Tensor Product: Theory and Applications

Podcast Learn & Fun [*]

January 1, 2025

Introduction

Tensor product plays a crucial role in various areas such as linear algebra, quantum mechanics, computer science, and more. We will explore the tensor product from the perspective of vector spaces, then extend it to more general cases.

1 What is the Tensor Product?

Basic Intuition

The tensor product is a way to combine two mathematical objects (often vector spaces) into a new one, which retains the structure of both objects. It generalizes the concept of the Cartesian product, which we are more familiar with in set theory, by taking into account the algebraic structure of the objects involved.

Think of the tensor product as a "product" of vectors that combines information from both spaces in a way that respects their linear structure.

Notation

We denote the tensor product of two objects A and B as $A \otimes B$, where the "$\otimes$" symbol is called the *tensor product symbol*.

Formal Definition

Given two vector spaces V and W over a field F, the *tensor product space* $V \otimes W$ is another vector space formed by taking all possible "bilinear combinations" of elements from V and W. This means that elements of $V \otimes W$ are formal sums of tensors of the form $v \otimes w$, where $v \in V$ and $w \in W$, and these tensors satisfy certain properties.

For instance, the tensor product space $V \otimes W$ satisfies:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$

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$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

and

$$\alpha(v \otimes w) = (\alpha v) \otimes w = v \otimes (\alpha w)$$

for any scalars $\alpha \in F$.

Independence

Tensors such as $v_1 \otimes w_1$ and $v_2 \otimes w_2$ are considered distinct unless $v_1 = v_2$ and $w_1 = w_2$, even though we are essentially combining them in a structured way.

2 Tensor Product of Vector Spaces

Constructing the Tensor Product Space

Let's see how the tensor product space is constructed more formally:

- **Vector Space of Tensors**: For vectors $v \in V$ and $w \in W$, the tensor product $v \otimes w$ is a new object that does not reside in either V or W, but it belongs to the tensor product space $V \otimes W$.
- **Basis for Tensor Products**: If $\{v_1, v_2, \ldots, v_n\}$ is a basis for V and $\{w_1, w_2, \ldots, w_m\}$ is a basis for W, then the set of all tensor products $\{v_i \otimes w_j\}$ (where $1 \leq i \leq n$ and $1 \leq j \leq m$) forms a basis for the tensor product space $V \otimes W$.

This means that any element in $V \otimes W$ can be written as a linear combination of these basis elements:

$$\sum_{i,j} \alpha_{ij}(v_i \otimes w_j)$$

where $\alpha_{ij} \in F$.

3 Properties of the Tensor Product

The tensor product has several important properties that make it a powerful tool:

- **Associativity**: The tensor product is associative, meaning that:

  $$(V \otimes W) \otimes X \cong V \otimes (W \otimes X)$$

  for any vector space X. This allows us to group tensor products without ambiguity.

- **Distributivity**: The tensor product distributes over direct sums. If we have vector spaces V, W, X, and Y, we can use:

$$(V \oplus W) \otimes X \cong (V \otimes X) \oplus (W \otimes X)$$

  This is similar to how multiplication distributes over addition in basic algebra.

- **Symmetry and Antisymmetry**:
  - If the vector spaces V and W are symmetric (as in quantum mechanics), the tensor product can be symmetric, meaning:

$$v \otimes w = w \otimes v$$

  - For antisymmetric tensor products (used in differential geometry), we can impose the relation:

$$v \otimes w = -w \otimes v$$

- **Tensor Product of Linear Maps**: If $A : V \to V'$ and $B : W \to W'$ are linear maps, then there exists a linear map:

$$A \otimes B : V \otimes W \to V' \otimes W'$$

  which acts on tensors in a natural way.

4 Examples

Let's go through a simple example to see how tensor products are constructed.

Consider two vector spaces:

- $V = \mathbb{R}^2$ with basis $\{e_1, e_2\}$
- $W = \mathbb{R}^3$ with basis $\{f_1, f_2, f_3\}$

The tensor product space $V \otimes W$ has the following basis:

$$\{e_1 \otimes f_1, e_1 \otimes f_2, e_1 \otimes f_3, e_2 \otimes f_1, e_2 \otimes f_2, e_2 \otimes f_3\}$$

This gives a 6-dimensional space since $\dim(V \otimes W) = \dim(V) \times \dim(W) = 2 \times 3 = 6$.

Let's now express a tensor like $(e_1 + e_2) \otimes (f_1 + f_2)$:

$$(e_1 + e_2) \otimes (f_1 + f_2) = e_1 \otimes f_1 + e_1 \otimes f_2 + e_2 \otimes f_1 + e_2 \otimes f_2$$

This demonstrates how tensor products combine the elements of both spaces.

5 Applications of Tensor Product

In Quantum Mechanics:

In quantum mechanics, the tensor product is used to describe composite quantum systems. If you have two quantum systems described by vector spaces V and W, the state of the combined system is described by the tensor product space $V \otimes W$.

In Computer Science:

Tensor products are used in fields like machine learning, especially in *tensor decomposition* methods. Tensor networks have gained importance in quantum computing and neural networks.

In Geometry and Differential Forms:

Tensor products are used to define *differential forms* and to study the geometric properties of manifolds.