Symmetry, Involutory, Unitary, and Hermitian Matrices

Introduction

We will discuss some important matrix properties in linear algebra. We will focus on the following key concepts:

- Symmetry in Matrices
- Involutory Matrices
- Unitary Matrices
- Hermitian Matrices

These concepts play a central role in various applications, including quantum mechanics, signal processing, and systems theory. Understanding these matrix properties is crucial for both theoretical and applied mathematics.

1 Symmetry in Matrices

Definition of Symmetric Matrices:

A matrix A is said to be *symmetric* if it is equal to its transpose. That is,

$$A = A^T$$

where A^T represents the transpose of matrix A, which is obtained by swapping rows and columns.

Properties of Symmetric Matrices:

- The entries of a symmetric matrix satisfy $a_{ij} = a_{ji}$ for all i, j.
- Symmetric matrices are always square matrices (i.e., they have the same number of rows and columns).
- The eigenvalues of a symmetric matrix are always real.
- Symmetric matrices are diagonalizable, meaning that there exists a basis of eigenvectors corresponding to distinct eigenvalues.

Example:

Consider the matrix A:

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

Here, $A^T = A$, so this is a symmetric matrix.

2 Involutory Matrices

Definition of Involutory Matrices:

A matrix A is called *involutory* if it satisfies the condition:

$$A^2 = I$$

where I is the identity matrix of the same size as A. In other words, when you multiply the matrix by itself, you get the identity matrix.

Properties of Involutory Matrices:

- An involutory matrix is always square.
- The eigenvalues of an involutory matrix must be either $+1$ or -1.
- If A is an involutory matrix, then its inverse is equal to itself: $A^{-1} = A$.

Example:

Consider the matrix A:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let's compute A^2:

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $A^2 = I$, matrix A is involutory.

3 Unitary Matrices

Definition of Unitary Matrices:

A matrix A is *unitary* if it satisfies:

$$A^\dagger A = AA^\dagger = I$$

where $A^\dagger$ is the conjugate transpose (also called Hermitian transpose) of A, and I is the identity matrix. The conjugate transpose is obtained by taking the transpose of the matrix and then taking the complex conjugate of each entry.

Properties of Unitary Matrices:

- Unitary matrices preserve lengths and angles, meaning that they represent rotations and reflections in complex vector spaces.
- Unitary matrices are always square matrices.
- The eigenvalues of a unitary matrix lie on the complex unit circle, meaning they have absolute value 1.
- A unitary matrix is invertible, and its inverse is its conjugate transpose: $A^{-1} = A^\dagger$.

Example:

Consider the matrix A:

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Let's compute $A^\dagger A$:

$$A^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Now,

$$A^\dagger A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $A^\dagger A = I$, matrix A is unitary.

4 Hermitian Matrices

Definition of Hermitian Matrices:

A matrix A is said to be *Hermitian* if it is equal to its conjugate transpose:

$$A = A^\dagger$$

This means that every element on the main diagonal of A must be real.

Properties of Hermitian Matrices:

- Hermitian matrices are always square.
- The eigenvalues of a Hermitian matrix are always real.
- A Hermitian matrix can be diagonalized by a unitary matrix, meaning there exists a unitary matrix U such that $U^{-1}AU$ is diagonal.
- The matrix is equal to its conjugate transpose: $a_{ij} = \overline{a_{ji}}$, where $\overline{a_{ji}}$ is the complex conjugate of a_{ji}.

Example:

Consider the matrix A:

$$A = \begin{pmatrix} 3 & 2+i \\ 2-i & 4 \end{pmatrix}$$

The conjugate transpose of A is:

$$A^\dagger = \begin{pmatrix} 3 & 2+i \\ 2-i & 4 \end{pmatrix}$$

Since $A = A^\dagger$, matrix A is Hermitian.

Conclusion

To summarize, we covered the following matrix types:

- **Symmetric Matrices**: $A = A^T$, real eigenvalues, diagonalizable.
- **Involutory Matrices**: $A^2 = I$, eigenvalues are ± 1, $A^{-1} = A$.
- **Unitary Matrices**: $A^\dagger A = I$, eigenvalues lie on the unit circle in the complex plane, preserves length and angles.
- **Hermitian Matrices**: $A = A^\dagger$, real eigenvalues, diagonalizable by a unitary matrix.

Understanding these matrix properties is foundational for advanced topics in linear algebra, quantum mechanics, and various engineering fields.