PMATH 333 Preview Notes

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June 30, 2019

Contents

Chapter 1

The Real Number System

This Chapter Will Be Skipped Due To The Easiness of The Chapter

Chapter 2

Sequences in R

2.1 Limits of Sequences

Infinite Sequence is a function whose domain is N. For example,

$1,\frac{1}{2},\frac{1}{4}$... represents the sequence$\{\frac{1}{2^{n-1}}\}_{n\in\mathbf{N}}$
-1,1,-1,1...represents the sequence $\{(-1)^n\}_{n\in\mathbf{N}}$

- It is important not to confuse a **Sequence** with the **Set**. For example,

Sequence $\{x_n\}_{n\in\mathbf{N}}$ is 1,2,3,4....
Set $\{x_n : n \in \mathbf{N}\}$ could be 2,1,3,4...

- Also, the **Sequence** 1,-1,1,-1, is infinite, but the **Set** $\{(-1)^n : n \in \mathbf{N}\}$ has only 2 points.

Definition 2.1.1

A sequence of real numbers x_n is said to *converge* to a real number $a \in \mathbf{R}$ $\Longleftrightarrow$ for every $\varepsilon > 0$,there is an $N \in \mathbf{N}$ (which in general depends on ε),such that

$$n \geq N \qquad \text{implies} \qquad |x_n - a| < \varepsilon$$

Interchangeable notation We shall use the following phrases and notation interchangeably

a)$\{x_n\}$ converges to a

b)x_n converges to a;

c)$a = \lim_{n\to\infty} x_n$;

d)$x_n \to a$ as $n \to \infty$

e)the *limit* of x_n exists and equals a

2.1.1 Holds for large

Let $\mathcal{P}_n$ be a property indexed by $\mathbf{N}$.We shall say that $\mathcal{P}_n$ *Holds for large* n if there is an $N \in \mathbf{N}$ such that $\mathcal{P}_n$ is true for all $n \geq N$.
Hence a loose summary of Definition 2.1 is that x_n converges to $a \iff |x_n - a|$ is small for large n

2.1.2 At Most One Limit

A sequence can have at most one limits

Proof Suppose that $\{x_n\}$ converges to both a and b. By definition, given $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies $|x_n - a| < \frac{\varepsilon}{2}$and $|x_n - b| < \frac{\varepsilon}{2}$. Thus it follows

$$|a - b| \leq |a - x_n| + |x_n - b| < \varepsilon$$

Since $|a - b| < \varepsilon$, we conclude that a = b[1]

2.1.3 Subsequence

Definition 2.1.2

By a *Subsequence* of a sequence $\{x_n\}_{n \in \mathbb{N}}$, we shall mean a sequence of the form $\{x_n k\}_{k \in \mathbb{N}}$, where each $n_k \in \mathbb{N}$ and $n_1 < n_2 < ...$

- Thus a subsequence $x_{n1}, x_{x2}, ...$ of $x_1, x_2, ...$ is obtained by "deleting" from $x_1, x_2, ...$ all x_n's except those such that $n = nk$

- Subsequence are sometimes used to correct a sequence that behaves badly or to speed up convergence of another that converges slowly.

2.1.4 Example

1. Prove that $\frac{1}{n} \to 0$ as $n \to \infty$
2. if $x_n \to 2$, prove that $\frac{(2x_n+1)}{x_n} \to \frac{5}{2}$ as $n \to \infty$

[1]By Theorem 1.9

Proof 1) Let $\varepsilon > 0$. Use the Archimedean Principle[2] to choose $N \in \mathbf{N}$ such that $N > \frac{1}{\varepsilon}$. By taking the reciprocal of this inequality. We see that $n \geq N$ implies $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$.
Since $\frac{1}{n}$ are all positive, it follows that $|\frac{1}{n} < \varepsilon|$ for all $n \geq N$

Proof 2) Let $\varepsilon > 0$
Since $x_n \to 2$, apply Definition 2.1 to this $\varepsilon > 0$ to choose $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $|x_n - 2| < \varepsilon$
Next, apply Definition 2.1 with $\varepsilon = 1$ to choose N_2 such that $n \geq N_2$ implies $|x_n - 2| < 1$
By Fundamental Theorem of Absolute Values,we have $n \geq N_2$ implies $x_n > 1$
Set $N = \max\{N_1, N - 2\}$ and suppose that $n \geq N$.
Since $n \geq N_1$, we have $|2 - x_n| = |x_n - 2| < \varepsilon$
Since $n \geq N_2$, we have $0 < \frac{1}{(2x_n)} < \frac{1}{2} < 1$ It follows that

$$|\frac{2x_n + 1}{x_n} - \frac{5}{2}| = \frac{|2 - x_n|}{2x_n} < \frac{\varepsilon}{2x_n} < \varepsilon$$

for all $x \geq N$

2.1.5 Example

1. Prove the the sequnece $\{(-1)^n\}_{n \in \mathbb{N}}$ has no limits

Proof Suppose that $(-1)^n \to a$ as $n \to \infty$ for some $a \in R$. Given $\varepsilon = 1$, there i s an $N \in \mathbb{N}$ such that $n \geq N$ implies $|(-1)^n - a| < \varepsilon$
For n odd, this implies $\boxed{|1 + a| = |-1 - a| < 1}$
For n even, this implies $\boxed{|1 - a| < 1}$

$$2 = |1 + 1| \leq |1 - a| + |1 + a| < 1 + 1 = 2$$

that is, 2< 2, a contradiction[3]

2.1.6

[2] Definition An ordered field F has the Archimedean Property if, given any positive x and y in F there is an integer n > 0 so that nx > y.
[3] Triangle Inequality

Chapter 3

Proof Technique

3.1 Limits

3.1.1 Limits Not Exist

1. Prove the the sequnece $\{(-1)^n\}_{n\in\mathbb{N}}$ has no limits

Proof Suppose that $(-1)^n \to a$ as $n \to \infty$ for some $a \in R$. Given $\varepsilon = 1$, there i s an $N \in \mathbb{N}$ such that $n \geq N$ implies $|(-1)^n - a| < \varepsilon$
For n odd, this implies $\boxed{|1 + a| = |-1 - a| < 1}$
For n even, this implies $\boxed{|1 - a| < 1}$

$$2 = |1 + 1| \leq |1 - a| + |1 + a| < 1 + 1 = 2$$

that is, 2<2, a contradiction[1]

[1] Triangle Inequality

Part I

newpart