# Raising and Lowering Operators

Podcast Learn & Fun *

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In quantum mechanics, especially in the context of quantum harmonic oscillators and angular momentum, *raising and lowering operators* are key tools used to manipulate the quantum states of a system. These operators, also known as *creation* (raising) and *annihilation* (lowering) operators, play a fundamental role in the algebra of quantum states and observables.

This lecture explores the definition, properties, and applications of raising and lowering operators, with particular focus on their role in quantum mechanics.

## Definition

A quantum system is typically described by a set of quantum states, which are often represented by vectors in a Hilbert space. Operators act on these states and correspond to physical observables. For a system like the quantum harmonic oscillator, the states are labeled by quantum numbers, and the operators help in changing the state without altering the physical content (such as energy) in an easily manageable way.

The quantum harmonic oscillator is described by the Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

where $\hat{p}$ is the momentum operator, $\hat{x}$ is the position operator, $m$ is the mass, and $\omega$ is the angular frequency. The eigenstates of the harmonic oscillator Hamiltonian, $|n\rangle$, form a discrete set corresponding to energy eigenvalues:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots$$

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These eigenstates are called *number states* or *Fock states*. The Hamiltonian eigenfunction is written as:

$$\hat{H}|n\rangle = E_n|n\rangle$$

where $|n\rangle$ is the $n$-th energy eigenstate of the harmonic oscillator.

In this system, we define the *raising* (creation) operator $\hat{a}^\dagger$ and the *lowering* (annihilation) operator $\hat{a}$, which can be written in terms of position and momentum as:

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p})$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p})$$

These operators satisfy the following commutation relation:

$$[\hat{a}, \hat{a}^\dagger] = 1$$

This commutation relation implies the following: (1) The operators $\hat{a}$ and $\hat{a}^\dagger$ do not commute, which is a feature that leads to quantization of certain physical properties like energy. (2) The commutation relation is fundamental in deriving the properties of the energy eigenstates and the corresponding ladder of quantum states. From these relations, we can also deduce properties such as the *uncertainty principle* between position and momentum, as they are linked through the operators.

In terms of the creation (raising) and annihilation (lowering) operators, the Hamiltonian can be expressed as:

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

The quantum states $|n\rangle$ are also eigenstates of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$, with eigenvalue $n$:

$$\hat{N}|n\rangle = n|n\rangle$$

The Number Operator $\hat{N} = \hat{a}^\dagger\hat{a}$ represents the occupation number of the state, with eigenvalues corresponding to the quantum number $n$.

The operators $\hat{a}^\dagger$ (raising) and $\hat{a}$ (lowering) act on the quantum states $|n\rangle$ in a specific way, which we will derive and prove below.

### Action of the Lowering Operator $\hat{a}$

When operating on the quantum eigenstate $|n\rangle$, the lowering operator $\hat{a}$ lowers the quantum number $n$ by 1:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

**Proof:** Consider the commutation relation between $\hat{a}$ and $\hat{N}$:

$$[\hat{N}, \hat{a}] = [\hat{a}^\dagger \hat{a}, \hat{a}] = -\hat{a}$$

This implies:

$$\begin{aligned} \hat{N}\hat{a}|n\rangle &= (\hat{a}^\dagger \hat{a})\hat{a}|n\rangle = \hat{a}(\hat{a}^\dagger \hat{a})|n\rangle - \hat{a}|n\rangle \\ &= \hat{a}\,(n|n\rangle) - \hat{a}|n\rangle = (n-1)\hat{a}|n\rangle \end{aligned}$$

Thus, $\hat{a}|n\rangle$ is an eigenstate of the number operator $\hat{N}$ with eigenvalue $n-1$, which means:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

The factor $\sqrt{n}$ arises from the normalization of the state. Therefore, the lowering operator $\hat{a}$ lowers the quantum number by 1, as desired.

The *vacuum state* $|0\rangle$ is defined as the state with the lowest energy level, which is annihilated by the lowering operator:

$$\hat{a}|0\rangle = 0$$

This is a key property of the annihilation operator.

### Action of the Raising Operator $\hat{a}^\dagger$

When operating on the quantum eigenstate $|n\rangle$, the raising operator $\hat{a}^\dagger$ raises the quantum number $n$ by 1:

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

**Proof:** Consider the commutation relation between $\hat{a}^\dagger$ and $\hat{N}$:

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$$

This implies:

$$\begin{aligned} \hat{N}\hat{a}^\dagger|n\rangle &= (\hat{a}^\dagger \hat{a})\hat{a}^\dagger|n\rangle = \hat{a}^\dagger(\hat{a}^\dagger \hat{a})|n\rangle + \hat{a}^\dagger|n\rangle \\ &= \hat{a}^\dagger\,(n|n\rangle) + \hat{a}^\dagger|n\rangle = (n+1)\hat{a}^\dagger|n\rangle \end{aligned}$$

Thus, $\hat{a}^\dagger|n\rangle$ is an eigenstate of the number operator $\hat{N}$ with eigenvalue $n+1$, which means:

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

The factor $\sqrt{n+1}$ ensures the correct normalization of the state. Therefore, the raising operator $\hat{a}^\dagger$ raises the quantum number by 1, as desired.

### Normalization Factors

The normalization factors $\sqrt{n}$ and $\sqrt{n+1}$ arise from the requirement that the quantum states remain properly normalized after applying the raising or lowering operators. To see this explicitly, let's compute the inner product $\langle n|\hat{a}^\dagger \hat{a}|n\rangle$:

$$\langle n|\hat{a}^\dagger \hat{a}|n\rangle = n$$

This inner product reflects the fact that the quantum states are normalized, and the factors $\sqrt{n}$ and $\sqrt{n+1}$ ensure that after applying the operators, the quantum states remain orthonormal.

For example, when applying the lowering operator $\hat{a}$ to $|n\rangle$, the state $\hat{a}|n\rangle$ satisfy:

$$\langle n|\hat{a}^\dagger \hat{a}|n\rangle = \langle n|\hat{a}^\dagger \left(\sqrt{n}|n-1\rangle\right) = \sqrt{n}\langle n|\hat{a}^\dagger|n-1\rangle = (\sqrt{n})^2\langle n|n\rangle = n$$

Similarly, when applying the raising operator $\hat{a}^\dagger$ to $|n\rangle$, the normalization condition holds as well.

### Applications of Raising and Lowering Operators

Raising and lowering operators have broad applications in various areas of quantum mechanics. These operators provide a powerful method for analyzing and constructing quantum states in different physical systems. In this section, we discuss some of the most important applications of these operators, with a focus on the quantum harmonic oscillator, angular momentum, and quantum field theory.

**Quantum Harmonic Oscillator** The quantum harmonic oscillator is one of the most fundamental models in quantum mechanics, and the raising and lowering operators play a central role in its solution. The eigenstates of the harmonic oscillator correspond to discrete energy levels that are labeled by the quantum number $n$. These energy eigenstates can be generated systematically from the ground state (or vacuum state) by repeatedly applying the raising operator $\hat{a}^\dagger$. In this way, the harmonic oscillator states form a ladder of energy levels, with each application of the raising operator increasing the energy by a fixed amount, while the lowering operator $\hat{a}$ decreases the energy by the same amount, until it reaches the ground state.

The harmonic oscillator's Hamiltonian can be written in terms of these operators, and the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ acts as a count of the number of quanta (or particles) in the system. The eigenvalues of the number operator are the quantum numbers $n$, and each quantum state corresponds to a specific energy level given by $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$. The raising and lowering operators are thus essential tools for understanding the energy structure of the harmonic oscillator, as they allow us to move between different energy eigenstates and explore the spectrum of the system.

**Angular Momentum** Raising and lowering operators also have significant applications in the theory of angular momentum. In the context of angular momentum, the operators $J_+$ (raising) and $J_-$ (lowering) act on the eigenstates of the angular momentum operators $J_z$, which have eigenvalues corresponding to the magnetic quantum number $m$. These operators allow us to manipulate the quantum numbers associated with angular momentum, leading to transitions between different eigenstates.

For a system with total angular momentum quantum number $l$, the magnetic quantum number $m$ takes integer values in the range $-l \leq m \leq l$. The raising and lowering operators act as follows:

$$J_+|l,m\rangle = \hbar\sqrt{(l-m)(l+m+1)}|l,m+1\rangle$$
$$J_-|l,m\rangle = \hbar\sqrt{(l+m)(l-m+1)}|l,m-1\rangle$$

Here, $J_+$ raises the value of $m$ by 1, and $J_-$ lowers it by 1, with the appropriate normalization factors ensuring that the quantum states remain properly normalized. The total angular momentum quantum number $l$ is fixed, while the magnetic quantum number $m$ can vary. The action of the raising and lowering operators is crucial in the study of angular momentum, as they help describe the transitions between different magnetic sublevels and the corresponding energy shifts in systems such as atoms in a magnetic field. These operators also play a central role in constructing the algebra of angular momentum, which is essential for understanding many aspects of quantum mechanics, including the behavior of atomic and subatomic systems.

**Quantum Field Theory (QFT)** In quantum field theory (QFT), the concept of raising and lowering operators extends beyond simple quantum mechanical systems to the creation and annihilation of particles in a quantum field. In QFT, fields are quantized, and the field operators $\hat{\phi}(x)$ and their conjugates correspond to the creation and annihilation operators. The annihilation operator $\hat{a}_k$ destroys a particle with momentum $k$, while the creation operator $\hat{a}_k^\dagger$ creates a particle with momentum $k$. These operators act on the quantum vacuum state to generate or annihilate particles in the field.

The creation and annihilation operators obey commutation relations similar to those of the raising and lowering operators in the harmonic oscillator. These operators are fundamental for constructing quantum states in the field, which correspond to various particle number states. For example, the vacuum state $|0\rangle$ is the state with no particles, and applying the creation operator $\hat{a}_k^\dagger$ creates a particle in the state $k$. By applying the creation operator repeatedly, we can generate multi-particle states, while applying the annihilation operator removes particles from the state.

These operators are not just mathematical tools; they are directly linked to physical quantities such as particle number and energy in QFT. In fact, the Hamiltonian of a quantum field can often be expressed in terms of the creation and annihilation operators, which allows for the description of particle interactions and the dynamics of the field. The concept of field quantization via raising and lowering operators is at the heart of QFT, and it plays a central role in the study of particle physics, including the description of fundamental forces and interactions.

## Conclusion

Raising and lowering operators provide a powerful toolset for solving quantum systems such as the harmonic oscillator and angular momentum problems. These operators allow for the generation of quantum states in a systematic and algebraically elegant way, which is crucial for both quantum mechanics and quantum field theory. Understanding their properties, such as the commutation relations, helps in unraveling deeper insights into the quantization of energy and other physical quantities in quantum systems.