TEK4050 Sotchastic Systems – Spring 2025
COMPULSORY ASSIGNMENT 2

Kjetil Bergh Ånonsen/Oddvar Hallingstad

Due date: **Monday, May 19, 2023**
Please submit your report in an email with 'TEK4050' in the subject field to **kjetil-bergh.anonsen@ffi.no.**

This assignment is an example of how to simulate deterministic and stochastic systems, run Monte Carlo simulations, make a Kalman filter, run covariance analyses and error budgets. The equations describe a cart with a DC engine that moves horizontally along a line.

Please make a document including the equations that you use (with derivations if they have not been derived in the lecture), matrices with numerical values and figures. Use Matlab, Python or another suited programming language. Include code listings. The description should be so detailed that is possible to check all equations and results.

1 Simulation Model $\mathcal{M}^{\mathcal{S}}$

Given the continuous-discrete system

$$\dot{\underline{x}} = F\underline{x} + Lu + Gv \tag{1}$$

$$z_k = H\underline{x}_k + w_k \tag{2}$$

$$\underline{x}_0 \sim \mathcal{N}(\underline{0}, \hat{P}_0),\ v \sim \mathcal{N}(\underline{0}, \tilde{Q}\delta(t-\tau)),\ w_k \sim \mathcal{N}(\underline{0}, R\delta_{kl}) \tag{3}$$

with associated matrices

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{1}{T_2} & \frac{1}{T_2} \\ 0 & 0 & -\frac{1}{T_3} \end{bmatrix},\ L = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_3} \end{bmatrix},\ G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \tag{5}$$

where x_1 denotes position, x_2 velocity and x_3 armature current.

The values to be used in the simulations are: $T_2 = 5\,\mathrm{s}$, $T_3 = 1\,\mathrm{s}$, $\hat{P}_0 = diag(1, 0.1^2, 0.1^2)$, $\tilde{Q} = 2 \cdot 0.1^2$, $R = 1$, $t_0 = 0$, $t_f = 100\,\mathrm{s}$.

Given a covariance matrix on the form

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \tag{6}$$

we define the standard deviations by

$$\underline{s} = diag(P)^{1/2} = \left[p_{11}^{1/2}; \quad p_{22}^{1/2}; \quad p_{33}^{1/2}; \right].$$

2 Discretization

Find the matrices in the discrete process equation

$$\underline{x}_{k+1} = \Phi \underline{x}_k + \Lambda \underline{u}_k + \Gamma \underline{v}_k; \; \underline{v}_k \sim \mathcal{N}(\underline{0}, \delta_{kl} Q) \tag{7}$$

when $\Delta t = 0.01\,\mathrm{s}$ by implementing the necessary algorithms. Make, e.g., the following functions: [La,Fi] = cp2dpLa(F,L,D), Ga=cp2dpGa(F,G,$\tilde{Q}$,d), S=cp2dpS(F,G,$\tilde{Q}$,d), where $S = \Gamma Q \Gamma^T$.

3 Simulation of stochastic system

We want to investigate how the velocity $x_2(k)$ evolves, both for a deterministic and stochastic process. Plot u and x_2 for the deterministic and stochastic process, respectively, in the same figure. Assume that $u \equiv 1$. Comment on the figure.

4 Optimal Kalman filter

We choose in this case to start with a time update. Run time updates with a frequency of 100 Hz and measurement updates at 1 Hz (every whole second). Write down the Kalman filter equations for the system and make three plots showing:

- Velocity: x_2, $\bar{x}_2$, $\hat{x}_2$, u
- Filtered velocity errors: $x_2 - \hat{x}_2$ and $\pm \hat{s}_2$
- Predicted velocity errors: $x_2 - \bar{x}_2$ and $\pm \bar{s}_2$.

Comment on the figures.

5 Monte Carlo simulation of optimal system

We will here use Monte Carlo simulations to estimate $E\{\underline{\hat{e}}_k\}$ and $\mathrm{Cov}\{\underline{\hat{e}}_k\}$. We denote these estimated $\underline{\hat{m}}_k^N$ and $\hat{P}_k^N$. The system and Kalman filter equations are estimated, and we draw $\underline{x}(0)$, $\underline{w}_k$ and $\underline{v}_k$ from the given distributions, to compute:

$$\underline{\hat{e}}_k;\ k = 0, 1, 2, \ldots, M. \tag{8}$$

By drawing new samples, we are able to generate at set of trajectories:

$$\underline{\hat{e}}_k^j = \underline{x}^j(k) - \underline{\hat{x}}^j(k);\ k = 0, 1, 2, \ldots, M;\ j = 1, 2, \ldots, N, \tag{9}$$

where k denotes the time step number and j trajectory (Monte Carlo run) number.

Note: $\underline{\hat{m}}_k^N$ and $\hat{P}_k^N$ can be computed recursively. $\underline{\hat{s}}^N = diag(\hat{P}^N)^{1/2}$.

Make four plots that include

- Plot 1: $N = 10$, each curve shows both $\hat{x}_2^i(k)$ and $\bar{x}_2^i(k)$
- Plot 2: $N = 10$, each curve shows both $x_2^i(k) - \hat{x}_2^i(k)$ and $x_2^i(k) - \bar{x}_2^i(k)$
- Plot 3: $N = 10$, $\hat{m}_2^N(k)$, $\hat{s}_2^N(k)$, $\hat{s}_2(k)$ (the mean of the velocity error, its standard deviation and the Kalman filter estimate for the velocity standard deviation)
- Plot 4: $N = 100$, $\hat{m}_2^N(k)$, $\hat{s}_2^N(k)$, $\hat{s}_2(k)$

Comment on the plots.

6 Error budget for optimal Kalman filter

In this example we want to run an error budget for the optimal Kalman filter, i.e for $\mathcal{M}^{\mathcal{S}}$. The equations below are valid for the case when the simulation model $\mathcal{M}^{\mathcal{S}}$ and the filter model $\mathcal{M}^{\mathcal{F}}$ are equal, except possibly for the noise description. To distinguish these models in the general case, asterisks are used for the matrices of $\mathcal{M}^{\mathcal{S}}$, but this can be disregarded here, since

$$\begin{aligned} F^* &= F \quad (\Delta F = 0) \\ H^* &= H \quad (\Delta H = 0) \end{aligned}$$

The equations used for the error budget are thus simplified to

$$\dot{\bar{P}}^e = F\bar{P}^e + \bar{P}^e F^T + G\tilde{Q}G^T,\ t \in [\hat{t}_k, \bar{t}_{k+1}] \quad \text{(TU)} \tag{10}$$

$$\hat{P}_k^e = (I - K_k H)\hat{P}_k^e(\bar{t}_k)(I - K_k H)^T + K_k R K_k^T \quad \text{(MU)} \tag{11}$$

$$\hat{P}_0^e = \hat{P}^*(\hat{t}_0) = \bar{P}_0^* \quad \text{(IV)} \tag{12}$$

where TU denotes time update, MU measurement update and IV initial value, respectively.

When the Kalman filter is used, all the error sources will contribute to the error of the filter estimate. An error budget is an overwiew that shows the contributions from each error source (or group of error sources) to the total error. The total error for a state is the root mean square (RMS) sum of the standard deviations of the error sources contributing to the state. This should be equal to the result of a simulation which includes all the error sources.

Make four plots that show

- Error budget for position
- Error budget for velocity
- Error budget for armature current
- The RMS sum of the standard deviations of the error contributions to the position errors, together with the position standard deviation computed by the Kalman filter.

Comment on the plots.

7 Suboptimal Kalman filter

We will here implement a suboptimal Kalman filter an analyze this.

7.1 Suboptimal Kalman filter designed from model reduction

We will here analyze a suboptimal Kalman filter using covariance analysis. Here we use both the simulation model $\mathcal{M}^{\mathcal{S}}$ and the filter model $\mathcal{M}^{\mathcal{F}}$. To distinguish between these, we attach asterisks to the matrices of $\mathcal{M}^{\mathcal{F}}$.

$$\mathcal{M}^{\mathcal{S}} \begin{cases} \dot{\underline{x}} = F\underline{x} + Lu + Gv \\ z_k = Hx_k + w_k \end{cases} \tag{13}$$

with associated matrices

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{1}{T_2} & \frac{1}{T_2} \\ 0 & 0 & \frac{-1}{T_3} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_3} \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v \tag{14}$$

$$z_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \underline{x}_k + w_k. \tag{15}$$

The filter model is found by assuming that the frequency of the armature current model is so high that it can be replaced by white noise. The filter model thus has two state variables:

$$\mathcal{M}^{\mathcal{F}} \begin{cases} \underline{\dot{x}}^* = F^* \underline{x}^* + Lu + G^* v^* \\ z_k = H^* x_k^* + w_k^* \end{cases} \tag{16}$$

with associated matrices

$$\underline{\dot{x}}^* = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T_2} \end{bmatrix} \underline{x}^* + \begin{bmatrix} 0 \\ \frac{1}{T_2} \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{1}{T_2} \end{bmatrix} v^*, \quad v^* \sim \mathcal{N}\big(0, \frac{T_3}{2} \tilde{Q} \delta(t-\tau)\big) \tag{17}$$

$$z_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}_k^* + w_k^*. \tag{18}$$

Find the matrices to use in the general equations for the covariance analysis. The state vector for the general error equations is defined by

$$\underline{x}_a(t) = \begin{bmatrix} \underline{e} \\ \underline{x} \end{bmatrix}. \tag{19}$$

Here we have

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{20}$$

Write a routine that makes the following plots ($\hat{s}_1^o$, $\hat{s}_1^*$, $\hat{s}_1^e$ correspond to optimal system, what the KF estimates and true value, respectively):

- The standard deviations for position $\hat{s}_1^o$, $\hat{s}_1^*$, $\hat{s}_1^e$
- The standard deviations for velocity $\hat{s}_2^o$, $\hat{s}_2^*$, $\hat{s}_2^e$
- The velocities x_2, $\hat{x}_2^*$, $\hat{x}_2^e$.

Comment on the plots.