CS245 Course Notes

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Chapter 1

TBD

1.1 Satisfaction of predicate formulas

Definition 1.1.1

An interpretation $\mathcal{I}$ and enviroment E **satisfy** a formula φ, denoted $\mathcal{I} \vDash_E \varphi$ if $\varphi^{\mathcal{I},E} = T$
They **do no satisfy** φ ($\mathcal{I} \nvDash \varphi$) if $\varphi^{\mathcal{I},E} = F$

1.1.1 Satisfaction Relation

We define the **Satisfaction Relation** $\mathcal{I} \vDash_E \varphi$ recursively defined as follows:

- If $\boxed{\varphi = P(t_1, ..., t_n)}$ for a predicate P and terms $t_1, t_2, t_3, ..., t_n$
  $\mathcal{I} \vDash_E \varphi \iff \langle t_1^{\mathcal{I},E}, ..., t_n^{\mathcal{I},E} \rangle \in P^{\mathcal{I}}$
  In other words,$\mathcal{I}$ and E satisfy $P(t_1, ..., t_n)$ if the terms t_i under $\mathcal{I}$ and E correspond to values in the domain that satisfy the predicate P under $\mathcal{I}$

- If $\boxed{\varphi = (\neg\alpha)}$, then $\mathcal{I} \vDash_E \varphi \iff \mathcal{I} \nvDash_E \alpha$

- If $\boxed{\varphi = (\alpha \land \beta)}$, then $\mathcal{I} \vDash_E \varphi \iff \mathcal{I} \vDash_E \alpha$ and $\mathcal{I} \vDash_E \beta$

- If $\boxed{\varphi = (\alpha \lor \beta)}$, then $\mathcal{I} \vDash_E \varphi \iff \mathcal{I} \vDash_E \alpha$ or $\mathcal{I} \vDash_E \beta$ or both

- If $\boxed{\varphi = (\alpha \to \beta)}$, then $\mathcal{I} \vDash_E \varphi \iff \mathcal{I} \nvDash_E \alpha$ and $\mathcal{I} \vDash_E \beta$ or both

- If $\boxed{\varphi = (\forall x \alpha)}$, then $\mathcal{I} \vDash_E \varphi \iff$ for every $a \in D^{\mathcal{I}}$, we have $\mathcal{I} \vDash_E [x \to \alpha]\alpha$

- If $\boxed{\varphi = (\exists x \alpha)}$, then $\mathcal{I} \vDash_E \varphi \iff$ there is some $a \in D^{\mathcal{I}}$, we have $\mathcal{I} \vDash_E [x \to \alpha]\alpha$

If $\mathcal{I} \vDash_E \varphi$ for every environment E, then $\mathcal{I}$ **Satisfy** φ and we write $\mathcal{I} \vDash \varphi$

1.1.2 Valid

Definition 1.1.2

A formula φ is

- **Valid** if $\mathcal{I} \vDash_E \varphi$ for ever interpretation $\mathcal{I}$ **and** every environment E
- **Satisfiable** if $\mathcal{I} \vDash_E \varphi$ for some Interpretation $\mathcal{I}$ and environment E
- **Unsatisfiable** if $\mathcal{I} \nvDash_E \varphi$ for every Interpretation $\mathcal{I}$ and environment E

Show Entailment DOES Hold For example $a \vDash b$, then assume a is true and b is false $\boxed{\text{seek for } \textbf{contradiction}}$

Show Entailment DOES NOT Hold For example $a \nvDash b$, then show that $\boxed{\text{a is true AND b is false}}$

Question 1.
Prove that the Predicate formula

$$((\forall x\ P(x)) \to (\exists x\ P(x)))$$

is valid.

Solution: For a formula to be valid in Predicate logic, it must be true under any interpretation and with any environment. There are no free variables here, so the choice of environment will not matter to the satisfaction (or not) of the given formula.
Let $\mathcal{I}$ be any interpretation and let E be any environment. Let D be the domain of $\mathcal{I}$. There are two cases for $(\forall x\ P(x))^{(\mathcal{I},E)}$:

(a) If $(\forall x\ P(x))^{(\mathcal{I},E)} = \mathtt{F}$, then by the $\to$-satisfaction rule we have $((\forall x\ P(x)) \to (\exists x\ P(x)))^{(\mathcal{I},E)} = \mathtt{T}$.

(b) If $(\forall x\ P(x))^{(\mathcal{I},E)} = \mathtt{T}$, then let $a \in D$ be arbitrary. Because $(\forall x\ P(x))^{(\mathcal{I},E)} = \mathtt{T}$, by the $\forall$-satisfaction rule, we have that $P(x)^{(\mathcal{I},E[x \mapsto a])} = \mathtt{T}$. Hence by the $\exists$-satisfaction rule, we have $(\exists x\ P(x))^{(\mathcal{I},E)} = \mathtt{T}$. Then by the the $\to$-satisfaction rule we have $((\forall x\ P(x)) \to (\exists x\ P(x)))^{(\mathcal{I},E)} = \mathtt{T}$.

In either case, we have shown that $((\forall x\ P(x)) \to (\exists x\ P(x)))^{(\mathcal{I},E)} = \mathtt{T}$. Since $\mathcal{I}$ and E were arbitrary, this completes the proof.

Not FINISHED

1.2 Natural Deduction for Predicate Logic

Definition 1.2.1

$$\frac{(\forall x \alpha)}{a[t/x]} \forall e$$

if we have a formula $(\forall x \alpha)$, then we can conclude that we can substitute anything for x in α

Definition 1.2.2

$$\frac{a[t/x]}{(\exists x \alpha)} \exists i$$

Chapter 2

Soundness and Completeness

2.1 Soundness

Definition 2.1.1

A proof system is $\boxed{\text{sound}}$ if whenever $\sum \vdash \varphi$, then $\sum \vDash \varphi$

Soundness says that if we have a proof in our proof system, then it says something about the relationship of the formulas involved under all valuations

Definition 2.1.2

A proof system is $\boxed{\text{complete}}$ if whenever $\sum \vDash \varphi$, then $\sum \vdash \varphi$

Chapter 3

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