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A UNIFIED APPROACH FOR SOLVING QUADRATIC, CUBIC AND QUARTIC EQUATIONS BY RADICALS

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1. INTRODUCTION

It seems in the literature that methods for solving algebracially quadratic, cubic and quartic equations have very little in common. The goal of this article is to present a unified method for obtaining the algebraic solution of these equations. Presentations of the solutions to the general quartic equation in standard references do not proceed beyond the point of reducing the quartic into a corresponding cubic equation. It is therefore interesting to realize that the unified method presented here leads to a simple, complete system of the four solutions to the general quartic equation. The simplicity of this system of solutions makes it worthwhile to revisit some old problems related to cubic and quartic equations. We therefore investigate when a real quartic has four real roots or two real roots or no real roots, and other similar problems. It turns out that the solution system to the quartic yields elegant answers to these old problems.

2. THE QUADRATIC EQUATION

Let the two roots, w_0 and w_1, of the complex quadratic equation

$$w^2 + a_1 w + a_2 = 0, \tag{2.1}$$

be represented by the complex numbers α and β,

$$\begin{aligned} w_0 &= \alpha + \beta, \\ w_1 &= \alpha - \beta. \end{aligned} \tag{2.2}$$

The two relations between the roots w_0, w_1 and the coefficients a_1, a_2,

$$\begin{aligned} w_0 + w_1 &= -a_1, \\ w_0 w_1 &= a_2, \end{aligned} \tag{2.3}$$

yield, by means of the representation (2.2),

$$\begin{aligned} 2\alpha &= -a_1, \\ \alpha^2 - \beta^2 &= a_2. \end{aligned} \tag{2.4}$$

Equations (2.4) determine the value of α, $\alpha = -a_1/2$, and provide a first degree equation for β^2 from which $\pm\beta$ can be obtained. We thus have the solution system of the general quadratic (2.1),

$$\begin{aligned} w_0 &= -\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4a_2}, \\ w_1 &= -\frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4a_2}, \end{aligned} \tag{2.5}$$

a well-known result. The same method, with an appropriate representation for the roots, applies for cubic and quartic equations as well. We should notice here for later convenience that the square root branch in system (2.5) is fixed but arbitrarily selected. The choice of the branch affects only the *order* of the solutions w_0 and w_1. The solution system (2.5) is therefore unordered, unless one specifies a particular choice for the square root branch.

3. THE CUBIC EQUATION

Let the three roots, w_0, w_1 and w_2, of the complex cubic equation

$$w^3 + a_1 w^2 + a_2 w + a_3 = 0, \tag{3.1}$$

be represented by the complex numbers α, β and γ according to the equations

$$\begin{aligned} w_0 &= \alpha + q_0\beta + q_0\gamma, \\ w_1 &= \alpha + q_1\beta + q_2\gamma, \\ w_2 &= \alpha + q_2\beta + q_1\gamma, \end{aligned} \tag{3.2}$$

where $q_k, k = 0, 1, 2,$ are the three cube roots of unity: $q_0 = 1$, $q_1 = (-1 + i\sqrt{3})/2$ and $q_2 = (-1 - i\sqrt{3})/2$.

The three relations between the roots $w_k, k = 0, 1, 2,$ and the coefficients $a_m, m = 1, 2, 3,$

$$\begin{aligned} w_0 + w_1 + w_2 &= -a_1, \\ w_0 w_1 + w_0 w_2 + w_1 w_2 &= a_2, \\ w_0 w_1 w_2 &= -a_3, \end{aligned} \tag{3.3}$$

yield, by means of the representation (3.2),

$$\begin{aligned} 3\alpha &= -\alpha_1, \\ 3\alpha^2 - 3\beta\gamma &= a_2, \\ \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma &= -a_3. \end{aligned} \tag{3.4}$$

From equations (3.4) one can find the sum and the product

$$3^3(\beta^3 + \gamma^3),$$
$$3^6\beta^3\gamma^3,$$

which are the coefficients of the quadratic equation (3.5) below, the two solutions of which are $(3\beta^3)$ and $(3\gamma^3)$,

$$(3v)^6 + (2a_1^3 - 9a_1 a_2 + 27a_3)(3v)^3 + (a_1^2 - 3a_2)^3 = 0. \tag{3.5}$$

Thus, equations (3.4) determine the value of α, $\alpha = -a_1/3$, and yield a second degree equation, equation (3.5), for β^3 and γ^3. The unknowns β^3 and γ^3 can now be found by the radical formula (2.5) for the roots of a quadratic equation.

Let

$$\begin{aligned} Q &= a_1^2 - 3a_2 \\ R &= -2a_1^3 + 9a_1 a_2 - 27a_3. \end{aligned} \tag{3.6}$$

Then the two solutions of the quadratic equation (3.5) are

$$\begin{aligned} (3\beta)^3 &= \frac{R + \sqrt{R^2 - 4Q^3}}{2} \\ (3\gamma)^3 &= \frac{R - \sqrt{R^2 - 4Q^3}}{2}, \end{aligned} \tag{3.7}$$

and hence,

$$\alpha = -\tfrac{1}{3}a_1,$$
$$\beta = \frac{1}{3}\sqrt[3]{R + \frac{\sqrt{R^2 - 4Q^3}}{2}},$$
$$\gamma = \frac{1}{3}\sqrt[3]{R - \frac{\sqrt{R^2 - 4Q^3}}{2}}, \tag{3.8}$$

The cube root branch in the second equation in equations (3.8) can be chosen arbitrarily while, by the second equation in equations (3.4), the cube root branch in the third equation in equations (3.8) is such that

$$\beta\gamma = Q/9. \tag{3.9}$$

The three roots of the cubic equation (3.1) are therefore w_0, w_1 and w_2 given in equations (3.2) where α, β and γ are given in equations (3.8) and the choice of the cube root branches in equations (3.8) is such that equation (3.9) is satisfied. The choice of the cube root branch affects the order of the solutions and hence the solutions (3.2) of the general cubic equation (3.1) form an unordered set. An analogous result to that of Sections 2 and 3, associated with the general quartic equation will be given in the next section.

4. THE QUARTIC EQUATION

Let the four roots w_k, $k = 0, 1, 2, 3$, of the complex quartic equation

$$w^4 + a_1 w^3 + a_2 w^2 + a_3 w + a_4 = 0, \tag{4.1}$$

be represented by the complex numbers α, β, γ and δ,

$$w_0 = \alpha + \beta + \gamma + \delta,$$
$$w_1 = \alpha + \beta - \gamma - \delta,$$
$$w_2 = \alpha - \beta + \gamma - \delta,$$
$$w_3 = \alpha - \beta - \gamma + \delta. \tag{4.2}$$

The four relations between the roots w_k, $k = 0, 1, 2, 3$, and the coefficients a_m, $m = 1, 2, 3, 4$,

$$w_0 + w_1 + w_2 + w_3 = -a_1,$$
$$w_0 w_1 + w_0 w_2 + w_0 w_3 + w_1 w_2 + w_1 w_3 + w_2 w_3 = a_2,$$
$$w_0 w_1 w_2 + w_0 w_1 w_3 + w_0 w_2 w_3 + w_1 w_2 w_3 = -a_3,$$
$$w_0 w_1 w_2 w_3 = a_4, \tag{4.3}$$

yield, by means of the representation (4.2),

$$4\alpha = -a_1,$$
$$6\alpha^2 - 2(\beta^2 + \gamma^2 + \delta^2) = a_2,$$
$$4\alpha^3 - 4\alpha(\beta^2 + \gamma^2 + \delta^2) + 8\beta\gamma\delta = -a_3,$$
$$\alpha^4 + (\beta^2 + \gamma^2 + \delta^2)^2 - 2\alpha^2(\beta^2 + \gamma^2 + \delta^2) - 4(\beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2) + 8\alpha\beta\gamma\delta = a_4. \tag{4.4}$$

From equations (4.4) one can find the expressions

$$4^2(\beta^2+\gamma^2+\delta^2),$$
$$4^4(\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2),$$
$$4^6\beta^2\gamma^2\delta^2,$$

which are the coefficients of the cubic equation (4.5) below, the three solutions of which are $(4\beta)^2$, $(4\gamma)^2$ and $(4\delta)^2$,

$$(16v^2)^3-(3a_1^2-8a_2)(16v^2)^2+(3a_1^4-16a_1^2a_2+16a_1a_3+16a_2^2-64a_4)(16v^2)$$
$$-(a_1^3-4a_1a_2+8a_3)^2=0. \quad (4.5)$$

Thus, equations (4.4) determine the value of α, $\alpha=-a_1/4$, and yield a third degree equation, equation (4.5), for β^2, γ^2 and δ^2. The method for obtaining the four roots of the quartic equation (4.1) by means of the cubic equation (4.5) and the representation (4.2) is straightforward, and the final result is given below.

The *four roots* of the quartic equation (4.1) are

$$\begin{aligned} w_0&=\alpha+\beta+\gamma+\delta,\\ w_1&=\alpha+\beta-\gamma-\delta,\\ w_2&=\alpha-\beta+\gamma-\delta,\\ w_3&=\alpha-\beta-\gamma+\delta, \end{aligned} \quad (4.6)$$

where if we use the notation

$$\begin{aligned} P&=a_1^3-4a_1a_2+8a_3,\\ Q&=12a_4+a_2^2-3a_1a_3,\\ R&=27a_1^2a_4-9a_1a_2a_3+2a_2^3-72a_2a_4+27a_3^2 \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} \alpha_0&=a_1^2-\tfrac{8}{3}a_2,\\ \beta_0&=\frac{4}{3}\sqrt[3]{R+\frac{\sqrt{R^2-4Q^3}}{2}},\\ \gamma_0&=\frac{4}{3}\sqrt[3]{R-\frac{\sqrt{R^2-4Q^3}}{2}}, \end{aligned} \quad (4.8)$$

then

$$\begin{aligned} \alpha&=-\tfrac{1}{4}a_1,\\ \beta&=\tfrac{1}{4}\sqrt{\alpha_0+\beta_0+\gamma_0},\\ \gamma&=\tfrac{1}{4}\sqrt{\alpha_0+q_1\beta_0+q_2\gamma_0},\\ \delta&=\tfrac{1}{4}\sqrt{\alpha_0+q_2\beta_0+q_1\gamma_0}, \end{aligned} \quad (4.9)$$

q_1 and q_2 being the two non-real cube roots of unity, the roles of which are interchangeable.

Any choice of the cube root branches in equations (4.8) for which

$$\beta_0\gamma_0=16Q/8, \quad (4.10)$$

is allowed and, similarly, any choice of the signs of the square roots in equations (4.9) for which

$$\beta\gamma\delta=-P/64 \quad (4.11)$$

is allowed. The restriction (4.11) for the choice of the signs of β, γ and δ in equations (4.9) follows from the first three equations in equations (4.4).

The choice of the cube root branch and the square root branches affects the order of the solutions. Hence, the solutions (4.6) of the general quartic equation (4.1) form an unordered set.

5. WHEN DOES A REAL QUARTIC EQUATION HAVE FOUR REAL ROOTS, TWO REAL ROOTS OR NO ROOTS?

In this section we use the notation of Section 4, but the coefficients a_m, $m = 1, 2, 3, 4$, of the quartic equation (4.1) are assumed to be real. Counting multiplicities, a quartic equation has four roots. Since complex roots of a real quartic equation appear in pairs, the number of real roots is either four, two or zero. These three possibilities are studied in the following subsections (a), (b) and (c), and are summarized in a theorem. In these subsections we study the quartic equation (4.1) for which the expressions $R^2 - 4Q^3$ of equations (4.8) is respectively positive, zero and negative.

(a) If

$$R^2 - 4Q^3 > 0,$$

then, by equations (4.8) with real cube roots, β_0 and γ_0 are unequal real numbers and hence, by (4.9), γ^2 and δ^2 are non-real complex conjugate to one another. Hence $\gamma\delta$ is real and $\gamma\delta \neq 0$. Therefore, by equations (4.11), β is real. Also α is real, as implied from the first equation in equations (4.9). Since γ is a non-real complex conjugate of either δ or $-\delta$, one of the two elements of the pair $\{\gamma + \delta, \gamma - \delta\}$ is purely imaginary and the other one is real. Hence, the representation (4.2) shows that two of the four roots w_k, $k = 0, 1, 2, 3$, are real and two are non-real.

(b) If

$$R^2 - 4Q^3 = 0,$$

then, from equation (4.8)

$$\beta_0 = \gamma_0 = \frac{4}{3}\sqrt[3]{\frac{R}{2}}, \tag{5.1}$$

where the root in equation (5.1) is chosen to be the real one, in agreement with equation (4.10).

By equations (4.9) and (5.1) we have

$$16\beta^2 = \alpha_0 + \frac{8}{3}\sqrt[3]{\frac{R}{2}},$$

$$16\gamma^2 = 16\delta^2 = \alpha_0 - \frac{4}{3}\sqrt[3]{\frac{R}{2}}. \tag{5.2}$$

The product $\gamma\delta$ is real since $\gamma = \pm\delta$ and γ is either real or purely imaginary. Hence, if $\gamma \neq 0$, then by equation (4.11), β is real. By continuity considerations, β is real also when $\gamma = 0$. Hence, β is always real.

By equations (4.8), (4.9) and (5.2), the three values of the triple-valued expression T,

$$T = 3a_1^2 - 8a_2 + \mathrm{Re}\sqrt[3]{\frac{R}{2}}, \tag{5.3}$$

obtained by taking the three possible cube roots, are $\beta^2/48$, $\gamma^2/48$ and $\delta^2/48$. Hence, γ and δ are purely imaginary if and only if two values of T are negative. Otherwise, γ and δ are real.

Since α and β are real and either $\gamma + \delta = 0$ or $\gamma - \delta = 0$, we see from the representation (4.2) that two roots of equation (4.1) are real. The other two roots of equation (4.1) are real if and only if $T \geqslant 0$ for all the three possible cube roots in equation (5.3).

The expression T in equation (5.3) and the discussion following thereafter can be simplified by considering only the real cube root. We thus may consider the expression

$$T_0 = 3a_1^2 - 8a_2 - \frac{4}{3}\sqrt[3]{\frac{R}{2}}, \tag{5.4}$$

where the cube root in the single valued expression T_0 is the real one. In terms of T_0, γ and δ are real if and only if $T_0 \geqslant 0$. Hence, two roots of equation (4.1) are real; and the remaining two roots of equation (4.1) are real if and only if $T_0 \geqslant 0$.

(c) If

$$R^2 - 4Q^3 < 0,$$

then, by equations (4.8) and (4.10), β_0 and γ_0 are non-real complex conjugate to one another. Hence, by equations (4.9), β^2, γ^2 and δ^2 are real,

$$\begin{aligned} \beta^2 &= (\alpha_0 + 2\mathrm{Re}(\beta_0))/16, \\ \gamma^2 &= (\alpha_0 + 2\mathrm{Re}(q_1\beta_0))/16, \\ \delta^2 &= (\alpha_0 + 2\mathrm{Re}(q_2\beta_0))/16. \end{aligned} \tag{5.5}$$

Since β_0 is non-real, any two of the three numbers $\{\beta^2, \gamma^2, \delta^2\}$ in equations (5.5) are *unequal.*

From equation (4.11) we see that at most two out of the three numbers $\{\beta, \gamma, \delta\}$ are purely imaginary [and *unequal*, by equations (5.5)].

Therefore, the four roots w_k, $k = 0, 1, 2, 3$, in equations (4.2) are real if and only if β, γ and δ are real, and are non-real if and only if at least one of the three numbers $\{\beta, \gamma, \delta\}$ is purely imaginary.

By equations (4.8), (4.9) and (5.5), β^2, γ^2 and δ^2 can be expressed by means of the triple valued expression T,

$$T = 3a_1^2 - 8a_2 + 8\mathrm{Re}\sqrt[3]{R + \frac{\sqrt{R^2 - 4Q^3}}{2}} \tag{5.6}$$

as follows: the three choices of the cube root in evaluating T give the three values $\beta^2/48$, $\gamma^2/48$ and $\delta^2/48$.

The four roots w_k, $k = 0, 1, 2, 3$, in equations (4.2) are therefore real if and only if $T \geqslant 0$ for all the three possible cube roots in equation (5.6). Otherwise, the four roots in equations (4.2) are non-real. In subsections (a), (b) and (c) we have thus proved the following theorem.

Theorem 5.1

Let

$$w^4 + a_1 w^3 + a_2 w^2 + a_3 w + a_4 = 0, \tag{A}$$

be a real quartic equation, and let

$$\begin{aligned} Q &= 12a_4 + a_2^2 - 3a_1 a_3, \\ R &= 27a_1^2 a_4 - 9a_1 a_2 a_3 + 2a_2^3 - 27a_2 a_4 + 27a_3^2 \end{aligned}$$

and

$$T = 3a_1^2 - 8a_2 + 8\mathrm{Re}\sqrt[3]{R + \frac{\sqrt{R^2 - 4Q^3}}{2}}. \tag{B}$$

With multiplicities counted,

(i) if $R^2 - 4Q^3 > 0$ then two and only two roots of equation (A) are real;

(ii) if $R^2 - 4Q^3 = 0$ then two roots of equation (A) are real and the remaining two roots of equation (A) are real if and only if $T \geqslant 0$ for *all* the three possible cube roots in equation (B);

(iii) if $R_2 - 4Q^3 < 0$ then

(1) the four roots of (A) are real if and only if $T \geqslant 0$ for all the three possible cube roots in equation (B)

and

(2) the four roots of equation (A) are non-real if and only if $T < 0$ for at least one of the three possible cube roots in equation (B).

6. WHEN DOES A COMPLEX QUADRATIC, CUBE OR QUARTIC EQUATION HAVE TWO EQUAL ROOTS?

If the two roots of the quadratic equation (2.1) are equal, $w_0 = w_1 = w$, then $a_1 = -2w$ and $a_2 = w^2$, implying that $a_1^2 - 4a_2 = 0$. The converse is also true, as we see from equations (2.5). In some sense, a remarkably similar situation is common to the general cubic and the general quartic equations.

Theorem 6.1

(i) The cubic equation (3.1) has two equal roots if and only if $R^2 - 4Q^3 = 0$; and has three equal roots if and only if $R = Q = 0$, where

$$R = -2a_1^3 + 9a_1 a_2 - 27a_3,$$

$$Q = a_1^2 - 3a_2.$$

(ii) The quartic equation (4.1) has two equal roots if and only if $R^2 - 4Q^3 = 0$; and has three equal roots if and only if $R = Q = 0$, where

$$R = 27a_1^2 a_4 - 9a_1 a_2 a_3 + 2a_2^3 - 27a_2 a_4 + 27a_3^2,$$

$$Q = 12a_4 + a_2^2 - 3a_1 a_3.$$

The results in Theorem 6.1 are not new; see for example Ref. [1]. Their proof, however, is immediate if based on the algebraic solutions of the cubic and the quartic equations as presented in Sections 3 and 4. The remaining possibilities of root equalities are now stated.

Theorem 6.2

(i) The quartic equation (4.1) has two pairs of equal roots if and only if

$$R^2 - 4Q^3 = 0$$

and

$$32R = 27\alpha_0^3.$$

(ii) The quartic equation (4.1) has four equal roots if and only if

$$R = Q = \alpha_0.$$

REFERENCE

1. L. E. Dickson, *Elementary Theory of Equations.* Wiley, New York (1914).