STAT 545, Spring 2026
Homework 6

Due: 2/26/26, 11:59pm, through Brightspace
Total points: 50

1. (Rejection sampler for truncated normals) We revisit the problem of truncated normal sampling based on Christian Robert's 1995 paper. Define the truncated normal pdf with left truncation point c as $\pi(x) = \exp(-x^2/2)\mathbf{1}\{x > c\}/\sqrt{2\pi}\Phi(-c)$. Consider the trial density $g(x) = \alpha\exp(-\alpha(x-c))\mathbf{1}\{x > c\}$.

A. [5 points] First show that g is easy to sample from. Specifically, show that if $X \sim \text{Exponential}(\alpha)$ and $Y = X + c$ then $Y \sim g$.

B. [5 points] Consider the ratio $r(x) = \pi(x)/Mg(x)$. Show that if we choose

$$M(\alpha) \quad = \quad \frac{1}{\alpha\sqrt{2\pi}[\Phi(-c)]}\exp(\alpha^2/2 - \alpha c) \quad \text{for} \quad \alpha \geq c,$$

then $r(x) \leq 1 \quad \forall x$.

C. [10 points] Find an expression for the expected acceptance probability of the samples $\mathrm{E}_g[r(X)]$ and show that when $\alpha > c$, the optimal value of $\alpha = \alpha^*$ that maximizes $\mathrm{E}_g[r(X)]$ is given by

$$\alpha^* \quad = \quad \frac{c + \sqrt{c^2+4}}{2}.$$

D. [10 points] Implement a rejection sampler in R based on the optimal choice for α. Compare the acceptance rate of this sampler with another rejection sampler based on a standard Gaussian trial density when you run both the samplers 10,000 times. Consider the cases where $c = 0.5, 1.0, 1.5, 2.0$.

(Note: It can be shown that all choices of $\alpha < c$ result in rejection samplers whose expected acceptance probabilities are suboptimal)

2. [20 points] Consider the following algorithm, consisting of steps A and B:

A. Generate
$$Y_1, Y_2 \overset{ind}{\sim} \text{Exponential}(1)$$
until $Y_2 > (1 - Y_1)^2/2$.

B. Generate $U \sim \text{Uniform}(0, 1)$ and set

$$X = \begin{cases} Y_1, & \text{if } U < 0.5, \\ -Y_1, & \text{otherwise.} \end{cases}$$

Show that X follows a standard normal distribution.