Invariance of the Trace of a Matrix

Podcast Learn & Fun *

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Introduction

The trace is a scalar value that has interesting properties and applications, especially in the context of linear transformations and matrix diagonalization. Let's start by defining what the trace of a matrix is and gradually build up to understanding why it remains invariant when we change the basis.

1 Definition of the Trace of a Matrix

To begin, let's define the trace of a matrix.

For a square matrix $A = [a_{ij}]$ of size $n \times n$, the *trace* of A, denoted $\mathrm{Tr}(A)$, is the sum of the diagonal elements of the matrix:

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

So, for example, if we have a matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The trace of A would be:

$$\mathrm{Tr}(A) = 1 + 5 + 9 = 15$$

This is the sum of the diagonal elements of the matrix A.

2 Basic Properties of the Trace

Before delving into the invariance property, let's go over some basic properties of the trace.

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- **Linearity**: The trace is a linear function. This means:

$$\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

for two square matrices A and B of the same size.

And also:

$$\mathrm{Tr}(cA) = c \cdot \mathrm{Tr}(A)$$

where c is a scalar.

- **Cyclic Property**: The trace has a very useful cyclic property:

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

for any two square matrices A and B of the same size.

This is true even if the matrices are not necessarily diagonal or symmetric. The cyclic property will be important when we show why the trace is invariant under a change of basis.

3 Change of Basis and Matrix Representation

To understand why the trace is invariant, we need to explore what happens when we change the basis of a vector space. A change of basis involves a transformation from one coordinate system to another.

Suppose we have a linear transformation T represented by a matrix A in some basis $\mathcal{B}$. Now, let's change to a new basis $\mathcal{B}'$. The matrix representing T in the new basis $\mathcal{B}'$ is denoted A', and it is related to A by the following equation:

$$A' = P^{-1}AP$$

Here, P is the matrix whose columns are the coordinate vectors of the new basis vectors expressed in terms of the old basis. P^{-1} is the inverse of that matrix.

4 Invariance of the Trace Under a Change of Basis

Now, let's prove that the trace of a matrix is invariant under a change of basis.

We want to show that:

$$\mathrm{Tr}(A') = \mathrm{Tr}(A)$$

Starting from the formula for A':

$$A' = P^{-1}AP$$

The trace of A' is:

$$\mathrm{Tr}(A') = \mathrm{Tr}(P^{-1}AP)$$

Using the cyclic property of the trace, we can reorder the matrices inside the trace:

$$\mathrm{Tr}(P^{-1}AP) = \mathrm{Tr}(APP^{-1})$$

Since PP^{-1} is the identity matrix I, we have:

$$\mathrm{Tr}(APP^{-1}) = \mathrm{Tr}(AI) = \mathrm{Tr}(A)$$

Thus, we have shown that:

$$\mathrm{Tr}(A') = \mathrm{Tr}(A)$$

This proves that the trace of the matrix A is invariant under a change of basis.

5 Intuition Behind the Invariance

Let's take a moment to build some intuition about why the trace is invariant.

- The trace is essentially the sum of the eigenvalues of the matrix when the matrix is diagonalizable (which is true for most matrices, although there are exceptions for non-diagonalizable matrices). This is because the trace is equal to the sum of the diagonal elements, which represent the action of the linear transformation on the basis vectors.
- When you change the basis, you are simply re-representing the same linear transformation in a new coordinate system. The eigenvalues (which are the diagonal elements of the matrix in a diagonalized form) do not change; only the coordinates of the vectors representing the linear transformation change.
- Since the trace is related to the sum of eigenvalues, and the eigenvalues do not change under a change of basis, the trace remains the same.

6 Example

Let's now go through an example to solidify our understanding. Consider a 2D vector space with a linear transformation represented by the matrix A:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

The trace of A is:

$$\mathrm{Tr}(A) = 3 + 2 = 5$$

Now, let's change the basis using a matrix P:

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

To find the matrix A' in the new basis, we compute:

$$A' = P^{-1}AP$$

First, compute P^{-1}, which is:

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Now calculate A':

$$A' = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

After performing the matrix multiplication, you'll find that A' is:

$$A' = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$$

Finally, compute the trace of A':

$$\mathrm{Tr}(A') = 4 + 1 = 5$$

As you can see, even though we changed the basis and the matrix representation of the linear transformation changed, the trace remains the same.

7 Conclusion

In summary, we have shown that the trace of a matrix is invariant under a change of basis. This result is a direct consequence of the properties of the trace, particularly its cyclic property, and the fact that a change of basis corresponds to a similarity transformation, which does not affect the eigenvalues of a matrix.

The invariance of the trace under a change of basis is a very useful property in various fields of mathematics and physics, particularly when working with linear transformations, diagonalization, and matrix invariants.