6.2 ACCELERATION OF A RIGID BODY

We now extend our analysis of rigid-body motion to the case of accelerations. At any instant, the linear and angular velocity vectors have derivatives that are called the linear and angular accelerations, respectively. That is,

$$ {}^{B}\dot{V}_{Q} = \frac{d}{dt}\,{}^{B}V_{Q} = \lim_{\Delta t \to o} \frac{{}^{B}V_{Q}(t+\Delta t) - {}^{B}V_{Q}(t)}{\Delta t}, \tag{6.1} $$

and

$$ {}^{A}\dot{\Omega}_{B} = \frac{d}{dt}\,{}^{A}\Omega_{B} = \lim_{\Delta t \to o} \frac{{}^{A}\Omega_{B}(t+\Delta t) - {}^{A}\Omega_{B}(t)}{\Delta t}. \tag{6.2} $$

As with velocities, when the reference frame of the differentiation is understood to be some universal reference frame, $\{U\}$, we will use the notation

$$ \dot{v}_{A} = {}^{U}\dot{V}_{AORG} \tag{6.3} $$

and

$$ \dot{\omega}_{A} = {}^{U}\dot{\Omega}_{A}. \tag{6.4} $$

Linear acceleration

We start by restating (5.12), an important result from Chapter 5, which describes the velocity of a vector ${}^{B}Q$ as seen from frame $\{A\}$ when the origins are coincident:

$$ {}^{A}V_{Q} = {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q. \tag{6.5} $$

The left-hand side of this equation describes how ${}^{A}Q$ is changing in time. So, because origins are coincident, we could rewrite (6.5) as

$$ \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q) = {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q. \tag{6.6} $$

This form of the equation will be useful when deriving the corresponding acceleration equation.

By differentiating (6.5), we can derive expressions for the acceleration of ${}^{B}Q$ as viewed from $\{A\}$ when the origins of $\{A\}$ and $\{B\}$ coincide:

$$ {}^{A}\dot{V}_{Q} = \frac{d}{dt}({}^{A}_{B}R\,{}^{B}V_{Q}) + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_{B} \times \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q). \tag{6.7} $$

Now we apply (6.6) twice—once to the first term, and once to the last term. The right-hand side of equation (6.7) becomes

$$ \begin{aligned} {}^{A}_{B}R\,{}^{B}\dot{V}_{Q} &+ {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q \\ &+ {}^{A}\Omega_{B} \times ({}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q). \end{aligned} \tag{6.8} $$

Combining two terms, we get

$$ {}^{A}_{B}R\,{}^{B}\dot{V}_{Q} + 2\,{}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q). \tag{6.9} $$

Finally, to generalize to the case in which the origins are not coincident, we add one term which gives the linear acceleration of the origin of $\{B\}$, resulting in the final general formula:

$$
\begin{aligned}
&{}^{A}\dot{V}_{BORG} + {}^{A}_{B}R\,{}^{B}\dot{V}_{Q} + 2\,{}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q \\
&\quad + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q).
\end{aligned}
\tag{6.10}
$$

A particular case that is worth pointing out is when ${}^{B}Q$ is constant, or

$$
{}^{B}V_{Q} = {}^{B}\dot{V}_{Q} = 0. \tag{6.11}
$$

In this case, (6.10) simplifies to

$$
{}^{A}\dot{V}_{Q} = {}^{A}\dot{V}_{BORG} + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q) + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q. \tag{6.12}
$$

We will use this result in calculating the linear acceleration of the links of a manipulator with rotational joints. When a prismatic joint is present, the more general form of (6.10) will be used.

Angular acceleration

Consider the case in which $\{B\}$ is rotating relative to $\{A\}$ with ${}^{A}\Omega_{B}$ and $\{C\}$ is rotating relative to $\{B\}$ with ${}^{B}\Omega_{C}$. To calculate ${}^{A}\Omega_{C}$, we sum the vectors in frame $\{A\}$:

$$
{}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C}. \tag{6.13}
$$

By differentiating, we obtain

$$
{}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + \frac{d}{dt}({}^{A}_{B}R\,{}^{B}\Omega_{C}). \tag{6.14}
$$

Now, applying (6.6) to the last term of (6.14), we get

$$
{}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + {}^{A}_{B}R\,{}^{B}\dot{\Omega}_{C} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}\Omega_{C}. \tag{6.15}
$$

We will use this result to calculate the angular acceleration of the links of a manipulator.

6.3 MASS DISTRIBUTION

In systems with a single degree of freedom, we often talk about the mass of a rigid body. In the case of rotational motion about a single axis, the notion of the *moment of inertia* is a familiar one. For a rigid body that is free to move in three dimensions, there are infinitely many possible rotation axes. In the case of rotation about an arbitrary axis, we need a complete way of characterizing the mass distribution of a rigid body. Here, we introduce the **inertia tensor**, which, for our purposes, can be thought of as a generalization of the scalar moment of inertia of an object.

We shall now define a set of quantities that give information about the distribution of mass of a rigid body relative to a reference frame. Figure 6.1 shows a rigid body with an attached frame. Inertia tensors can be defined relative to any frame, but we will always consider the case of an inertia tensor defined for a frame

6.5 ITERATIVE NEWTON–EULER DYNAMIC FORMULATION

We now consider the problem of computing the torques that correspond to a given trajectory of a manipulator. We assume we know the position, velocity, and acceleration of the joints, $(\Theta, \dot{\Theta}, \ddot{\Theta})$. With this knowledge, and with knowledge of the kinematics and the mass-distribution information of the robot, we can calculate the joint torques required to cause this motion. The algorithm presented is based upon the method published by Luh, Walker, and Paul in [2].

Outward iterations to compute velocities and accelerations

In order to compute inertial forces acting on the links, it is necessary to compute the rotational velocity and linear and rotational acceleration of the center of mass of each link of the manipulator at any given instant. These computations will be done in an iterative way, starting with link 1 and moving successively, link by link, *outward* to link n.

The "propagation" of rotational velocity from link to link was discussed in Chapter 5 and is given (for joint $i+1$ rotational) by

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}. \tag{6.31}$$

From (6.15), we obtain the equation for transforming angular acceleration from one link to the next:

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\, {}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\, {}^{i}\omega_i \times \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}. \tag{6.32}$$

When joint $i+1$ is prismatic, this simplifies to

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i. \tag{6.33}$$

The linear acceleration of each link-frame origin is obtained by the application of (6.12):

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R[{}^{i}\omega_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i], \tag{6.34}$$

For prismatic joint $i+1$, (6.34) becomes (from (6.10))

$$\begin{aligned} ^{i+1}\dot{v}_{i+1} = {}& {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i) \\ & + 2\, {}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}. \end{aligned} \tag{6.35}$$

We also will need the linear acceleration of the center of mass of each link, which also can be found by applying (6.12):

$$^{i}\dot{v}_{C_i} = {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times ({}^{i}\omega_i + {}^{i}P_{C_i}) + {}^{i}\dot{v}_i, \tag{6.36}$$

Here, we imagine a frame, $\{C_i\}$, attached to each link, having its origin located at the center of mass of the link and having the same orientation as the link frame, $\{i\}$. Equation (6.36) doesn't involve joint motion at all and so is valid for joint $i+1$, regardless of whether it is revolute or prismatic.

Note that the application of the equations to link 1 is especially simple, because $^{0}\omega_0 = {}^{0}\dot{\omega}_0 = 0$.

$$V_{N+1} = V_N + \omega_N \otimes P_{N,N+1} + \vec{\dot{d}}_{N+1}$$

$$\frac{d}{dt} V_{N+1} = \dot{V}_N + \dot{\omega}_N \otimes P_{N,N+1} + \omega_N \otimes \underline{\dot{P}_{N,N+1}} + \vec{\ddot{d}}_{N+1}$$

$$\boxed{\dot{P} = \left(\omega_N \otimes P_{N,N+1} + \vec{\dot{d}}_{N+1} \right)}$$

$$= \dot{V}_N + \dot{\omega}_N \otimes P_{N,N+1} + \omega_N \otimes \omega_N \otimes P_{N,N+1} + \underline{\omega_N \otimes \vec{\dot{d}}_{N+1}} + \vec{\ddot{d}}_{N+1}$$

$$\frac{d}{dt}\, {}^{N+1}\dot{V}_{N+1} = {}^{N+1}_{N}R \left[{}^{N}\dot{V}_N + {}^{N}\dot{\omega}_N \otimes {}^{N}P_{N,N+1} + {}^{N}\omega_N \otimes {}^{N}\omega_N \otimes {}^{N}P_{N,N+1} + {}^{N}\omega_N \otimes {}^{N}_{N+1}R \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{N+1} \end{bmatrix} \right] + \begin{bmatrix} 0 \\ 0 \\ \ddot{d}_{N+1} \end{bmatrix}$$

2 ?

don't we need this?