Fermi's Golden Rule on Quantum Transition Rates

Podcast Learn & Fun *

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The Fermi Golden Rule provides a quantitative description of the transition rate between energy states in a quantum system subjected to a time-dependent perturbation. It plays a crucial role in various fields of physics, including quantum mechanics, statistical mechanics, and quantum field theory. The rule is particularly useful in the study of phenomena such as atomic transitions, radioactive decay, scattering processes, and interaction between particles and fields.

Preliminaries and Basic Setup

Consider a quantum system with a Hamiltonian H, which consists of two parts: (1) The unperturbed Hamiltonian H_0, which describes the system in the absence of any external influences. (2) The perturbation H', which represents the external influence or interaction that causes transitions between energy states. Thus, the total Hamiltonian is written as:

$$H = H_0 + H'$$

We assume that the system is initially in a stationary state, say $|i\rangle$, corresponding to an eigenstate of H_0 with energy E_i, i.e., $H_0|i\rangle = E_i|i\rangle$. This means that initially, the probability of the system being in state $|i\rangle$ is unity, and the probability of being in any other state $|n\rangle$ is zero. The perturbation is applied at $t = 0$, and we wish to calculate the probability of the system transitioning from the initial state $|i\rangle$ to a final state $|f\rangle$ with energy E_f, due to the perturbation.

Time-Dependent Perturbation Theory

In time-dependent perturbation theory, the time evolution of the quantum state is governed by the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

where $|\psi(t)\rangle$ is the state vector of the system at time t.

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For small perturbations, we can use perturbation theory to find the first-order approximation to the state vector. The total state $|\psi(t)\rangle$ is expressed as a superposition of the unperturbed eigenstates:

$$|\psi(t)\rangle = \sum_n c_n(t)|n\rangle$$

where $|n\rangle$ are the eigenstates of H_0, and $c_n(t)$ are the time-dependent coefficients.

In the presence of a weak time-dependent perturbation, the coefficients $c_n(t)$ can be expanded in powers of the perturbation. For a first-order approximation, we assume that the system's state at time t is only weakly perturbed, meaning that the coefficients $c_n(t)$ deviate slightly from their initial values.

The solution for $c_n(t)$ at first order in perturbation theory is given by:

$$c_n(t) \approx \delta_{ni} - \frac{i}{\hbar}\int_0^t e^{i(E_n - E_i)t'/\hbar}\langle n|H'(t')|i\rangle dt'$$

where δ_{ni} is the Kronecker delta, ensuring that initially, $c_n(0) = \delta_{ni}$, meaning that the system starts in the state $|i\rangle$ at time $t = 0$.

The perturbation $H'(t)$ introduces a coupling between the initial state $|i\rangle$ and the other eigenstates of the unperturbed Hamiltonian $|n\rangle$. As time progresses, the coefficients $c_n(t)$ evolve, leading to a time-dependent probability that the system is found in each state $|n\rangle$. The term $\langle n|H'(t')|i\rangle$ represents the strength of the coupling between the initial and final states induced by the perturbation. If this matrix element is large, the transition between the states will be more likely. If it is small, the transition will be less likely.

The factor $e^{i(E_n - E_i)t/\hbar}$ arises due to the energy eigenvalues of the unperturbed Hamiltonian H_0. It introduces oscillatory behavior in the time evolution. The frequency of these oscillations depends on the energy difference $E_n - E_i$, which determines how rapidly the system transitions between states due to the perturbation.

Transition Probability

The first-order approximation describes how the system evolves over time under the influence of a small perturbation. The state of the system is a superposition of unperturbed eigenstates, with the amplitudes $c_n(t)$ determined by the interaction with the perturbation. The probability of finding the system in a final state $|f\rangle$ at time t, denoted $P_{i\to f}(t)$, is given by the squared modulus of the corresponding coefficient:

$$P_{i\to f}(t) = |c_f(t)|^2$$

Substituting the expression for $c_f(t)$ from the first-order perturbation theory, we obtain:

$$P_{i\to f}(t) = \left|\frac{-i}{\hbar}\int_0^t e^{i(E_f - E_i)t'/\hbar}\langle f|H'|i\rangle dt'\right|^2$$

This expression gives the transition probability in the time domain. The transition from the initial state $|i\rangle$ to other states $|n\rangle$ occurs with a probability that depends on both the perturbation strength and the energy differences between the initial and final states.

Transition Rate and Fermi Golden Rule

In time-dependent perturbation theory, we are interested in the transition probability from an initial state $|i\rangle$ to a final state $|f\rangle$ due to a perturbation $H'(t)$. The transition probability $P_{i\to f}(t)$ describes the likelihood that the system, initially in state $|i\rangle$, will be found in state $|f\rangle$ after time t.

As time progresses, the probability $P_{i\to f}(t)$ evolves. In the limit of long times, i.e., as $t \to \infty$, the transition probability approaches a steady state. This steady-state value is known as the transition rate, $W_{i\to f}$, which represents the probability per unit time that the system undergoes the transition from state $|i\rangle$ to state $|f\rangle$.

The transition rate $W_{i\to f}$ is formally defined as the time derivative of the transition probability:

$$W_{i\to f} = \lim_{t\to\infty} \frac{P_{i\to f}(t)}{t}$$

where $P_{i\to f}(t)$ is the probability of the system being in the final state $|f\rangle$ after time t. This definition assumes that the transition probability reaches a steady value as $t \to \infty$, and the transition rate characterizes this steady-state behavior.

For simplicity, assume that the perturbation $H'(t)$ is approximately constant over the time interval of interest, and that the integral is dominated by long times, which leads to the following approximation:

$$P_{i\to f}(t) \approx \frac{1}{\hbar^2} \left|\langle f|H'|i\rangle\right|^2 \left| \int_0^t e^{i(E_f - E_i)t'/\hbar} dt' \right|^2$$

Now, focus on the integral:

$$I(t) = \int_0^t e^{i(E_f - E_i)t'/\hbar} dt'$$

This integral represents the time evolution of the transition between two energy states due to the perturbation H'. To evaluate this integral, we can compute it explicitly:

$$I(t) = \frac{\hbar}{i} \left(\frac{e^{i(E_f - E_i)t/\hbar} - 1}{E_f - E_i} \right)$$

For large t, the exponential term oscillates rapidly unless $E_f = E_i$. This rapid oscillation means that the integral will effectively "cancel out" for most values of t', except when $E_f = E_i$, where the oscillations are absent. Therefore, as $t \to \infty$, this integral behaves like a delta function, as the oscillations give rise to a sharp peak at $E_f = E_i$.

To find the transition rate $W_{i\to f}$, we have:

$$W_{i\to f} = \lim_{t\to\infty} \frac{P_{i\to f}(t)}{t} = \lim_{t\to\infty} \frac{1}{t}\frac{1}{\hbar^2} \left|\langle f|H'|i\rangle\right|^2 \left| \int_0^t e^{i(E_f - E_i)t'/\hbar} dt' \right|^2$$

By using the L'Hôpital's Rule and the definition of a Delta function, we obtain:

$$W_{i\to f} = \frac{2\pi}{\hbar} \left|\langle f|H'|i\rangle\right|^2 \delta(E_f - E_i)$$

This is the **Fermi Golden Rule**. It tells us that the transition rate between an initial state $|i\rangle$ and a final state $|f\rangle$ due to a perturbation H' is proportional to the square of the matrix element $\langle f|H'|i\rangle$, and it is weighted by a delta function $\delta(E_f - E_i)$ that ensures energy conservation during the transition.

Applications of Fermi Golden Rule

The Fermi Golden Rule is a powerful tool in quantum mechanics that is widely applied in various fields of physics, particularly when dealing with transitions between quantum states induced by perturbations. Below are some key areas where the rule is commonly used.

Atomic Transitions One of the most well-known applications of the Fermi Golden Rule is in the description of atomic transitions, specifically the absorption and emission of photons by atoms. When an atom interacts with an electromagnetic field, it can transition between different energy levels, either absorbing or emitting a photon in the process. The Fermi Golden Rule provides the transition rate between these energy levels, allowing for the calculation of spectroscopic features such as absorption and emission lines. This is crucial in atomic spectroscopy, where the rates of photon absorption or emission can be measured experimentally to deduce various properties of the atom, including its energy structure and the nature of the interaction with the electromagnetic field.

Scattering Processes In scattering theory, the Fermi Golden Rule plays a central role in describing the transition rates of particles undergoing collisions or scattering events. The rule is often used in high-energy physics to compute cross sections for particle interactions, such as those occurring in particle accelerators. In these processes, an incoming particle (or set of particles) interacts with an external potential (e.g., another particle or field), and the rule gives the probability per unit time for the system to transition from an initial state to a final state. The Fermi Golden Rule thus allows the determination of the likelihood of different scattering outcomes, such as elastic or inelastic scattering, and is instrumental in the analysis of experimental results in particle physics.

Radiative Decay The Fermi Golden Rule is also crucial in describing radiative decay processes, where an excited quantum state of an atom, molecule, or nucleus decays to a lower energy state by emitting a photon. This is a fundamental process in quantum electrodynamics (QED) and is commonly encountered in atomic, molecular, and nuclear physics. For instance, when an atom in an excited state undergoes radiative decay, the Fermi Golden Rule can be used to calculate the rate at which the photon is emitted and the energy spectrum of the emitted radiation. This application is particularly important in the study of spontaneous emission and fluorescence, as well as in understanding the decay lifetimes of excited states.

Nuclear Reactions In nuclear physics, the Fermi Golden Rule is applied to describe the rates at which nuclear reactions occur. These reactions may involve the interaction of atomic nuclei through strong nuclear forces or electromagnetic interactions. The perturbation could be due to a variety of nuclear processes, such as fission, fusion, or the emission of gamma rays. By using the Fermi Golden Rule, the transition rates for these reactions can be determined, providing insights into the probabilities of various nuclear processes under different conditions. This is especially important in the study of nuclear energy production, radioactive decay, and the mechanisms involved in the synthesis of elements in stars.

In all these applications, the Fermi Golden Rule helps us understand the rates at which transitions occur between quantum states under the influence of perturbations. It is a versatile tool that applies across a wide range of physical systems, from atomic to nuclear scales, and is essential for predicting and analyzing experimental outcomes in these areas.

Extensions and Generalizations

While the Fermi Golden Rule is a powerful tool for describing transitions between quantum states induced by a perturbation, it is also adaptable and can be generalized to accommodate more complex systems. These extensions include modifications for systems with continuous spectra, time-dependent perturbations, and interactions involving multiple quantum states, such as phonons or photons. Below are some key extensions and generalizations of the Fermi Golden Rule.

Time-Dependent Perturbations In its basic form, the Fermi Golden Rule assumes that the perturbation H' is time-independent. However, in many physical situations, the perturbation can vary with time. For example, when an atom is exposed to an external time-dependent electromagnetic field, the perturbation is not constant but oscillates as a function of time. In such cases, we need to modify the expression for the transition probability to account for the time dependence of the perturbation. This can be done by solving the time-dependent Schrödinger equation with the time-dependent Hamiltonian $H(t) = H_0 + H'(t)$,

where H_0 is the unperturbed Hamiltonian of the system and $H'(t)$ represents the time-varying perturbation. In this context, the Fermi Golden Rule is generalized by incorporating the time-dependent interaction, and the transition rate becomes a function of both the energy differences and the time dependence of the perturbation. This approach is crucial for describing processes like photon absorption or emission in atoms under oscillating electromagnetic fields, as well as in the study of systems driven by time-varying forces.

Multi-Phonon and Multi-Photon Transitions Another important extension of the Fermi Golden Rule involves systems interacting with multiple quantum states, such as multi-phonon or multi-photon transitions. In simple systems, the Fermi Golden Rule typically considers single-photon or single-phonon interactions, where the perturbation involves the absorption or emission of a single quantum of energy. However, in more complex scenarios, the system may interact with multiple quanta. For instance, in solid-state systems, an electron can absorb or emit multiple phonons during a transition between energy states, or in atomic or molecular systems, transitions can involve the absorption or emission of several photons simultaneously. The Fermi Golden Rule can be generalized to describe such processes by considering the transition probabilities for multi-phonon or multi-photon absorption and emission. This extension requires adjusting the perturbation Hamiltonian to include terms that represent the interactions with multiple quanta, and the transition rate involves calculating the matrix elements corresponding to these multi-particle interactions. These generalized forms of the Fermi Golden Rule are important in fields such as nonlinear optics, where phenomena like multi-photon absorption and stimulated Raman scattering are observed, and in solid-state physics, where phonon-mediated interactions play a key role in thermal transport and electron-phonon coupling.

These generalizations of the Fermi Golden Rule allow for a more accurate description of systems with time-varying perturbations and complex interactions involving multiple quantum states. They provide a framework for studying a wide range of physical phenomena, from time-dependent processes in atomic and molecular systems to multi-particle interactions in condensed matter systems. By extending the Fermi Golden Rule in these ways, we can address more intricate scenarios that go beyond the simple case of a constant perturbation acting on a discrete set of energy levels, thus making it a versatile tool for theoretical and experimental analysis in various branches of physics.

Conclusion

The Fermi Golden Rule provides a simple and powerful framework for calculating transition rates in quantum systems under perturbation. It has widespread applications across different branches of physics, from atomic physics to high-energy particle physics. By understanding the rule's derivation and physical implications, one gains insight into the dynamics of quantum systems interacting with external forces or fields.