STAT 545, Spring 2026
Homework 4

Due: 2/12/26, 11:59pm, through Brightspace
Total points: 50

Note: Submit your homework through Brightspace as a `.zip` archive consisting of the following files: (1) your written report as a `.pdf` file, (2) your R source code as a `.R` file and (3) any other supporting files as necessary. The numerical results presented in your report must be reproducible from your submitted R code.

1. [20 + 15 + 15 points] Consider a regression model with $p < n$:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where $\mathbf{y} \in \mathbb{R}^n, \mathbf{X} \in \mathbb{R}^{n \times p}$ and $\boldsymbol{\epsilon} \sim \mathrm{N}(0, \sigma^2 \mathrm{I}_n)$ and $\mathrm{rank}(\mathbf{X}) = p$. Recall that the ridge regression estimate is given by

$$\hat{\boldsymbol{\beta}}_{ridge} = (\mathbf{X}^T\mathbf{X} + \lambda \mathrm{I})^{-1}\mathbf{X}^T\mathbf{y},$$

for given $\lambda > 0$. Answer the following:

A. Let the SVD of $\mathbf{X}$ be given by $\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{V}^T$. Show that the fitted value is

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}}_{ridge} = \sum_{j=1}^{p} \mathbf{u}_j \frac{d_j^2}{d_j^2 + \lambda} \mathbf{u}_j^T \mathbf{y}.$$

where $\mathbf{u}_j$ are the columns of $\mathbf{U}$ and $\mathbf{D} = \mathrm{diag}(d_j)$.

B. When the fitted value $\hat{\mathbf{y}}$ has a form $\hat{\mathbf{y}} = \mathbf{M}\mathbf{y}$ where $\mathbf{M}$ is a matrix that does not depend on $\mathbf{y}$ then the "degrees of freedom" is given by the trace of the matrix $\mathbf{M}$. Derive an expression (a scalar quantity) for the degrees of freedom of ridge regression.

C. Take $n = 70, p = 100$. Simulate $\mathbf{X}$ and $\boldsymbol{\epsilon}$ from i.i.d. standard normals. Take $\boldsymbol{\beta}$ to be a vector of all ones and set $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$. Take $\lambda = 1$ and verify that (a) that the ridge estimate is well-defined and (b) both ways of calculating the fitted value above (via matrix inverse and via SVD) indeed result in the same $\hat{\mathbf{y}}$ in `R`. The commands `svd()` for calculating SVD may be useful.