Time Series Models 2019
Assignments

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Introduction

- All assignments are collected in this document (will be updated)
- Implementation tips in tutorials (Fridays)
- Deadlines as communicated in course outline
- In groups of 3-5 students (same for the exercises)
- "Any" programming language but no use of packages
- Email solutions (.pdf and code) to Ilka at `i.vande.werve@vu.nl`

Assignment 1

Assignment 1.1: LLM (start)

Consider the Local Level model.

(a) Simulate time series from a Local Level Model with initial conditions $\mu_1 \sim \mathcal{N}(a_1 = 0, p_1 = 10^7)$ and variances $\sigma_\varepsilon^2 = 1$ and $\sigma_\eta^2 = q$ with $q \in \{10, 1, 0.1, 0.001\}$;

(b) Implement the Kalman Filter for the four time series;

(c) Plot the observations and filtered estimates for the four time series and interpret;

(d) Present the weight functions for the four time series and interpret;

(e) Repeat this assignment by initialising the Kalman filter at time $t = 2$ with $a_2 = y_1$ and $p_2 = \sigma_\varepsilon^2 + \sigma_\eta^2$ for $q = 0.1$. Do the numerical results change very much? Explain.

Assignment 1.2: Nile data (start)

Consider Chapter 2 of DK-book, there are 8 figures.

(a) Write computer code that can reproduce all these figures (except Figure 2.4 and optionally Figures 2.5 - 2.8, see course outline on Canvas for details);

(b) Repeat this assignment by using the AR(1)+noise model for the *mean adjusted* Nile data;

(c) Write a short documentation of your computer code and explain shortly what you obtained in the different figures and how they relate to each other.

Assignment 2

Assignment 2.1: Nile data (continued)

Finish Assignment 1.2, that is

(a) Reproduce Figures 2.5 - 2.8 if you haven't done so already (LLM for Nile data and AR(1)+noise model for *mean adjusted* Nile data);
(b) Reproduce Figure 2.4 (LLM for Nile data and AR(1)+noise model for *mean adjusted* Nile data);
(c) Write a short documentation of your computer code and explain shortly what you obtained in the different figures and how they relate to each other.

Assignment 2.2: LLM (continued)

In Assignment 1.1 you have simulated time series from the local level model (slide 4) with parameters $\sigma_\varepsilon^2 = 1$ and $\sigma_\eta^2 = q$ and for four different signal-to-noise ratios $q \in \{10, 1, 0.1, 0.001\}$. We revisit this assignment for the local level model.

For each value of q, simulate 100 univariate time series of three different lengths, $T = \{50, 100, 200\}$, and estimate the two variance parameters by the method of maximum likelihood (ML). Then analyze the properties of these ML estimators by, for example, inspection of the empirical distribution of the estimates, the average bias, the sample standard deviation, Normality, etc.

Discuss your results and include comments on possible numerical problems that you have encountered.

Assignment 3

Introduction

Choose **one** of the assignments

- Stochastic volatility model
  - ▷ Also discussed in lectures of week 6
  - ▷ Return data from DK-book
  - ▷ Slides 11 - 13 intro and slides 14 - 16 questions
  - ▷ Hints are updated

- Dynamic Nelson-Siegel model
  - ▷ Also discussed in lectures of week 6
  - ▷ Interest rate data on Canvas
  - ▷ Slides 17 - 29 intro and slides 30/31 questions

Groups with former TSE-students will work on DNS-assignment.

Assignment Stochastic Volatility Model (intro)

Background : denote closing price at trading day t by P_t with its return

$$r_t = \log(P_t \,/\, P_{t-1}) = \Delta \log P_t = \Delta p_t, \qquad t = 1, \ldots, n.$$

The price p_t can be regarded as a discretisized realisation from a continuous-time log-price process $\log P(t)$, that is

$$\mathrm{d} \log P(t) = \mu \, \mathrm{d}\, t + \sigma(t) \, \mathrm{d}\, W(t),$$

where μ is the mean-return, $\sigma(t)$ is a continuous volatility process and $W(t)$ is standardised Brownian motion. We concentrate on the volatility process and we let $\log \sigma(t)^2$ follow a so-called Ornstein-Uhlenbeck process

$$\log \sigma(t)^2 = \xi + H(t), \qquad \mathrm{d}\, H(t) = -\lambda H(t) \, \mathrm{d}\, t + \sigma_\eta \, \mathrm{d}\, B(t),$$

where ξ is constant, $0 < \lambda < 1$, σ_η is the "volatility-of-volatility" coefficient (strictly positive) and $B(t)$ is standardised Brownian motion, independent of $W(t)$.

Assignment SV Model (intro)

The general framework can lead to a statistical model for the daily returns y_t. By applying the Euler-Maruyama discretisation method, we obtain

$$y_t = \mu + \sigma_t \varepsilon_t, \qquad \log \sigma_t^2 = \xi + H_t, \qquad H_{t+1} = \phi H_t + \sigma_\eta \eta_t,$$

where $\phi = 1 - \lambda$ so that $0 < \phi < 1$. Since both σ_t and u_t are stochastic processes, we have a nonlinear time series model.

However, after data transformation $x_t = \log(y_t - \mu)^2$ and some redefinitions, we obtain

$$x_t = h_t + u_t, \qquad h_{t+1} = \omega + \phi h_t + \sigma_\eta \eta_t,$$

where $u_t = \log \varepsilon_t^2$, $\omega = (1 - \phi)\xi$ and $h_t = H_t + \xi$.

We obtain the linear AR(1) plus noise model, but the disturbance u_t is not necessarily Gaussian.

This is the basis of QML for the stochastic volatility (SV) model.

Assignment SV Model (intro)

The SV model for $x_t = \log(y_t - \mu)^2$ is

$$x_t = h_t + u_t, \qquad h_{t+1} = \omega + \phi h_t + \sigma_\eta \eta_t,$$

where $u_t = \log \varepsilon_t^2$. When we assume ε_t is Gaussian, u_t is generated from a $\log \chi^2$ distribution from which the mean and variance are well-defined.

The quasi-Maximum Likelihood (QML) method adopts the Kalman filter to compute the likelihood; do as if ε_t is Gaussian with mean and variance corresponding to those of the $\log \chi^2$ distribution.

The analysis above can be regarded as an approximate analysis.

Assignment SV Model (1/3)

(a) Make sure that the financial series is in returns (transform if needed). Present graphs and descriptives.

(b) The SV model can be made linear by transforming the returns data to $x_t = \log y_t^2$. This is the basis of the QML method. Compute x_t and present a graph. Hint: avoid taking logs of zeros, you can do so by demeaning y_t.

(c) The disturbances in the model for x_t will not be normally distributed. But we can *assume* that they are normal with mean and variance corresponding to those of the $\log \chi^2(1)$ distribution. Estimate the unknown coefficients by the QML method using the Kalman filter and present the results in a Table. Hint: the $\log \chi^2(1)$ distribution has mean -1.27 and variance $\pi^2/2 = 4.93$ (mean adjustment and fixed variance).

(d) Take the QML estimates as your final estimates. Compute the smoothed mean of h_t based on the approximate model for x_t by using the Kalman filter and smoother.

Assignment SV Model (2/3)

We have the SV model given by

$$y_t = \mu + \sigma_t \varepsilon_t, \qquad \log \sigma_t^2 = \xi + H_t, \qquad H_{t+1} = \phi H_t + \sigma_\eta \eta_t.$$

(e) By adopting the QML estimates for the unknown coefficients obtained from the linear model for $x_t = \log(y_t - \bar{y})^2$, consider the SV model for y_t as given before and compute the smoothed mode of H_t using Kalman filter smoothing methods. Hint: if you want to relate it to the model of the DK-book, first rewrite it in the form of equation (9.26).

Assignment SV Model (3/3)

We have the SV model given by

$$y_t = \mu + \sigma_t \varepsilon_t, \qquad \log \sigma_t^2 = \xi + H_t, \qquad H_{t+1} = \phi H_t + \sigma_\eta \eta_t.$$

Now consider the SV model with Student's t disturbances with $\varepsilon_t \sim t(\nu)$, where $t(\nu)$ refers to the *standardized* Student's t distribution with ν degrees of freedom

(f) Reconsider questions (a) - (e) but now for the Student's t SV model. Compare your results between the Gaussian and Student's t SV models, and comment. Hint: in (c), the errors are now normal with mean and variance corresponding to those of the $\log F(1, \nu)$ distribution, estimate these as additional parameters (mean adjustment and variance).

Assignment Dynamic Nelson-Siegel Model (intro)

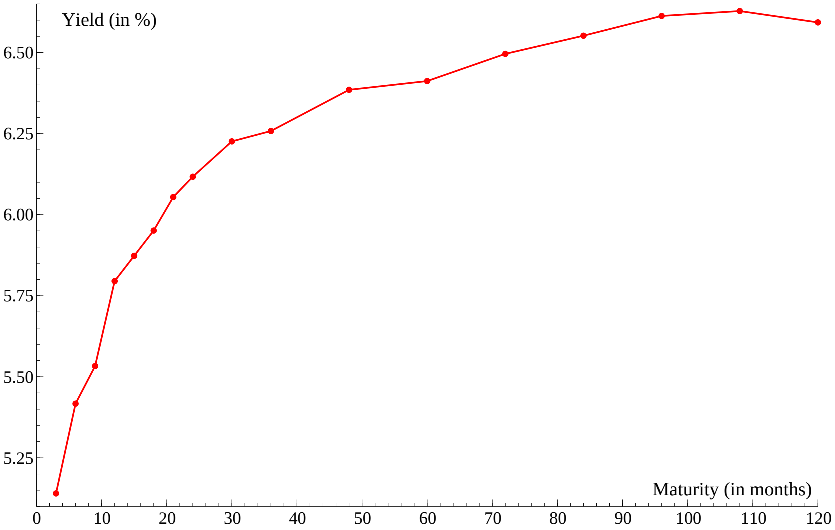
This assignment allows you to study multivariate filtering and estimation for the modelling and analysis of US interest rates (yield curve) by means of the dynamic Nelson-Siegel (DNS) model.

Good references for the DNS model are :

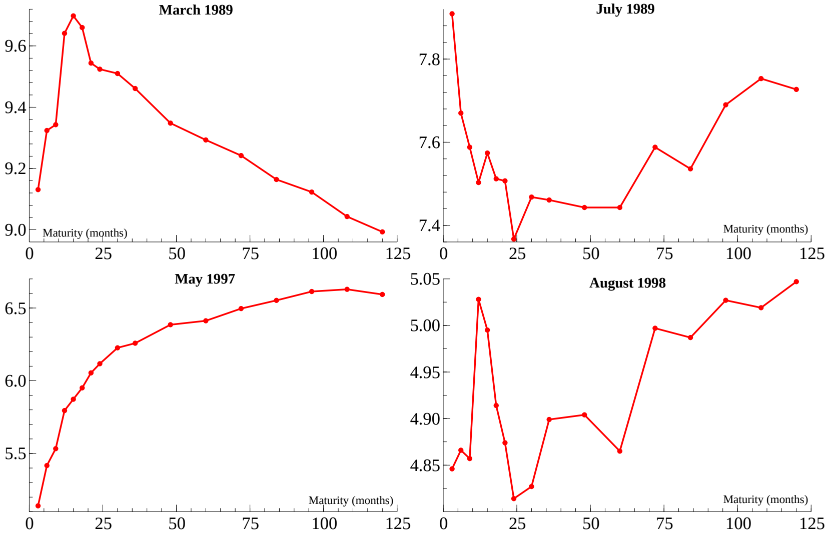
- Nelson and Siegel (J of Business, 1987)
- Diebold and Li (J of Econometrics, 2006)
- Diebold, Rudebusch and Aruoba (J of Econometrics, 2006)
- Koopman, Mallee and van der Wel (KMW, J Business Economics Statistics, 2010)

The dataset of KMW will be made available to you (Fama-Bliss data).

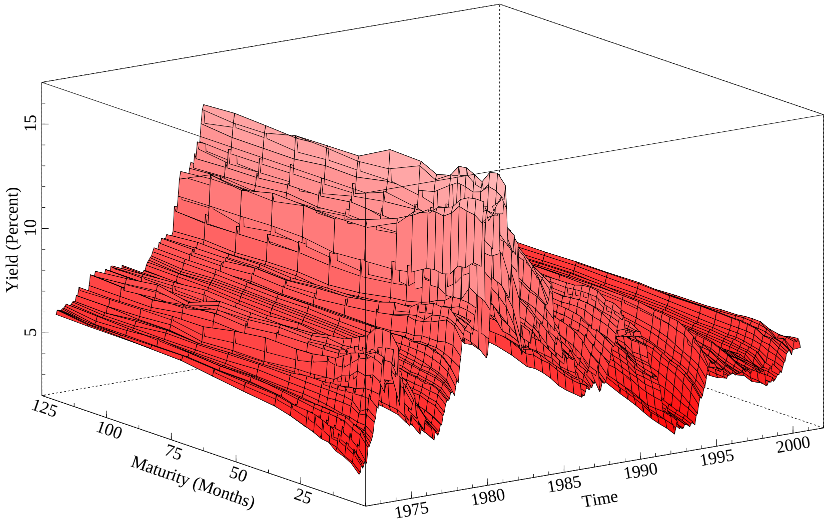
U.S. Yield Curve May 1997



U.S. Yield Curves for Four Months



U.S. Yield Curves over Time



Assignment DNS Model (intro)

- Nelson and Siegel (1987) provide a statistical model for the Yield curve with some economic interpretation.
- At month t, the yield is a function of maturity:

$$\theta(\tau; \lambda, \beta_t) = \beta_{1t} + \beta_{2t} \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \beta_{3t} \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right),$$

  with τ time to maturity (in months), λ a coefficient, and β_{1t}, β_{2t} and β_{3t} (time-varying) factors.
- At each month the yield curve is given by three underlying factors and their associated factor loadings.
- Each of the factors carries an interpretation:
  - First Factor (β_{1t}) – Level,
  - Second Factor (β_{2t}) – Slope,
  - Third Factor (β_{3t}) – Curvature.

Assignment DNS Model (intro)

- When observing a series of interest rates $y_t(\tau_i)$ for a set of N maturities, $\tau_1 < \ldots < \tau_N$, at times $t = 1, \ldots, n$, for some λ value, we can use OLS to estimate the factors:

$$\begin{aligned} y_t(\tau_i) &= \theta_t(\tau_i; \lambda, \beta_t) + \varepsilon_{it} \\ &= \beta_{1t} + \beta_{2t}\left(\frac{1 - e^{-\lambda\tau_i}}{\lambda\tau_i}\right) + \beta_{3t}\left(\frac{1 - e^{-\lambda\tau_i}}{\lambda\tau_i} - e^{-\lambda\tau_i}\right) + \varepsilon_{it}, \end{aligned}$$

  with $\varepsilon_{1t}, \ldots, \varepsilon_{Nt}$ assumed independent, mean zero and constant variance σ^2 for a given t.
- Diebold and Li (2006) recognize that the factors β_{1t}, β_{2t} and β_{3t} are serially correlated and thus forecastable.
- The forecasts of these factors can be used to obtain yield curve forecasts, which outperform other methods such as the random walk and univariate autoregressive models.

Assignment DNS Model (intro)

- Diebold, Rudebusch and Aruoba (2006) go a step further and treat the Nelson-Siegel framework as a state space model or a dynamic factor model.
- The general state space model is given by:

$$\begin{aligned} y_t &= Z_t\alpha_t + \varepsilon_t, & \varepsilon_t &\sim NID(0, H_t), \\ \alpha_{t+1} &= T_t\alpha_t + R_t\eta_t, & \eta_t &\sim NID(0, Q_t), \end{aligned}$$

  with α_t the unobserved state, initial condition $\alpha_1 \sim \mathrm{N}(a_1, P_1)$ and with system matrices Z_t, H_t, T_t, R_t and Q_t.
- The model is linear Gaussian. From the multivariate normal distribution properties, a filtering algorithm (Kalman Filter) can be derived to compute the likelihood function.

Assignment DNS Model (intro)

Dynamic Nelson-Siegel model as a state space model is given by:

$$
\begin{aligned}
y_t &= \Gamma(\lambda)\beta_t + \varepsilon_t, & \varepsilon_t &\sim NID(0, \Sigma_\varepsilon), \\
\beta_{t+1} &= (I - \Phi)\mu + \Phi\beta_t + \eta_t, & \eta_t &\sim NID(0, \Sigma_\eta),
\end{aligned}
$$

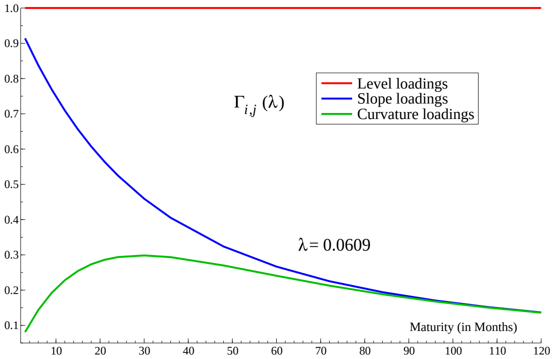
with

$$
\begin{aligned}
y_t &= (y_t(\tau_1), \ldots, y_t(\tau_N))', \\
\Gamma_{ij}(\lambda) &= \begin{cases} 1, & j = 1, \\ \left(1 - e^{-\lambda \cdot \tau_i}\right) \ / \ \lambda \cdot \tau_i, & j = 2, \\ \left(1 - e^{-\lambda \cdot \tau_i} - \lambda \cdot \tau_i e^{-\lambda \cdot \tau_i}\right) \ / \ \lambda \cdot \tau_i, & j = 3, \end{cases} \\
\beta_t &= (\beta_{1t}, \beta_{2t}, \beta_{3t})', \\
\varepsilon_t &= (\varepsilon_{1t}, \ldots, \varepsilon_{Nt})', \\
\eta_t &= (\eta_{1t}, \eta_{2t}, \eta_{3t})'.
\end{aligned}
$$

Here τ_i is the maturity of the i-th interest rate series.
Typical choices for λ are 0.0609 and 0.077.

Assignment DNS Model (intro)

- The parameter λ determines the shape of the factor loadings



- In most studies assumptions are made about λ:
  - Diebold and Li (2006) assume λ is fixed at 0.0609
  - Diebold, Rudebusch and Aruoba (2006) estimate λ

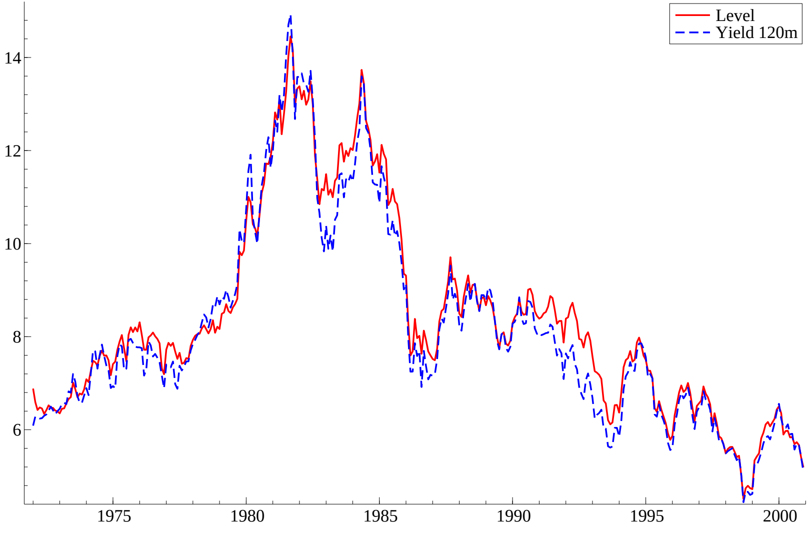
Assignment DNS Model (intro)

DNS Model : Φ and μ estimated by MLE				
	Level_{t-1}	Slope_{t-1}	Curvature_{t-1}	Constant (μ)
Level_t	0.997^{**} 0.00811	0.0271^{**} 0.00889	-0.0216^{*} 0.0105	8.03^{**} 1.27
Slope_t	-0.0236 0.0167	0.942^{**} 0.0176	0.0392 0.0212	-1.46^{**} 0.527
Curvature_t	0.0255 0.023	0.0241 0.0257	0.847^{**} 0.0312	-0.425 0.537

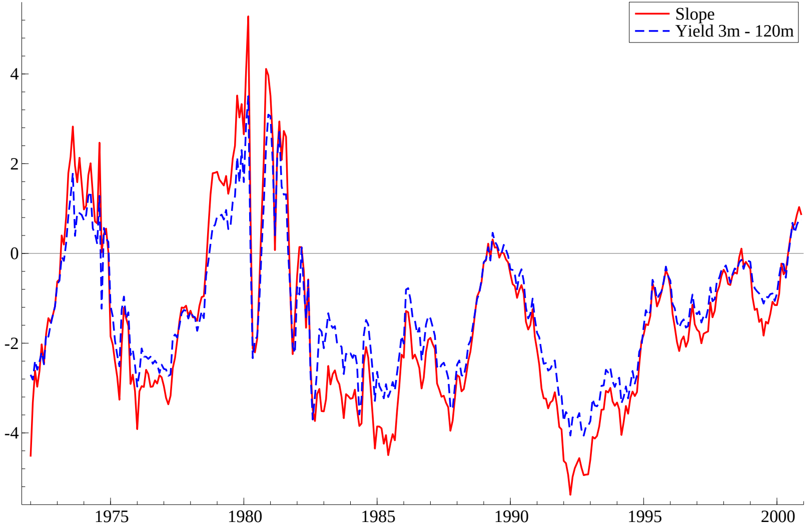
DNS Model : Σ_η estimated by MLE			
	Level_t	Slope_t	Curvature_t
Level_t	0.0949^{**} 0.00841	-0.014 0.0113	0.0439^{*} 0.0186
Slope_t		0.384^{**} 0.0306	0.00927 0.0344
Curvature_t			0.801^{**} 0.0812

Choice for λ is 0.0609.

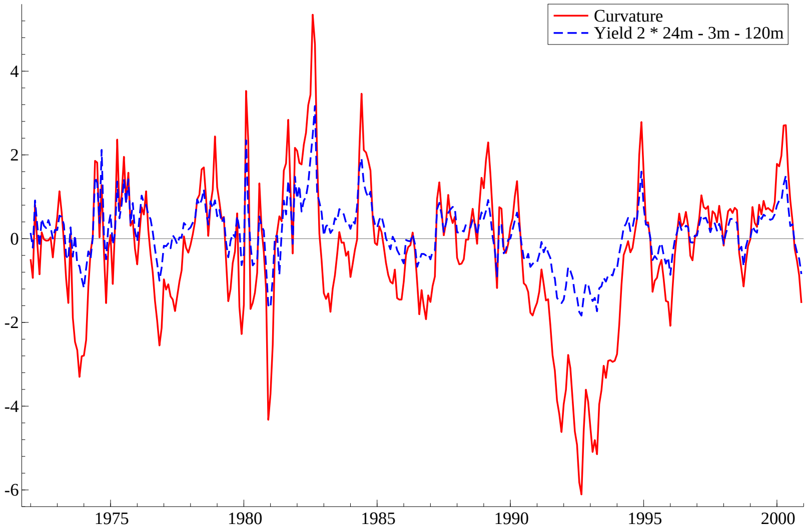
U.S. Yield Curve : estimated "Level" factor



U.S. Yield Curve : estimated "Slope" factor (spread)



U.S. Yield Curve : estimated "Curvature" factor



Assignment DNS Model (1/2)

(a) Provide the details of state space model formulation of the Dynamic Nelson-Siegel (DNS) model.

(b) Develop computer code that computes the factor loadings $\Lambda_{i,j}(\lambda)$ as a function of λ. This procedure enables you to plot the factor loadings for level, slope and curvature, as above. How different are these plots when different values of λ are used ?

(c) The factors are modelled as a VAR(1) process. However, for this assignment, you can consider a trivariate random walk (RW) for the factors :

$$\beta_{t+1} = \beta_t + \eta_t, \qquad \eta_t \sim \text{NID}(0, \Sigma_\eta),$$

where Σ_η is a 3×3 variance matrix. Implement the Kalman filter for the DNS model with $\beta_t \sim$ RW and $\Sigma_\eta = 0.05 \times I_3$.

Assignment DNS Model (2/2)

(d) Implement the equation by equation Kalman filter. Optional: implement the collapsed Kalman filter.

(e) Develop computer code that calculates the Gaussian loglikelihood function for the DNS model with $\beta_t \sim$ RW and for the parameters in Σ_η. Initially, you can keep λ at a fixed value of your choice.

(f) Estimate Σ_η by MLE and report the results. How do your estimates compare with those reported above ?

(g) Given the ML estimate of Σ_η, produce plots of the smoothed estimates of β_t. How do they compare with the plots presented above ?

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