Introduction to Eigenvectors and Eigenvalues

Podcast Learn & Fun *

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Introduction

We are going to explore a fundamental concept in mathematics called **eigenvectors** and **eigenvalues**. While these terms might sound complicated at first, we'll take a step-by-step approach to break them down and understand them in an intuitive way, using both simple ideas and mathematical definitions.

What Are Eigenvectors and Eigenvalues?

Let's begin by thinking of a **transformation**. A transformation is simply a rule that changes things in a specific way. Imagine you have a piece of paper with some points on it, and you apply a transformation — for example, stretching, shrinking, rotating, or reflecting. When you do this, the points might change their position, but they will all follow the transformation rules.

Now, consider vectors. In simple terms, a **vector** is just an arrow — a direction and a length. For example, an arrow pointing to the right, or an arrow pointing up at a 45-degree angle, represents vectors.

The key idea here is that when you apply a transformation to most vectors, they **move** — they change direction or get longer/shorter. However, there are special vectors (called **eigenvectors**) that, when the transformation is applied, **don't change direction**. They might stretch or shrink, but they stay in the same direction.

Eigenvectors are the special vectors that stay along the same line or direction after the transformation.

The **eigenvalue** is the amount by which the eigenvector is stretched or shrunk.

Example in Simple Terms

Imagine you have a rubber band lying flat on a table. When you pull on it, you stretch it. The rubber band still lies along the same line, it just becomes longer.

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The line the rubber band follows is like an **eigenvector**, and the amount the rubber band stretches is the **eigenvalue**.

Mathematical Definition of Eigenvectors and Eigenvalues

We can now formally define eigenvectors and eigenvalues mathematically.

Eigenvector

A vector $\mathbf{v}$ is an eigenvector of a matrix A if, when we multiply the matrix A by the vector $\mathbf{v}$, the result is a scalar multiple of $\mathbf{v}$. In other words, $\mathbf{v}$ only gets stretched or shrunk but doesn't change direction.

Mathematically, this is written as:

$$A\mathbf{v} = \lambda\mathbf{v}$$

where:

- A is the matrix representing the transformation.
- $\mathbf{v}$ is the eigenvector.
- λ is the **eigenvalue**, a scalar that represents how much the eigenvector is stretched or shrunk.

Eigenvalue

The eigenvalue λ is the scalar that tells you how much the eigenvector is stretched or shrunk during the transformation. If $\lambda > 1$, the eigenvector is stretched. If $\lambda < 1$, the eigenvector is shrunk. If $\lambda = 1$, the eigenvector is unchanged in length.

Geometric Intuition

- **Eigenvectors** are directions that **stay the same** after the transformation.
- **Eigenvalues** are numbers that tell you how much those directions are stretched or shrunk.

Example 1: A 2D Transformation

Let's consider a simple example with a 2×2 matrix that transforms vectors in a 2D space. We'll find the eigenvectors and eigenvalues for this matrix.

Let's take the matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

We want to find the **eigenvalues** and **eigenvectors** for this matrix.

Step 1: Eigenvalue Equation

The eigenvalue equation is:

$$A\mathbf{v} = \lambda\mathbf{v}$$

or equivalently:

$$(A - \lambda I)\mathbf{v} = 0$$

where I is the identity matrix. The goal is to find the values of λ (the eigenvalues) and the corresponding vectors $\mathbf{v}$ (the eigenvectors).

To find the eigenvalues, we need to solve the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

For our matrix A, the characteristic equation becomes:

$$\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$$

Let's calculate the determinant:

$$(2-\lambda)(2-\lambda) - (1)(1) = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$(2-\lambda)^2 = 1$$

Taking the square root of both sides:

$$2 - \lambda = \pm 1$$

So, we have two possible eigenvalues:

$$\lambda_1 = 2 - 1 = 1 \quad \text{and} \quad \lambda_2 = 2 + 1 = 3$$

Step 2: Finding the Eigenvectors

For each eigenvalue, we substitute λ back into the equation $(A - \lambda I)\mathbf{v} = 0$ and solve for the corresponding eigenvector.

- For $\lambda_1 = 1$, we substitute into $(A - \lambda_1 I)\mathbf{v} = 0$:

  $$\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  This gives us the system of equations:

  $$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  This simplifies to:

  $$x + y = 0$$

  So, the eigenvector for $\lambda_1 = 1$ is any vector of the form:

  $$\mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- For $\lambda_2 = 3$, we substitute into $(A - \lambda_2 I)\mathbf{v} = 0$:

$$\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  This gives us the system of equations:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  Which simplifies to:

$$-x + y = 0 \quad \text{or} \quad x = y$$

  So, the eigenvector for $\lambda_2 = 3$ is any vector of the form:

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Conclusion: What Have We Learned?

To recap:

- **Eigenvectors** are special vectors that, when a transformation is applied to them, don't change direction. They might get longer or shorter, but they stay on the same line.
- **Eigenvalues** are the numbers that tell you how much the eigenvectors stretch or shrink when the transformation happens.

In our example, the transformation matrix A had two eigenvalues: 1 and 3. The eigenvectors corresponding to these eigenvalues were $\mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively.

By understanding eigenvectors and eigenvalues, we gain powerful tools to analyze and simplify many real-world problems, such as understanding physical systems, analyzing data, or solving complex mathematical problems.