Time Evolution of Operators

Podcast Learn & Fun *

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In quantum mechanics, understanding how observables and operators evolve over time is crucial for describing the dynamics of quantum systems. The time evolution of operators provides insight into the relationship between the system's states and how measurements evolve in time.

There are two main pictures used to describe quantum dynamics: *The Schrödinger Picture:* In this picture, the state vectors $|\psi(t)\rangle$ evolve in time, but the operators $\hat{O}$ are time-independent. *The Heisenberg Picture:* In this picture, the state vectors are constant (fixed), and the operators evolve in time. The relationship between these pictures is crucial for understanding time evolution in quantum mechanics.

The Schrödinger Picture

In the Schrödinger picture, the state vector $|\psi(t)\rangle$ evolves according to the Schrödinger equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

where $\hbar$ is the reduced Planck's constant, $\hat{H}$ is the Hamiltonian operator, which represents the total energy of the system, and $|\psi(t)\rangle$ is the state of the system at time t. The formal solution to this equation is:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \hat{U}(t)|\psi(0)\rangle$$

where $|\psi(0)\rangle$ is the state at $t = 0$, and the operator $e^{-i\hat{H}t/\hbar}$ is called the time evolution operator, denoted by $\hat{U}(t)$. It is unitary, as we can check that $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \mathbf{I}$. The unitary nature of time evolution ensures the conservation of probability, meaning that the total probability of all possible outcomes remains 1. This preservation is crucial because the probabilities of

*YouTube Channel: `https://www.youtube.com/@Podcast_Learn_Fun` quantum events are given by the squared magnitudes of state vectors, and unitary transformations preserve these magnitudes.

In the Schrödinger picture, the operators $\hat{O}$ do not explicitly depend on time. If $\hat{O}$ is time-independent, its form remains the same throughout the evolution. For any observable $\hat{O}$, the time evolution of the expectation value is given by:

$$\langle \hat{O} \rangle(t) = \langle \psi(t) | \hat{O} | \psi(t) \rangle$$

This expectation value is computed by evolving the state $|\psi(t)\rangle$ according to the time evolution operator $\hat{U}(t)$ on the state vector.

The Heisenberg Picture

In the Heisenberg picture, the state vectors $|\psi(t)\rangle \equiv |\psi(0)\rangle$ are time-independent, and the operators evolve over time. The time evolution of an operator $\hat{O}$ in the Heisenberg picture is given by:

$$\hat{O}_H(t) = \hat{U}^\dagger(t) \hat{O} \hat{U}(t)$$

The expectation value of an observable $\hat{O}$ in the Heisenberg picture is:

$$\langle \hat{O}(t) \rangle = \langle \psi(0) | \hat{O}_H(t) | \psi(0) \rangle$$

This expression gives the same physical result as the Schrödinger picture, but the time evolution of the operator is tracked rather than that of the state.

The time evolution of operators in the Heisenberg picture is governed by the Heisenberg equation of motion. To find the time derivative of $\hat{O}_H(t)$, we differentiate:

$$\frac{d}{dt} \hat{O}_H(t) = \frac{d}{dt} \left(\hat{U}^\dagger(t) \hat{O}(t) \hat{U}(t) \right)$$

Here we use $\hat{O}(t)$ for a general case where the operator itself can be time-dependent even in the Schrödinger picture. For example, if a potential field changes with time, then the Hamiltonian will also have an explicit time dependence. Applying the product rule:

$$\frac{d}{dt} \hat{O}_H(t) = \frac{d}{dt} \hat{U}^\dagger(t) \hat{O}(t) \hat{U}(t) + \hat{U}^\dagger(t) \frac{d}{dt} \hat{O}(t) \hat{U}(t) + \hat{U}^\dagger(t) \hat{O}(t) \frac{d}{dt} \hat{U}(t)$$

Next, we compute the time derivatives of $\hat{U}(t)$ and $\hat{U}^\dagger(t)$: From the definition of $\hat{U}(t)$, we have:

$$\frac{d}{dt} \hat{U}(t) = -\frac{i}{\hbar} \hat{H} \hat{U}(t) \,, \quad \frac{d}{dt} \hat{U}^\dagger(t) = \frac{i}{\hbar} \hat{H} \hat{U}^\dagger(t)$$

Substitute these results into the derivative of $\hat{O}_H(t)$:

$$\frac{d}{dt}\hat{O}_H(t) = \frac{i}{\hbar}\hat{H}\hat{U}^\dagger(t)\hat{O}(t)\hat{U}(t) + \hat{U}^\dagger(t)\frac{d}{dt}\hat{O}(t)\hat{U}(t) - \frac{i}{\hbar}\hat{U}^\dagger(t)\hat{O}(t)\hat{H}\hat{U}(t)$$

We can now combine the terms involving $\hat{U}^\dagger(t)$ and $\hat{U}(t)$. Note that $\hat{H}$ and $\hat{U}^\dagger$ are commutable because $\hat{U}^\dagger$ is a function of $\hat{H}$. We then have:

$$\frac{d}{dt}\hat{O}_H(t) = \hat{U}^\dagger(t)\left(\frac{i}{\hbar}\hat{H}\hat{O}(t) - \frac{i}{\hbar}\hat{O}(t)\hat{H}\right)\hat{U}(t) + \hat{U}^\dagger(t)\frac{d}{dt}\hat{O}(t)\hat{U}(t)$$

The first term inside the parentheses is the commutator of $\hat{H}$ and $\hat{O}(t)$:

$$\frac{d}{dt}\hat{O}_H(t) = \hat{U}^\dagger(t)\left(\frac{i}{\hbar}[\hat{H}, \hat{O}(t)]\right)\hat{U}(t) + \hat{U}^\dagger(t)\frac{d}{dt}\hat{O}(t)\hat{U}(t)$$

Since $\hat{O}_H(t) = \hat{U}^\dagger(t)\hat{O}(t)\hat{U}(t)$, we can rewrite the equation as:

$$\frac{d}{dt}\hat{O}_H(t) = \frac{i}{\hbar}[\hat{H}, \hat{O}_H(t)] + \hat{U}^\dagger(t)\frac{d}{dt}\hat{O}(t)\hat{U}(t)$$

The second term on the right-hand side accounts for the explicit time dependence of $\hat{O}(t)$ in the Schrödinger picture. In the Heisenberg picture, this term becomes:

$$\left(\frac{\partial \hat{O}}{\partial t}\right)_H = \hat{U}^\dagger(t)\frac{d}{dt}\hat{O}(t)\hat{U}(t)$$

Thus, the full Heisenberg equation of motion, including both the commutator with the Hamiltonian and the explicit time dependence of the operator, is:

$$\frac{d}{dt}\hat{O}_H(t) = \frac{i}{\hbar}[\hat{H}, \hat{O}_H(t)] + \left(\frac{\partial \hat{O}}{\partial t}\right)_H$$

The first term on the right-hand side represents the change in the operator due to the Hamiltonian dynamics, while the second term accounts for any explicit time dependence of the operator itself.

For time-independent operators, the Heisenberg equation simplifies to:

$$\frac{d}{dt}\hat{O}_H(t) = \frac{i}{\hbar}[\hat{H}, \hat{O}_H(t)]$$

This is a first-order differential equation that governs the time evolution of the operator in the Heisenberg picture.

Example: Time Evolution of Position and Momentum Operators
To illustrate the Heisenberg equation of motion, consider the position $\hat{x}$ and momentum $\hat{p}$ operators for a particle in one dimension, where the Hamiltonian is given by:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

For the position operator:

$$\frac{d}{dt}\hat{x}_H(t) = \frac{i}{\hbar}[\hat{H}, \hat{x}_H(t)] = \frac{\hat{p}_H(t)}{m}$$

For the momentum operator:

$$\frac{d}{dt}\hat{p}_H(t) = \frac{i}{\hbar}[\hat{H}, \hat{p}_H(t)] = -\frac{\partial V(\hat{x}_H)}{\partial \hat{x}_H}$$

For a free particle, where $V(\hat{x}) = 0$, these equations simplify to:

$$\frac{d}{dt}\hat{x}_H(t) = \frac{\hat{p}_H(t)}{m}, \quad \frac{d}{dt}\hat{p}_H(t) = 0$$

Thus, the momentum $\hat{p}_H(t)$ remains constant, and the position evolves linearly with time.

For a particle in a potential $V(\hat{x})$, the second equation shows that the momentum changes in time depending on the force derived from the potential.

Conclusion

The time evolution of operators in quantum mechanics is a key concept that allows us to understand how physical observables evolve over time. The Heisenberg picture offers a powerful framework for studying this evolution, especially in systems with time-independent Hamiltonians. The Schrödinger picture, on the other hand, focuses on the evolution of the state vectors. Both pictures provide the same physical predictions, but their form and utility depend on the specific problem being studied.

Understanding the time evolution of operators and applying the Heisenberg equation of motion is essential for analyzing quantum dynamics, including the behavior of particles, fields, and other quantum systems.