information theory, quantum computing, and quantum optics.