Contribution Title*

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Abstract. The abstract should briefly summarize the contents of the paper in 15–250 words.

Keywords: First keyword · Second keyword · Another keyword.

1 First Section

1.1 A Subsection Sample

Please note that the first paragraph of a section or subsection is not indented. The first paragraph that follows a table, figure, equation etc. does not need an indent, either.

Subsequent paragraphs, however, are indented.

Sample Heading (Third Level) Only two levels of headings should be numbered. Lower level headings remain unnumbered; they are formatted as run-in headings.

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2 Group decomposition based on 4D approximation

We select the better group among the points through the approximation hypervolume of each pair of points in the same layer. Exactly, we will select the pair of points which has the abstract zone with smaller hypervolume. Because the value of the hypervolume exactly is a iterval, so the smaller hypervolume of a pair of points means the over-approximation of the hypervolume is smaller than under-approximation of others or the hypervolume has a smaller over-approximation

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than others which has overlapping with the interval of the hypervolume. At first, we assume that all of the hypervolume and volume mentioned following are convex and all of the situations dicussed following under the $k = 2$.
We will introduct how to expand convex volume calculation formular from three dimension to four dimension, how to obtain the under- and over- approximation of four-dimensional hypervolume in order.

2.1 Expand convex volume calculation formula to four dimensiton

Theorem 1. *Making a three-dimensional prismatic cone as the base volume which volume is V, like the base area of a three-dimensional prismatic cone, integrating the volume of the three-dimensional prismatic cone alone the forth dimension, we will get a hypervolume of four-dimensional prismatic cone* $V^{(4)}$*, which equals to*

$$V^{(4)} = \frac{1}{4} V^{(3)} h^{(4)} \tag{1}$$

In equation 1, $h^{(4)}$ *is the value of the forth dimension, like the 'height' in three-dimensional situation and* $V^{(3)}$ *is the volume as the base of the four-dimensional prismatic cone. Before deriving 1, we will first review the derivation of volume for a three-dimensonal prismatic cone.*

Proof. The main idea is to consider a three-dimensional prismatic cone as a pile of cubes [1]. For example, in Fig. 1, ABCD is a convex polygon on the yOz plane and its area is S. The three-dimensional prismatic cone is consisted of several cubes with base areas similar to the base area ABCD. That means if we use a plane paralleling to ABCD to intercept the prismatic cone J-ABCD, we will get a intersection area $A'B'C'D'$which is similar to ABCD. As a result, there exists a similarity between the areas of $A'B'C'D$(the area is defined as S') and S:

$$S' = \frac{S \cdot x^2}{h^2} \tag{2}$$

x is the distance between $A'B'C'D'$ and ABCD , h is the whole height of the three-dimensional prismatic cone J-ABCD.

For every cube as component of the J-ABCD, its volume definded as V_i, i=1,2,...,n.According to the similarity theorem:

$$V_i = \lim_{n\to\infty} \frac{h}{n} \cdot \frac{S \cdot x_i^2}{h^2} \tag{3}$$

As mentioned before, a three-dimensonal prismatic cone can be considered as a pile of cubes, so the whole volume of J-ABCD can be expressed as:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{h}{n} \cdot \frac{S \cdot x_i^2}{h^2} \tag{4}$$

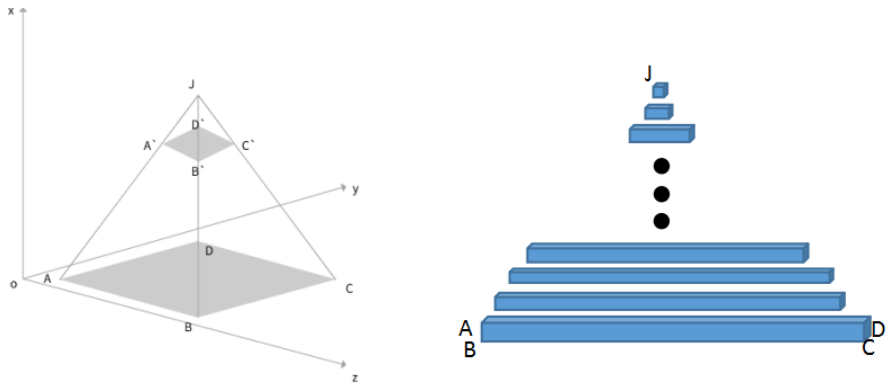

Fig. 1. The right picture of Fig. 1 means if we use a plane paralleling to ABCD to intercept the prismatic cone J-ABCD, we will get a intersection area $A'B'C'D'$which is similar to ABCD. The right one means that the three-dimensional prismatic cone is consisted of several cubes with base areas similar to the base area ABCD.

For $\frac{h}{n} \in (0, 1]$ and $x_i \in [0, h]$, equation 4 and the conception of definite integration can be expressed as:

$$V = \int_0^h \frac{S \cdot x^2}{h^2} dx = \frac{1}{3} Sh \tag{5}$$

equation 5 is the general volume formula of a three-dimensional prismatic cone. Now we have reviewed the expansion from two-dimensional area to three-dimensional volume, it's time to expand from three-dimensional volume to four-dimensional prismatic cone. Comparing to the situation from two-dimensional area to three-dimensional volume,a four-dimensional prismatic cone as a pile of hypercubes(assume that there are n such hypercubes, marked as H_i, i=1,2,...,n), and if we use a three-dimensional space paralleling to the base of the four-dimensional prismatic cone to intercept the four-dimensional hypervolum, the intersection volume(which is three-dimensional) is similar to the base three-dimensional volume [2], like Fig. 2.

If we mark the volume of base three-dimension volume as $V^{(3)}$, the value of the forth dimension as $h^{(4)}$, the intersection volumes as $V_i^{(3)}$, i=1,2,...,n, like the derivation of equation 2 according to the similarity theorem:

$$V_i^{(3)} = \frac{V^{(3)} \cdot x_i^{(4)3}}{h^{(4)3}} \tag{6}$$

In equation 6, $x_i^{(4)}$ is the distence from the base volume to each of the hypercubes H_i. As same as the derivation above, the hypervolum of each four-dimensional hypercubes can be expressed as:

$$V_i^{(4)} = \lim_{n \to \infty} \frac{h^{(4)}}{n} \cdot \frac{V^{(3)} \cdot x_i^{(4)3}}{h^{(4)3}}, i = 1, 2, ..., n \tag{7}$$

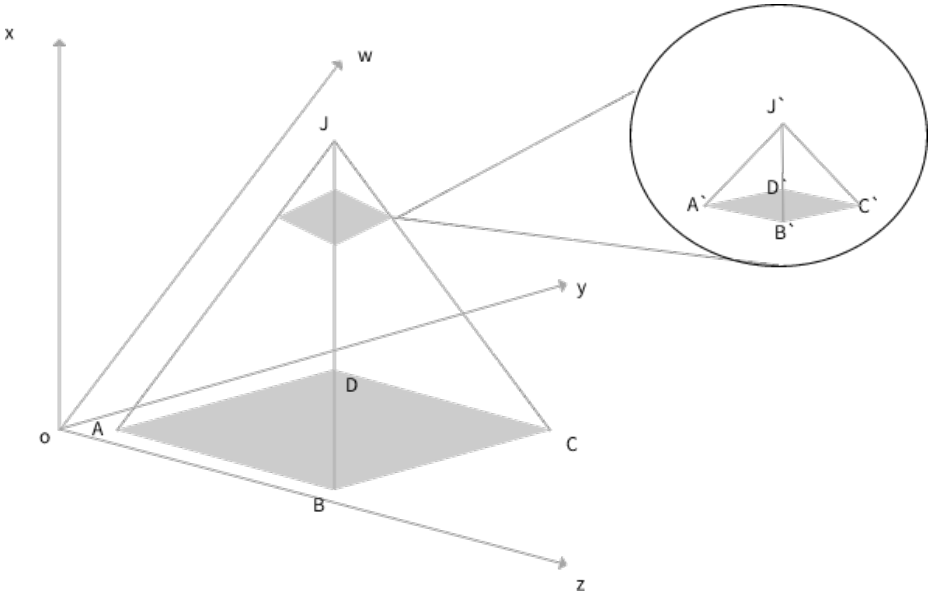

Fig. 2. The picture of Fig. 2 means if we use a three-dimensional space paralleling to the base of the four-dimensional prismatic cone to intercept the four-dimensional hypervolum JABCD, the intersection volume $J' - A'B'C'D'$(which is three-dimensional) is similar to the base three-dimensional volume

So the whole hypervolume defined as $V_{(4)}$ can be expressed as:

$$V^{(4)} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{h^{(4)}}{n} \cdot \frac{V^{(3)} \cdot x_i^{(4)3}}{h^{(4)3}} \tag{8}$$

Because of $\frac{h^{(4)}}{n} \in (0, 1)$

$$V^{(4)} = \int_0^{h^{(4)}} \frac{V^{(3)} \cdot x^{(4)3}}{h^{(4)3}} dx^{(4)} = \frac{1}{4} V^{(3)} h^{(4)} \tag{9}$$

2.2 Obtain approximations of four-dimensional convex hypervolume

Because the state-of-the art algorithms for hypervolume calculation are time-consuming, the runtime is more than $O(d^4)$ [4], d is the number of edges in a hypervolume, or preform far from pretty while facing the value of the hypervolume is a interval with perturbation in neuron net although they can get a approximation of the hypervolume more accurate than ours [5].

Definition 1. *Assume that there is a fully connected net N with n pre-activation layers and there are n_s neurons(or points) in the s th pre-activation layer of N, $s \in \{1, 2, ..., n\}$. Point i and j mean the i th and j th points among the n_s points*

in the s th pre-activation layer of N, $i, j \in \{1, 2, ..., n_s\}, i \neq j$*. A point also means a tuple* (x, y)*,* x *is the output of the point as the pre-activation value to ReLU layer,* y *means the result of ReLU(x).*

When calculating 2-relu, $k = 2$*, assume that the pre-activation value and ReLU result of point i are* x_1 *and* y_1*, similarly we have* x_2 *and* y_2 *of point j. And we start spliting quadrants from* x_1 *dimention, as a result, the* y_2 *will be the additional forth dimention.*

Given the set of three-demensional constraints $C^{(3)}$*,* $c^{(3)}_{(i,j)} \in C^{(3)}$ *is defined as three-denmensional constraints generated from point i and j. Given the set of four-demensional constraints* $C^{(4)}$*,* $c^{(4)}_{(i,j)} \in C^{(4)}$ *is defined as four-denmensional constraints generated from point i and j. Define that* $c^{(3)}_{(i,j),\{x\leq 0\}}$ *be the part of* $c^{(3)}_{(i,j)}$ *satisfying the constrain* $x \leq 0$ *and* $x \in \{x_1, x_2, y_1\}$*. Similarly,* $c^{(3)}_{(i,j),\{x\geq 0\}}$ *be the part of* $c^{(3)}_{(i,j)}$ *satisfying the constrain* $x \geq 0$*. In addition, define* $c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)} \wedge c^{y_2}\right\}}$ *as a* $c^{(4)}_{(i,j)}$ *whose four-dimensional constrains are consisted of* $c^{(3)}_{(i,j)} \in C^{(3)}$ *and a unitary linear constraint* c^{y_2} *about the forth dimension* y_2*, we can find that the constrains of* $c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)} \wedge c^{y_2}\right\}}$ *equals to the ones of* $c^{(3)}_{(i,j)} \wedge c^{y_2}$*. For example,* $c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x\leq 0\}} \wedge y_2=0\right\}}$ *means the part of* $c^{(3)}_{(i,j)} \in C^{(3)}$ *satisfying the constrain* $x \leq 0$ *and a unitary linear constraint* $y_2 = 0$ *about* y_2 *make up a set of four-dimensional constraints* $c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x\leq 0\}} \wedge y_2=0\right\}} \in C^{(4)}$*.*

Definition 2. *According to Def. 1, we can define volume of the abstract zone of* $c^{(3)}_{(i,j)} \in C^{(3)}$ *as* $V^{(3)}_{\left\{c^{(3)}_{(i,j)}\right\}}$*, and hypervolume of the abstract zone of* $c^{(4)}_{(i,j)} \in C^{(4)}$ *as* $V^{(4)}_{\left\{c^{(4)}_{(i,j)}\right\}}$ *whose upper and lower approxiations are defined as* $V^{(4)}_{\left\{c^{(4)}_{(i,j)}\right\},upper}$ *and* $V^{(4)}_{\left\{c^{(4)}_{(i,j)}\right\},lower}$ *respectively.*

Now, we have defined some symbols, it is time to get the formula of the four-dimentional approximations. For example, assume that one of the three-dimensional abstract zones generated by PRIMA is a cube, shown by the left picture of Fig. 3. Then we add the forth dimension y_2 to expand the three-dimensional abtract zone to a four-dimensional one, the result shown by the middle of Fig. 3. Finally, the blue part of the right one is the abstrat of $conv(c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}} \wedge y_2=x_2\right\}}, c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\leq 0\}} \wedge y_2=0\right\}})$ which is the final four-dimensional convex hull $c^{(4)}_{(i,j)}$ obtained by PRIMA.

According to the properties of the four-dimensional abstract zone generated by PRIMA, we use $\frac{1}{4}V^{(3)}_{\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\right\}}u_2$ as the under-approximation of hypervolume and $\frac{1}{4}V^{(3)}_{\left\{c^{(3)}_{(i,j)}\right\}}u_2$ as the over-approximation of hypervolume, of which u_4 is the upper bound of y_2. Here is a example in Fig. 4 about the under- and over-

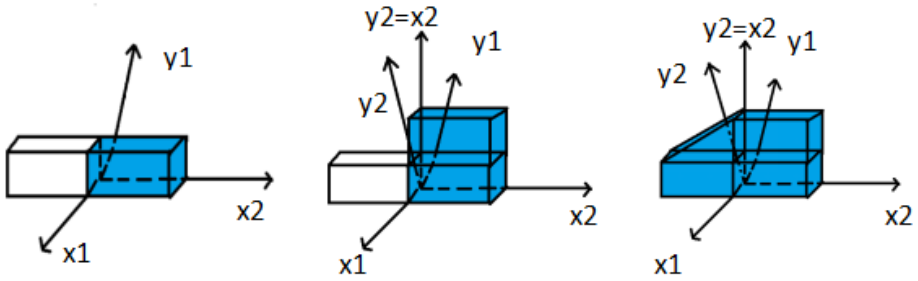

Fig. 3. We assume that the left picture is one of the three-dimensional abstract zones $c^{(3)}_{(i,j)}$ generated by PRIMA, the shape of the abstract zone is a cube, its blue part is the abstract zone of $c^{(3)}_{(i,j),\{x_2\geq 0\}}$. The blue part of the middle one is the abstract zone of $c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\wedge y_2=x_2\right\}}$. the blue part of the right one is the abstrat of $conv(c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\wedge y_2=x_2\right\}}, c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\leq 0\}}\wedge y_2=0\right\}})$ which is the final four-dimensional convex hull $c^{(4)}_{(i,j)}$ obtained by PRIMA.

approximation based on Fig. 3 and we will give the proof about approximation in the following part.

Before giving the proof about approximation, give Theorem 2 first as the front theorem for the proof of approximation.

Theorem 2. *According to the geometric significance and properties of eterminant [3] ,the volume of a three-dimensional convex polyhedra can be expressed as a sum of volumes of some three-dimensional prismatic cones, and the hypervolume of a four-dimensional convex polyhedron can be expressed as a sum of hypervolumes of some four-dimensional prismatic cones. The hypervolume of a four-dimensional convex polyhedron can be expressed by a sum of volumes of some three-dimensional prismatic cones.*

Proof. Assume that the hypervolume of a four-dimensional convex pollyhdron is $V^{(4)}$*,we have known that* $V^{(4)}$ *can be expressed as a sum of hypervolumes of several four-dimensional prismatic cones marked as* $V^{(4)}_1, V^{(4)}_2, ..., V^{(4)}_n$*, the base of each of the* $V^{(4)}_i, i = 1, 2, ..., n$ *is marked as* $V^{(3)}_{base_i}$*,i=1,2,...,n. We have known that* $V^{(3)}_{base_i}$ *can be expresses as a sum of volumes of several three-dimensional prismatic cones marked as* $V^{(3)}_{i,1}, V^{(3)}_{i,2}, ..., V^{(3)}_{i,n_i}$*,* n_i *is the number of three-dimensional prismatic cones consisting* $V^{(3)}_{base_i}$*, with the help of Theorem. 1, we can get the relationships between* $V^{(4)}$ *and* $V^{(3)}_{i,j}, i = 1, 2, ..., n, j = 1, 2, ..., n_i$*,*$h^{(4)}_i$ *is the forth dimension 'height' of each of* $V^{(4)}_i$*:*

$$V^{(4)} = \sum_{i=1}^{n} V^{(4)}_i = \sum_{i=1}^{n} \frac{1}{4} h^{(4)}_i V^{(3)}_{base_i} = \sum_{i=1}^{n} \frac{1}{4} h^{(4)}_i \sum_{j=1}^{n_i} V^{(3)}_{i,j} \tag{10}$$

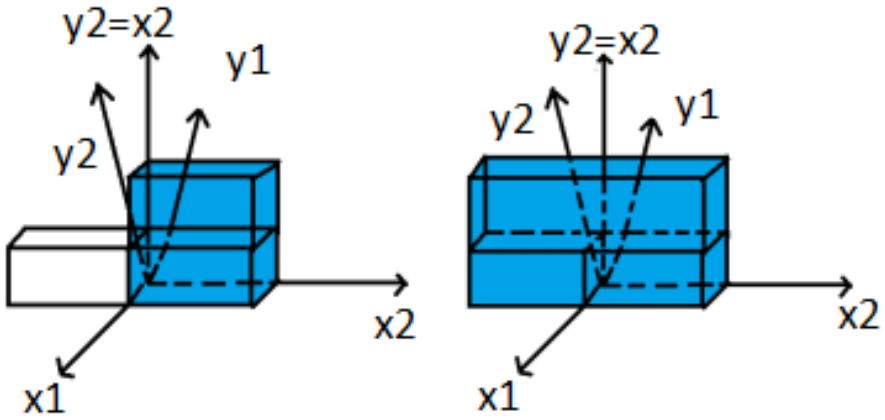

Fig. 4. In the left part of this picture, we directly use the $V^{(3)}_{\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\right\}}$ as the base, the upper bound of x_2, u_2, as the four-dimensional 'height' to structure the under-approximation $V^{(4)}_{\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\wedge y_2=x_2\right\}}$. In the right part of this picture, we use the $V^{(3)}_{\left\{c^{(3)}_{(i,j)}\right\}}$ as the base, the upper bound of x_2, u_2, as the four-dimensional 'height' to structure the over-approximation $V^{(4)}_{\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}}$

According to Theorem.ẽfT2, we are able to find relationships between three-dimensional convex hull and four-dimensional convex hull in order to obtain the under- and over- approximation mentioned before.

Proof. For two selected points i and j, their precise constrains formed by PRIMA is marked as $c^{(4)}_{(i,j),precise} \in C^{(4)}$, so with the help of Def. 1 and Def. 2, the hyper-volume of the abstract zone from $c^{(4)}_{(i,j),precise}$ can be marked as $V^{(4)}_{\left\{c^{(4)}_{(i,j),precise}\right\}}$. There has a equation according to the process of PRIMA or k-ReLU(assume that split about x_1 firstly so that the y_2 will be the forth dimention):

$$c^{(4)}_{(i,j),precise} = conv(c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\leq 0\}}\wedge y_2=0\right\}}, c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\wedge y_2=x_2\right\}}) \tag{11}$$

Previously there has inequality relationships:

$$V^{(4)}_{\left\{c^{(4)}_{(i,j),precise}\right\}} \geq V^{(4)}_{\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\wedge y_2=x_2\right\}} \tag{12}$$

Otherwise, it will have contradictions with equation 11. Because of euaqtion 12, we select $V^{(4)}_{\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\wedge y_2=x_2\right\}}$ as the $V^{(4)}_{\left\{c^{(4)}_{(i,j)}\right\},lower}$:

$$V^{(4)}_{\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\wedge y_2=x_2\right\}} = V^{(4)}_{\left\{c^{(4)}_{(i,j)}\right\},lower} = \frac{1}{4}V^{(3)}_{\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\right\}}h_{(4)} \tag{13}$$

In equation 13, $h_{(4)}$ is the maximum value of the forth dimention y_2, because the constrains $y_2 = x_2$ and $x_2 \in [l_2, u_2]$, with the help of Theorem. 1 and Theorem. 2, the finall equation for $V^{(4)}_{\left\{c^{(4)}_{(i,j)}\right\},lower}$ is:

$$V^{(4)}_{\left\{c^{(4)}_{(i,j)}\right\},lower} = \frac{1}{4}V^{(3)}_{\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\right\}}u_2 \tag{14}$$

Proof. At First, mark $c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\leq 0\}}\wedge y_2=0\right\}}$ as con_1, $c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\wedge y_2=x_2\right\}}$ as con_2 for convenience. As mentioned before, the precise four-dimensional constrains is $c^{(4)}_{(i,j),precise}$ in equation 11. Because of $x_2 \in [l_2, u_2]$, we have:

$$con_1 = c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\leq 0\}}\wedge y_2=0\right\}} \subseteq c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}} \tag{15}$$

$$con_2 = c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j),\{x_2\geq 0\}}\wedge y_2=x_2\right\}} \subseteq c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}} \tag{16}$$

Next, we will use rebuttal method to prove that $conv(con_1, con_2) \subseteq c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}}$:

if $conv\,(con_1, con_2)$ is not belongs to $c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}}$, exsits a point α. Because $conv(con_1, conv2)$ is a convex hull, so there exsits $\beta_1, \beta_2 \in \{con_1, con_2\}$ and $\lambda_0 \geq 0$ s.t.

$$\lambda_0\beta_1 + (1-\lambda_0)\beta_2 = \alpha \tag{17}$$

Because $\beta_1, \beta 2 \in \{con_1, con_2\}$ and equation 16 and equation /refD2, there exsits $\gamma_1, \gamma_2, \theta_1, \theta_2 \in conv(con_1, con_2) \subseteq c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}}$ and $\lambda_1, \lambda_2 \in (0, 1)$ s.t.

$$\beta_1 = \lambda_1\gamma_1 + (1-\lambda_1)\gamma_2 \tag{18}$$

$$\beta_2 = \lambda_2\theta_1 + (1-\lambda_2)\theta_2 \tag{19}$$

Because $\gamma_1, \gamma_2, \theta_1, \theta_2 \in c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}}$ and $c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}}$ is a convex hull formed by PRIMA or k-ReLU, $\beta_1, \beta_2 \in c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}}$ as the same as $\alpha \in c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}}$. Here is a contradiction.

Therefore, we have used rebuttal method to prove that $conv(con_1, con_2) \subseteq c^{(4)}_{(i,j),\left\{c^{(3)}_{(i,j)}\wedge y_2=x_2\right\}}$.

As a result, $V^{(4)}_{\{conv(con_1,con_2)\}} \leq V^{(4)}_{\left\{c^{(4)}_{(i,j)},\left\{c^{(3)}_{(i,j)} \wedge y_2 = x_2\right\}\right\}} = V^{(4)}_{\left\{c^{(3)}_{(i,j)} \wedge y_2 = x_2\right\}}$

Finally with the help of Theorem. 1 and Theorem. 2, we get the over-approximation of $V^{(4)}_{\left\{c^{(4)}_{(i,j)}\right\},upper}$:

$$V^{(4)}_{\left\{c^{(4)}_{(i,j)}\right\},upper} = \frac{1}{4} V^{(3)}_{\left\{c^{(3)}_{(i,j)}\right\}} u_2 \tag{20}$$

Then, if we can find a method to get the volume of the three-dimensional abstract zone generated from point i and j, we will get a better group of points to make k-ReLU for reducing time without losing a lot of accuracy.

2.3 Obtain the volume of three-dimensional volume

At first, because we only discuss the situation that $k = 2$, so it is not quiet difficult for us to use enumeration method to get the whole situations of points which will form the three-dimensional convex hull according to the process of PRIMA or k-ReLU. Then,make the points we get in the first step as the input of Melkman's Algorithm [6] to get the vertex-representation of the convex hull. Finally, we use three-order determinant [3], [7] to calculate the volume of the three-dimensional abstarct zone about i and j. Through equation 14 and equation 20, we can know that the four-dimensional approximation is in direct proportion to their three-dimensional approximation, so we only need to compare the three-dimensional .Algorithm. 1 shows the process of our algorithm.

3 Experiments

In this section, we will evaluate the effectiveness of Our selection strategy and show that it improves over state-of-the-art verifiers on a range of challenging perturbations on ReLU-based networks. Furter, we will also give the consparison among the robust radius of verifiers with variours selecting strategies on MNIST networks under $k = 2$.

3.1 Experimental setup

The neural network certification benchmarks for fully connected networks were worked on a 8 cores 2.90GHz Intel(R) Core(TM) i7-10700 CPU with 12GB of main memory and use Gurobi 9.5.1 for solving LP problems [8].

3.2 Image Classification with ReLU activation

For our experiments, we use the same setup as PRIMA in ERAN Toolbox [9], before multi-verificaiton, we use DEEPOLY to do backpropagation once to make the three-dimensional octahedral inputs required by PRIMA. All constrains generated from PRIMA are added to the LP encoding of the network, after all layers

Algorithm 1: Our algorithm

Input: the neuron net information *nn*, the number of present pre-activation layer *layerdepth*, the lower and upper bound of the present pre-activation layer *lb*, *ub*,the lower and upper bound of the previous pre-activation layer *lbi*, *ubi*

Output: a list record selected points pairs *kactargs* and points haven't pair *restpoints*

```
split0 ← unkownact(lb,ub);
w ← nn.weight[layerdepth];
b ← nn.bias[layerdepth];
record = -np.ones((4,len(split0)));
for i in range(len(split0)-1) do
    for j in range(i+1,len(split0)) do
        V_lower,V_upper ← 3Dcalculation(w,b,lb,ub,lbi,ubi,i,j);
        if record[0][i] == -1 then
            if V_upper and V_lower then
                record[i]← V_lower,V_upper,split0[j],False
            end
        end
        if record[0][j] == -1 then
            if V_upper and V_lower then
                record[j]← V_lower,V_upper,split0[i],False
            end
        end
        if V_upper ≤ record[1][i] and V_lower ≥ record[0][i] then
            record[i]← V_lower,V_upper,split0[j]
        end
        if V_upper ≤ record[1][j] and V_lower ≥ record[0][j] then
            record[j]← V_lower,V_upper,split0[i]
        end
        if V_upper ≤ record[0][i] then
            record[i]← V_lower,V_upper,split0[j],True
        end
        if V_upper ≤ record[0][J] then
            record[J]← V_lower,V_upper,split0[I],True
        end
    end
end
restpoints.append[i for i in split0 if record[3][i] == Flase];
kactargs ←getpair(record,split0);
return kactargs,restpoints;
```

are processed, an LP solver will prove the property. *PRIMA-h* is the PRIMA using heuristic selecting strategy in ERAN Toolbox. *PRIMA-all* is the PRIMA considering all of the C_n^2 constrains while verification, n is the number of the points with lower bound $l \leq 0$ and upper bound $u \geq 0$ in one pre-activation layer. *PRIMA-o* is the PRIMA using our selecting strategy mentioned in Sec. 2. Our experiment process on MNIST 10×80 network.

Netwrok	ε	PRIMA-h			PRIMA-all			**PRIMA-o**		
MNIST 10×80		Ver	T(s)	Acc.	Ver	T(s)	Acc.	Ver	T(s)	Acc.
	0.026	43753	3243.3	94	54034	3311.4	94	4226	3160.2	94
	0.05	137331	4447.4	90	168460	4551.5	91	10544	4183.3	90
	0.1	865177	9703.8	47	1049824	10240.6	49	125906	8233.3	49
	0.1125	1225209	11970.9	24	1510080	12729.5	30	97636	10133.1	29

Fig. 5. In the this table, *Ver* is the number of constrains generated by the strategy. *T* is the whole time for the program of the strategy. *Acc.* is the number of verified images among 100 pictures in MNIST test set.

According to Fig. 5, on the one hand, we can find that the number of verified images obtained by PRIMA-o is always more than the one obtained by PRIMA-h, and the execution time of ours is always less than the one of PRIMA-h. The reason is that the heuristic selection strategy of PRIMA-h ignore some points while doing multi-verification in order to save time, the ignored part bring the loss of accuracy. On the other hand, because PRIMA-all consider all of the possible groups to generate constrains, the constrains generated from our strategy are included in it. This experiment prove that our strategy can save a lot of computer sources with losing a little of accuacy through comparing with PRIMA-all. For example, when $\epsilon = 0.1125$, PRIMA-o work about 20.3% faster than PRIMA-all with losing about 3.3% verified pictures on the 10×80 MNIST network.

3.3 Improvement for precision on high-dimensional networks

In this experiment, we take the first 10 images from test set of MNIST as input. Respectively use PRIMA-all, PRIMA-o, and PRIMA-h to calculate the robust radius of these 10 images on 10*times* 80 MNIST network in order to comspare the difference on accuracy among these three strategies, under $k = 2$. The method of calculating robust radius is dichotomy [10] and the iterations is 15, accurating to the extreme position, the result shown by Fig. 6.

In the histogram in Fig 6, the blue part means the robust radius obtained by PRIMA-all, the cyan one means the robust radius obtained by PRIMA-o,

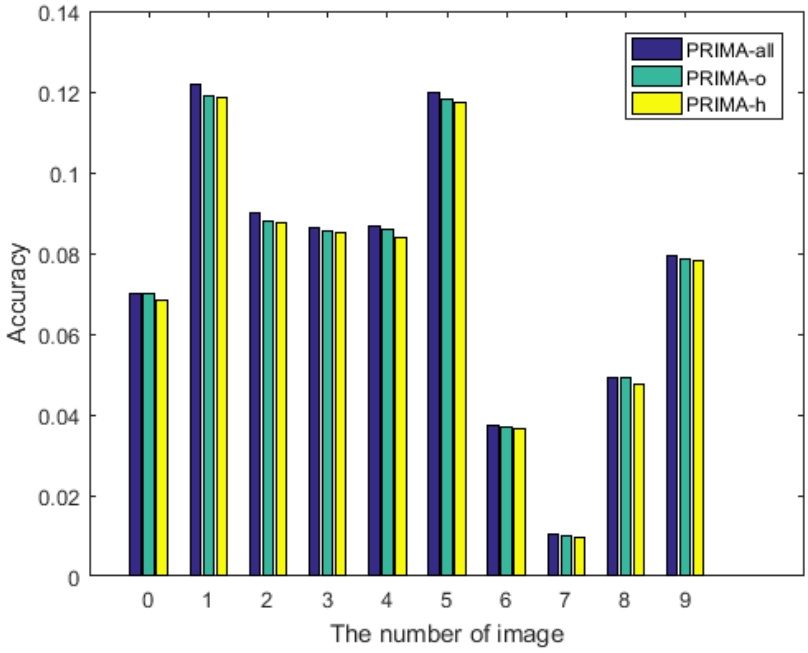

Fig. 6. Comparison of verified robust radius among PRIMA-all, PRIMA-h, and PRIMA-o.

the yellow one means the robust radius obtained by PRIMA-h. PRIMA-all cost XXXs to get only about 0.0017 advance than PRIMA-h, however, PRIMA-o just spending XXXs will have about 0.0015 adcanve than PRIMA-h. According to the experiment, PRIMA-o fills the accuracy gap between PRIMA-all and PRIMA-h and cost less time than others.

4 Conclusion

Among the severals of NN verification methods, considering the relationships and constrains among points in the same pre-activation layer is a new and promising direction. It provide a novel and effeciive way to overcome the *Delta* barrier produced by just considering the constrains of one point. In this paper, according to some experments fact from other work and mathematical theories, we propose a new method and thinking to select which pairs of points is better to generate multi-neuron approximations. The experimental result shows that our method can economize a lot of time and computer resources with losing a little of accuracy under $k = 2$.
It is worth noticing that our method can be expanded to higher dimensional situation to solve the situation when $k \geq 2$ and can used under sigmoid or tanh instead of ReLU, and these will be the research directions of our team. On the other hand, the most interesting thing is that, although it has work that uses three-dimensional convex hull to do the sybolic propagation, it doesn't expand to higher dimension. It is possible to use the approximations mentioned in our method to achieve multi-neuron propagation with high accuracy and lower time cost in the future.

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Proof. Proofs, examples, and remarks have the initial word in italics, while the following text appears in normal font.

For citations of references, we prefer the use of square brackets and consecutive numbers. Citations using labels or the author/year convention are also acceptable. The following bibliography provides a sample reference list with entries for journal articles [11], an LNCS chapter [12], a book [13], proceedings without editors [14], and a homepage [15]. Multiple citations are grouped [11,12,13], [11,13,**?**,15].

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