Quantum Entanglement

Podcast Learn & Fun *

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Quantum entanglement is one of the most fascinating phenomena in quantum mechanics. Mathematically, entanglement describes a situation where the quantum state of a composite system cannot be written as a product of the states of its individual subsystems. This concept becomes clearer when we consider the formalism of quantum states, operators, and their tensor product spaces.

1 State Vectors and Hilbert Spaces

For a single particle, the state is described by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, where the vector represents the state of the system.

If we have two particles (or subsystems), each with their own Hilbert space, say $\mathcal{H}_A$ and $\mathcal{H}_B$ for particles A and B, then the total Hilbert space of the two-particle system is the **tensor product** of the two individual Hilbert spaces:

$$\mathcal{H}_{\text{total}} = \mathcal{H}_A \otimes \mathcal{H}_B$$

A state vector in this combined space represents the state of the entire system.

2 Product States

A **separable state** (or **product state**) of two subsystems is a state where the total system can be described as a product of the individual states of the subsystems. Mathematically, this is written as:

$$|\psi_{\text{separable}}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

where $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$.

3 Entangled States

An **entangled state** is a state that cannot be factored into a product of individual states of the subsystems. Instead, the state of the system must be described as a superposition of multiple possible product states.

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In the simplest case, if both particles are qubits (two-state systems), the total state is a superposition of all possible combinations of the two qubits. The general form of such an entangled state for two particles A and B can be written as:

$$|\psi_{\text{entangled}}\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

where $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ are the basis states of the combined Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, and $\alpha, \beta, \gamma, \delta$ are complex coefficients. A well-known example of an entangled state is the **Bell state**:

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

This is an entangled state because it cannot be written as a product of states of particle A and particle B. The particles A and B are correlated in such a way that measuring one particle's state will instantaneously determine the state of the other, even if they are spatially separated.

4 Measurement and Entanglement

When a measurement is made on an entangled system, the outcome is not independent of the other subsystem. Suppose we measure the state of particle A in some basis (say, the computational basis $|0\rangle, |1\rangle$). In the Bell state example, if we measure particle A and find it in the state $|0\rangle$, then particle B will instantly be in the state $|1\rangle$, and vice versa.

The entanglement means that the measurement on one part of the system "collapses" the state of the other part, regardless of the spatial separation between the two.

5 Entanglement of General States

In a more general sense, a state can be written as a superposition of multiple product states. If a state cannot be factored into a product of states of the individual subsystems, it is entangled.

For two particles A and B, the general form of a state $|\psi_{\text{total}}\rangle$ in the combined Hilbert space is:

$$|\psi_{\text{total}}\rangle = \sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B$$

where c_{ij} are complex coefficients, and $|i\rangle_A$ and $|j\rangle_B$ are basis vectors in the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.

6 Partial Trace and Reduced Density Matrix

To describe the state of a subsystem, we can trace out the degrees of freedom of the other subsystem. This leads to the **reduced density matrix**.

The **density matrix** ρ_{total} for the total system is given by:

$$\rho_{\text{total}} = |\psi_{\text{total}}\rangle\langle\psi_{\text{total}}|$$

To obtain the reduced density matrix of one subsystem, we trace out the other subsystem:

$$\rho_A = \text{Tr}_B(\rho_{\text{total}}), \quad \rho_B = \text{Tr}_A(\rho_{\text{total}})$$

The reduced density matrix describes the state of a subsystem when the other subsystem is not measured or is ignored.

7 Entanglement Measures

There are various measures for the degree of entanglement in a quantum state. One common measure is the **concurrence**, which is used for two-qubit systems.

The **entanglement of formation** quantifies how much entanglement is needed to form a given quantum state, and the **von Neumann entropy** is often used to measure the mixedness or uncertainty of a quantum state.

8 Entanglement and Non-locality

The most famous property of entanglement is that measurements on entangled particles exhibit **non-locality**, meaning the measurement on one particle affects the state of the other particle instantaneously, regardless of distance. This is a key feature of quantum mechanics that is not explainable by classical physics.