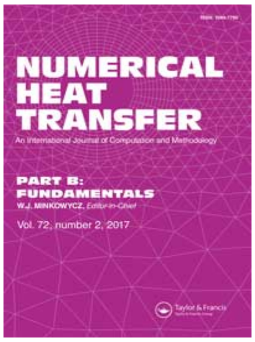

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ABSTRACT
In this article, we consider the two-level stabilized nonconforming finite element methods for the stationary conduction–convection equations based on local Gauss integration. The proposed methods are applied to solve conduction–convection equations with a special relationship of coarse mesh and fine mesh $h = H/3$ to avoid the coarse-to-fine intergrid operator. The methods involves three different corrections: Stokes correction, Oseen correction, and Newton correction. Moreover, the stability and convergence of the proposed methods are deduced. Finally, numerical results are shown to validate the theory analysis and demonstrate the effectiveness of the given methods.

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1. Introduction

In this article, we consider the stationary conduction–convection equations

$$\begin{cases} -\nu\Delta u + (u \cdot \nabla u) + p = \lambda \mathbf{j} T \quad \text{in} \quad \Omega, \\ \nabla \cdot u = 0 \quad \text{in} \quad \Omega, \\ -\Delta T + \lambda u \cdot \nabla T = 0 \quad \text{in} \quad \Omega, \\ u = 0, \quad T = T_0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$, $u = (u_1(x), u_2(x))$ represents the velocity vector, $p = p(x)$ the pressure, $T = T(x)$ the temperature, $\lambda > 0$ the Grashoff number, $j = (0,1)$ the two-dimensional vector, and $\nu > 0$ the viscosity.

The stationary conduction–convection problem is a coupling equations of steady viscous incompressible fluid flow and heat transfer process. On the one hand, this problem is very important on dissipative nonlinear system in atmospheric dynamics and it includes the velocity vector field, pressure field, and temperature field. On the other hand, fluid motion must have the transformation temperature, speed, and pressure from the point of thermodynamics. Therefore, our research is indispensable on it.

Now, we know some methods, numerical analysis and results of natural convection equations by looking in the literature by Boland and Layton [1, 2] and Çıbık and Kaya [3]. Finite element method is widely used in computational fluid dynamics. On the other hand, we obtain some methods to solve the conduction–convection problems by finding in the literature by Luo [4], Shi and Ren [5, 6], Manzari [7], Huang [8, 9].

The inf–sup condition play an important role in solving conduction–convection equations by the hybrid finite element method, because it ensures the stability and accuracy of numerical solution [10].

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In all conditions of conduction–convection equations, the function of it is to make two finite element space of some correlation. Therefore, the research on conduction–convection equations stability has received enough attention. There are some studies with regard to the lowest equal-order finite element pair P_1-P_1 (linear functions) or Q_1-Q_1(bilinear functions) using the projection of the pressure onto the piecewise constant space [9, 11]. In this stabilization technique, stabilization parameters are discretionary and it do not need any calculation of high order derivatives or edge-based data structures. Therefore, the method in computational fluid dynamics has been more and more popular and paid more attention.

However, nonconforming finite element methods for incompressible flows are more popular than conforming finite element methods, because of theirs implicity and small support sets of basis functions. And then, they seem much easier to fulfill the discrete inf–sup condition and they can easily relax the high order continuity requirement for conforming finite element. Therefore, the nonconforming finite element methods seem superior to the conforming finite element methods in practice.

Some research on the two-level approach can look in the works of Ayhn [12], Layton [13, 14], Xu [15], Ervin [16], and He [17]. From these, we find that the two-level discretization can save a lot of time and that is the biggest advantage. Besides, the core concept of it is to compute an initial approximation on a very coarse mesh. The fine structure to solve linear systems is to perform it on a fine mesh.

In this paper, we propose three two-level stabilized nonconforming finite element algorithms for the conduction–convection equations which are based on the locally stabilized method with Gaussian quadrature rule. Three algorithms are Stokes correction, Oseen correction, and Newton correction. Specially, we choose certain relationship $h = H/3$ between coarse mesh and fine mesh to avoid the coarse-to-fine intergrid operator. Hence, our algorithms are more efficient and simpler than the standard one.

The content of this paper is organized as follows. In Section 2, an abstract functional setting of the conduction–convection problem and some well-known results used through out this paper. In Section 3, a stabilized nonconforming finite element strategy is recalled. In Section 4, two-level stabilized nonconforming finite element algorithms are given. Then in Section 5, the results of numerical experiments to complete validation theory. Finally, we end with a short conclusion.

2. Preliminaries

For the mathematical setting of the conduction–convection equations (1), we introduce the Hilbert space:

$$X = (H_0^1(\Omega))^2, \qquad W = H^1(\Omega), \qquad W_0 = H_0^1(\Omega),$$
$$Y = L^2(\Omega)^2, \qquad M = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}.$$

The spaces $L^2(\Omega)^m, m = 1, 2, 4$, are equipped with the L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $||\cdot||_0$. The space X is endowed with the usual scalar product $(\nabla u,\nabla v)$ and the norm $||\nabla u||_0$. Standard definitions are used for the Sobolev spaces $W^{m,p}(\Omega)$, with the norm $||\cdot||_0,m,p\geq0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $||\cdot||_m$ for $||\cdot||_{m,2}$.

Then, let the closed subset V of X be given by

$$V = \{v \in X, \operatorname{div} v = 0\},$$

and denote by H the closed subset of Y, that is

$$H = \{v \in Y, \operatorname{div} v = 0, v \cdot n|_{\partial\Omega} = 0\}.$$

We define two continuous bilinear forms and a trilinear term as follows,

$$
\begin{aligned}
a(u,v) &= (\nabla u, \nabla v), \quad \forall u, v \in X,\\
d(v,q) &= (q, \operatorname{div} v), \quad \forall (v,q) \in (X,M),\\
b(u;v,w) &= ((u\cdot\nabla)v + \frac{1}{2}(\operatorname{div} u)v, w) = \frac{1}{2}b_1(u;v,w) - \frac{1}{2}b_1(u;w,v), \quad \forall u,v,w \in X,
\end{aligned}
$$

where $b_1(u;v,w) = ((u\cdot\nabla)v, w)$. If fixed u, then $b(u;v,w)$ is the skew-symmetric part of $b_1(u;v,w)$.

We also define two kinds of continuous bilinear forms $\bar{a}(\cdot,\cdot)$ and $\bar{b}(\cdot;\cdot,\cdot)$ on (W,W) and (X,W,W), as follows

$$
\begin{aligned}
\bar{a}(T,s) &= (\nabla T, \nabla s), \quad \forall T, s \in W,\\
\bar{b}(u;T,s) &= ((u\cdot\nabla)T, s) + \frac{1}{2}((\operatorname{div} u)T, s)\\
&= \frac{1}{2}\bar{b}_1(u;T,s) - \frac{1}{2}\bar{b}_1(u;s,T), \quad \forall (u,T,s) \in (X,W,W),
\end{aligned}
$$

where $\bar{b}_1(u;T,s) = ((u\cdot\nabla)T, s)$.

The trilinear forms $b(\cdot;\cdot,\cdot)$ and $\bar{b}(\cdot;\cdot,\cdot)$ satisfy

$$
\begin{aligned}
b(u;v,w) &= -b(u;w,v),\\
\bar{b}(u;T,s) &= -\bar{b}(u;s,T),
\end{aligned}
\tag{2}
$$

and from [18], we know that the trilinear form satisfies

$$
\begin{aligned}
|b(u;v,w)| &\le N\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0, \qquad \forall u,v,w \in X,\\
|\bar{b}(u;s,T)| &\le \overline{N}\|\nabla u\|_0\|\nabla T\|_0\|\nabla s\|_0, \qquad \forall (u,T,s) \in (X,W,W),
\end{aligned}
\tag{3}
$$

where

$$
N = \sup_{u,v,w\in X}\frac{|b(u;v,w)|}{\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0}, \quad \overline{N} = \sup_{u\in X, T,s\in W}\frac{|\bar{b}(u;T,s)|}{\|\nabla u\|_0\|\nabla T\|_0\|\nabla s\|_0}.
$$

With the above notations, the variational formulation of problem (1) as follow: Finding $(u,p,T)\in(X,M,W)$ such that for all $(v,q,s)\in(X,M,W_0)$,

$$
\begin{cases}
B((u,p);(v,q)) + b(u;u,v) \ = \ \lambda(\mathbf{j}T, v),\\
\bar{a}(T,s) + \lambda\bar{b}(u;T,s) = 0.
\end{cases}
\tag{4}
$$

Moreover, the bilinear form $d(\cdot,\cdot)$ satisfies the inf–sup condition [18] for all $q\in M$,

$$
\sup_{v\in X}\frac{|d(v,q)|}{\|\nabla v\|_0} \ge \beta_1\|q\|_0,
$$

where β_1 is a positive constant only depending on Ω.

Next, we introduce a generalized bilinear form on $(X,M)\times(X,M)$ given by

$$
B((u,p);(v,q)) = \nu a(u,v) - d(v,p) + d(u,q), \quad \forall (u,p),(v,q) \in (X,M).
$$

It satisfies the following continuity property and inf–sup condition [16]:

$$
|B((u,p);(v,q))| \le c(\|\nabla u\|_0 + \|p\|_0)(\|\nabla v\|_0 + \|q\|_0), \quad \forall (u,p),(v,q) \in (X,M), \tag{5}
$$

$$
\sup_{(v,q)\in(X,M)}\frac{|B((u,p);(v,q))|}{\|\nabla v\|_0 + \|q\|_0} \ge \beta_2(\|\nabla u\|_0 + \|p\|_0), \quad \forall (u,p),(v,q) \in (X,M), \tag{6}
$$

where $c > 0$ and c is a generic positive constant, which is independent of mesh size μ. And $\beta_2 > 0$ only depends on Ω and ν.

Moreover, we need a further assumption on T_0 proved in [18].

(**A**). Assume that $\partial\Omega \in C^{k,\alpha}(k \geq 0, \alpha > 0)$, then, for $T_0 \in C^{k,\alpha}(\partial\Omega)$, there exists a prolongation of $T_0 \in C_0^{k,\alpha}(R^2)$(still marked as T_0) such that

$$\|T_0\|_{k,q} \leq \varepsilon, \qquad k \geq 0, 1 \leq q \leq \infty,$$

where $\varepsilon > 0$ is a sufficiently small constant.

Existence, uniqueness, and the stability of (6) are as follow.

Theorem 2.1 ([18]). *Assume that* $\widetilde{A} = 2\nu^{-1}\lambda(1+3c)\|T_0\|_1$ *and* $\widetilde{B} = 2\|\nabla T_0\|_0 + c^{-2}(1+3c)\|T_0\|_1$. *Let* δ_1 *and* δ_2 *be two constants and* $0 < \delta_1, \delta_2 \leq 1$, *such that*

$$\nu^{-1}N\widetilde{A} \leq 1 - \delta_1, \qquad \delta_1^{-1}\nu^{-1}\lambda^2 c^2 \overline{N}\widetilde{B} \leq 1 - \delta_1.$$

Then, under the assumption (**A**), *there is a unique solution* $(u,p,T) \in (X,M,W)$, *satisfying*

$$\|\nabla u\|_0 \leq \widetilde{A}, \quad \|\nabla T\|_0 \leq \widetilde{B}.$$

Theorem 2.2. *Under the assumption* (**A**). *Then the solution* $u \in H^1(D(A))$ *of problem (6) satisfies the following regularity*

$$\|Au\|_0 \leq 2C_0\nu^{-1}\lambda\widetilde{B} + \nu^{-2}\widetilde{A}^3.$$

Proof. Taking $v = Au$ and $q = 0$ in the first equation of (4), we will obtain

$$\nu a(u, Au) = \lambda(jT, Au) - b(u, u, Au).$$

Then, using the Cauchy–Schwarz inequality and (3), we can get

$$\|Au\|_0 \leq 2C_0\nu^{-1}\lambda\|\nabla T\|_0 + \nu^{-2}\|\nabla u\|_0^3 \leq 2C_0\nu^{-1}\lambda\widetilde{B} + \nu^{-2}\widetilde{A}^3,$$

where C_0 is defined in [18].

3. Stabilized nonconforming finite element method based on local Gauss integration

From now on, the finite element subspace (NC_H, M_H, W_H) is characterized by T_H, a partitioning of Ω into triangles K with the mesh size H. And we bring in finite element space (NC_h, M_h, W_h) based on T_h. Let T_μ $(\mu = h, H)$ be a finite element partition with mesh size μ. Denote the boundary edge by $\Gamma_j = \partial\Omega_j \cap \partial K_j$ and the interior boundary by $\Gamma_{jk} = \Gamma_{kj} = \partial K_j \cap \partial K_k$. Set the centers of Γ_j and Γ_{jk} by ζ_j and ζ_{jk}, respectively.

The nonconforming finite element space NC_μ for the velocity is defined as

$$NC_\mu = \left\{ v : v_j = v|_{K_j} \in P_1(K_j)^2, v_j(\zeta_{jk}) = v_j(\zeta_{kj}), v_j(\zeta_j) = 0, K_j \in T_\mu, \forall j, k \right\},$$

where $P_1(K_j)$ means the space of piecewise linear functions on the set K_j. Note that the nonconforming finite element space NC_μ is not a subspace of X. The pressure and temperature will be approximated by the piecewise linear functions

$$M_\mu = \left\{ q \in M \cap C^0(\overline{\Omega}) : q|_{K_j} \in P_1(K), \forall K_j \in T_\mu, \forall j \right\},$$
$$W_\mu = \left\{ s \in W \cap C^0(\overline{\Omega}) : s|_{K_j} \in P_1(K), \forall K_j \in T_\mu, \forall j \right\}.$$

We define $W_{0,\mu} = W_\mu \cap W_0$, and the subspace V_μ of X_μ is given by

$$V_\mu = \{ v_\mu \in X_\mu : d(v_\mu, q_\mu) = 0, \forall q_\mu \in M_\mu \},$$

and the energy semi-norm is given by

$$\|v\|_{1,h} = \left(\sum_j |v|^2_{1,K_j} \right)^{\frac{1}{2}}, \quad \forall v \in NC_\mu.$$

Let $(\cdot,\cdot)_j = (\cdot,\cdot)_{K_j}$ and $\langle \cdot,\cdot \rangle_j = (\cdot,\cdot)_{\partial K_j}$, and define the discrete counterparts of the bilinear and trilinear forms as follows

$$a_h(u,v) = \sum_j (\nabla u, \nabla v)_j, \quad d_h(v,q) = \sum_j (\operatorname{div} v, p)_j, \quad \forall u, v \in NC_\mu, \forall p \in M_\mu,$$

$$b_{1,h}(u; v, w) = \sum_j \left(\sum_{i=1}^{2} u_i \partial_i v, w \right)_j, \quad \forall u, v, w \in NC_\mu.$$

For any $u,v,w \in NC_\mu$, integration by parts on each element gives

$$b_{1,h}(u; v, w) = -b_{1,h}(u; w, v) - \sum_j ((\operatorname{div} u)v, w)_j + \sum_j \langle (u \cdot n_j)v, w \rangle_j,$$

where n_j is the unit outward normal to ∂K_j. Therefore,

$$b_{1,h}(u; v, w) = b_h(u; v, w) - \frac{1}{2} \sum_j ((\operatorname{div} u)v, w)_j + \frac{1}{2} \sum_j \langle (u \cdot n_j)v, w \rangle_j,$$

where $b_h(u; v, w) = \frac{1}{2} \Big(b_{1,h}(u; v, w) - b_{1,h}(u; w, v) \Big)$.

Also, the trilinear terms satisfy [5]:

$$b_h(u; v, w) = -b_h(u; v, w), \quad |b_h(u; v, w)| \le N_0 \|u\|_{1,h} \|v\|_{1,h} \|w\|_{1,h}, \quad \forall u, v, w \in NC_\mu, \tag{7}$$

$$\bar{b}(u; T, s) = -\bar{b}(u; s, T), \quad |\bar{b}_h(u; T, s)| \le \overline{N}_0 \|u\|_{1,h} \|\nabla T\|_0 \|\nabla s\|_0, \quad \forall (u, T, s) \in \big(NC_\mu, W_\mu W_{0,\mu} \big), \tag{8}$$

where

$$N_0 = \sup_{u,v,w \in NC_\mu} \frac{|b_h(u; v, w)|}{\|u\|_{1,h} \|v\|_{1,h} \|w\|_{1,h}}, \quad \overline{N}_0 = \sup_{u \in NC_\mu, T \in W_\mu, s \in W_{0,\mu}} \frac{|\bar{b}(u; T, s)|}{\|u\|_{1,h} \|\nabla T\|_0 \|\nabla s\|_0}.$$

Note that (NC_μ, M_μ) does not satisfy the discrete inf–sup condition [16]:

$$\sup_{v \in NC_\mu} \frac{d_h(v,q)}{\|v\|_{1,h}} \ge \beta_3 \|q\|_0, \quad q \in M_\mu,$$

where the constant $\beta_3 > 0$ is independent of μ.

To fulfill this condition, a stabilized term [19] is used:

$$B_h((u_\mu, p_\mu); (v, q)) = \nu a_h(u_\mu, v) - d_h(v, p_\mu) + d_h(u_\mu, q) + G_h(p_\mu, q),$$

where $G_h(p_\mu, q)$ can be defined by

$$G_h(p_\mu, q) = (p_\mu - \Pi_\mu p_\mu, q - \Pi_\mu q),$$

and Π_μ is a L^2-projection operator, which is defined by

$$(p, q_\mu) = (\Pi_\mu p, q_\mu), \quad (p, q) \in (L^2(\Omega), R_\mu).$$

Here, $R_\mu \subset L^2(\Omega)$ denotes the piecewise constant space associated with the triangulation T_μ. The following properties of the projection operator can be proved [21]:

$$\|\Pi_\mu p\|_0 \le c\|p\|_0, \quad \forall p \in L^2(\Omega), \quad \|p - \Pi_\mu\|_0 \le c\mu\|p\|_1, \quad \forall p \in H^1(\Omega). \tag{9}$$

Then, we analyze the following stabilized of conduction–convection problem: Find $(u_\mu, p_\mu, T_\mu) \in (NC_\mu, M_\mu, W_\mu)$ such that

$$\begin{cases} B_h((u_\mu, p_\mu); (v, q)) + b_h(u_\mu; u_\mu, v) = \lambda \sum_j (\mathbf{j}T_\mu, v), \\ \bar{a}(T_\mu, s) + \lambda \bar{b}(u_\mu; T_\mu, s) = 0. \end{cases} \tag{10}$$

The following theorem establishes the weak coercivity of the bilinear form $B_h((\cdot, \cdot);(\cdot, \cdot))$ for the finite element pair (NC_μ, M_μ).

Theorem 3.1 ([19]). *For all (u_μ, p_μ), $(v, q) \in (NC_\mu, M_\mu)$, the bilinear form $\mathrm{B_h}((\cdot, \cdot);(\cdot, \cdot))$ satisfies the continuity property*

$$\left|B_h((u_\mu, p_\mu); (v, q))\right| \le c\Big(\nu\|u_\mu\|_{1,h} + \|p_\mu\|_0\Big)\Big(\|v\|_{1,h} + \|q\|_0\Big), \tag{11}$$

and the weak coercivity property

$$\sup_{(v,q)\in(NC_\mu, M_\mu)} \frac{\left|B_h((u_\mu, p_\mu); (v, q))\right|}{\|v\|_{1,h} + \|q\|_0} \ge \beta\Big(\nu\|u_\mu\|_{1,h} + \|p_\mu\|_0\Big), \tag{12}$$

where $\beta>0$ is independent of μ.

Moreover, we need following lemmas.

Lemma 3.1 ([5, 6]). *For any $v\in \mathrm{NC}_\mu$, the following discrete embedding over NC_μ holds*

$$\|v\|_0 \le c_1\|v\|_{1,h}.$$

Lemma 3.2 ([20]). *The trilinear form $\mathrm{b_h}(\cdot;\cdot,\cdot)$ satisfies the following estimates:*

$$\begin{aligned} \left|b_h(u_\mu; v_\mu, w_\mu)\right| &\le c\|u_\mu\|_{1,h}\|A_\mu u_\mu\|_0^{\frac{1}{2}}\|v_\mu\|_{1,h}\|w_\mu\|_{1,h}, \\ \left|b_h(u_\mu; v_\mu, w_\mu)\right| &\le c\|u_\mu\|_0\|v_\mu\|_{1,h}\|w_\mu\|_{1,h}^{\frac{1}{2}}\|A_\mu w_\mu\|_0^{\frac{1}{2}}, \\ \left|b_h(u_\mu; v_\mu, w_\mu)\right| &\le c|\ln\mu|^{\frac{1}{2}}\|u_\mu\|_{1,h}\|v_\mu\|_{1,h}\|w_\mu\|_0, \end{aligned}$$

for any $u_\mu, v_\mu, w_\mu \in NC_\mu$.

Lemma 3.3 ([21]). *Under the assumption of (**A**), Theorem 2.1, and let ε be a sufficiently small positive constants such that*

$$\frac{2N_0\lambda(1+3c_1)\varepsilon}{\nu^2} \le 1, \quad \frac{\overline{N}_0\lambda^2(2c_1^2+3c_1+1)}{\nu}\varepsilon \le 1, \quad \|\nabla T_0\|_0 \le \frac{\varepsilon}{4}.$$

Then, (u_μ, p_μ, T_μ) *defined by (10) satisfies the following stability:*

$$\|u_\mu\|_{1,h} \le \frac{c_1^2\lambda}{\nu}\varepsilon, \quad \|\nabla T_\mu\|_0 \le \varepsilon.$$

Theorem 3.2. *If the assumption (**A**) holds and under the assumption of Lemma 3.3, then*

$$\|A_\mu u_\mu\|_0 \le \frac{c_1^4\lambda^2\varepsilon^2}{\nu^3} + \frac{c_1\varepsilon}{\nu}.$$

Proof. Taking $(v, q) = (A_\mu u_\mu, 0)$ in (10), we have

$$\nu a_h(u_\mu, A_\mu u_\mu) + b_h(u_\mu; u_\mu, A_\mu u_\mu) = \lambda(\mathbf{j}T_\mu, A_\mu u_\mu),$$

with Cauchy–Schwarz inequality, Lemma 3.3 and (3), we obtain

$$\|A_\mu u_\mu\|_0 \le \nu^{-1}\Big(\|u_\mu\|_{1,h}^2 + c_1\|T_\mu\|_0\Big) \le \frac{c_1^4\lambda^2\varepsilon^2}{\nu^3} + \frac{c_1\varepsilon}{\nu}.$$

To derive error estimates for the finite element solution (u_μ, p_μ), we define

$$B_h^*((u,p);(v_\mu,q_\mu)) = B_h((u,p);(v_\mu,q_\mu)) - G_h(p,q_\mu), \quad \forall (v_\mu,q_\mu) \in (NC_\mu, M_\mu),$$

and introduce the projection operator (R_μ, Q_μ): $(X, M) \to (NC_\mu, M_\mu)$ by

$$B_h(R_\mu(v,q), Q_\mu(v,q));(v_\mu,q_\mu)) = B_h^*((v,q);(v_\mu,q_\mu)), \quad \forall (v_\mu,q_\mu) \in (NC_\mu, M_\mu), \tag{13}$$

which are well satisfy the following approximation properties.

Lemma 3.4 ([19]). *It holds that*

$$\|v - R_\mu(v,q)\|_0 + \mu\Big(\|v - R_\mu(v,q)\|_{1,h} + \|q - Q_\mu(v,q)\|_0\Big) \le c\mu^2(\|v\|_2 + \|q\|_1), \tag{14}$$

for all $(v,q) \in (H^2(\Omega)^2 \cap X, W \cap M)$.

Furthermore, we need the following conclusions in [22, 23] about the bounds of sums of some integrals.

Lemma 3.5.

$$\left|\sum_j \langle\frac{\partial w}{\partial n_j}, \psi\rangle_j\right| \le c\mu^{k+s-1}\|w\|_{k+1}\|\psi\|_{s,h}, \quad \forall w,\psi \in X \cap H^{k+1}(\Omega)^2,$$

$$\left|\sum_j \langle(w\cdot n_j, \psi)\rangle_j\right| \le c\mu^{k+s-1}\|w\|_{k+1,h}\|v\|_{s-1}\|\psi\|_{s,h}, \quad \forall v \in H^{k+1}(\Omega)^2,$$

$$\left|\sum_j \langle q, \psi_\mu\cdot n_j\rangle_j\right| \le c\mu^{k+s-1}\|q\|_{k+1}\|\psi\|_{s,h}, \quad \forall q \in H^{k+1}(\Omega).$$

Proof. Combing the results of [22] and [23], the proof is finished.

Lemma 3.6 ([18]). *There exists* $r_\mu : W \to W_\mu$, *such that for all* φ

$$\begin{aligned}&(\nabla(\varphi - r_\mu\varphi), \nabla\varphi_\mu) = 0, \quad \forall \varphi_\mu \in W_\mu,\\ &\int_\Omega(\varphi - r_\mu\varphi)dx = 0, \quad \|\nabla r_\mu\varphi\|_0 \le \|\nabla\varphi\|_0,\end{aligned} \tag{15}$$

and that when $\varphi \in W^{r,q}(\Omega)(1 \le q \le \infty)$, *we have*

$$\|\varphi - r_\mu \varphi\|_{-s,q} \le c\mu^{r+s}|\varphi|_{r,q}, \quad -1 \le s \le m,\ 0 \le r \le m+1. \tag{16}$$

Theorem 3.3. [24] *If the assumption* (**A**) *holds and* $(u, p, T) \in ((H^3(\Omega))^2 \cap X, H^1(\Omega) \cap M, W)$ *and* (u_μ, p_μ, T_μ) *are the solution of problems (4) and (10), respectively. Then, there holds that*

$$\|u - u_\mu\|_0 + \|T - T_\mu\|_0 + \mu(\|u - u_\mu\|_{1,h} + \|\nabla(T - T_\mu)\|_0 + \|p - p_\mu\|_0) \le c\mu^2. \tag{17}$$

4. Two-level stabilized nonconforming finite element algorithms

Let τ_H be the coarse mesh and we get the fine mesh τ_h by refining τ_H and it satisfies the conditions of $h = H/3$ (Figure 1). Meanwhile, with the certain relationship between coarse mesh and fine mesh, there is no need to establish a coarse-to-fine intergrid operator. Naturally, the nonconforming finite element space pair $(NC_H, M_H, W_H) \subset (NC_h, M_h, W_h)$ based on the triangulations $\tau_H(\Omega)$ and $\tau_h(\Omega)$. Then, we consider the two-level nonconforming stabilized finite element methods.

Algorithm 4.1 (Stokes correction).

Step I. Solve a conduction–convection problem on coarse mesh: Find $(u_H, p_H, T_H) \in (NC_H, M_H, W_H)$ such that for all $(v, q, s) \in (NC_H, M_H, W_{0,H})$,

$$\begin{cases} B_h((u_H, p_H); (v, q)) + b(u_H; u_H, v) = \lambda \sum_j (\mathbf{j} T_H, v), \\ \bar{a}(T_H, s) + \lambda \bar{b}(u_H; T_H, s) = 0. \end{cases} \tag{18}$$

Step II. Solve a linerazied conduction–convection problem on fine mesh: Find $(u^h, p^h, T^h) \in (NC_h, M_h, W_h)$ such that for all $(v, q) \in (NC_h, M_h, W_{0,h})$,

$$\begin{cases} B_h((u^h, p^h); (v, q)) + b_h(u_H; u^H, v) = \lambda \sum_j (\mathbf{j} T^h, v), \\ \bar{a}(T^h, s) + \lambda \bar{b}(u_H; T^H, s) = 0. \end{cases} \tag{19}$$

Theorem 4.1. *Under the assumptions of Lemma 3.3 with* $\mu = h$, (u^h, T^h) *defined by scheme (30), satisfies*

$$\|\nabla u^h\|_0 \le \frac{7c_1^2 \lambda \varepsilon}{6\nu}, \quad \|\nabla T^h\|_0 \le \varepsilon.$$

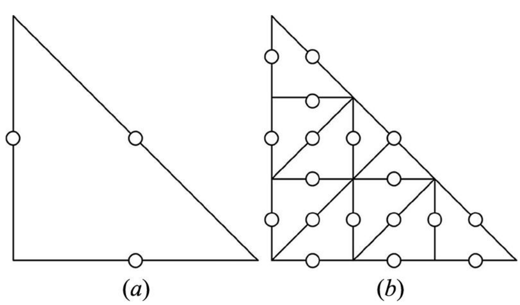

Figure 1. (*a*) τ_H mesh; (*b*) τ_h mesh.

Proof. Taking $(v, q, T^h) = (u^h, p^h, \omega^h + T_0)$ in the first equation of (19), we get

$$\nu\|u^h\|_{1,h} \le N_0\|u_H\|_{1,h}^2 + c_1^2\lambda\|\nabla\omega^h\|_0 + c_1^2\lambda\|\nabla T_0\|_0. \tag{20}$$

Then, letting $(s, T^h) = (\omega^h, \omega^h + T_0)$ in the second equation of (19), we obtain

$$\|\nabla\omega^h\|_0 \le \lambda\overline{N}_0\|u_H\|_{1,h}\|\nabla T_H\|_0 + \|\nabla T_0\|_0 \le \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \le \frac{3}{4}\varepsilon. \tag{21}$$

Next, with (21) we get

$$\|\nabla T^h\|_0 \le \|\nabla\omega^h\|_0 + \|\nabla T_0\|_0 \le \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} \le \varepsilon.$$

Then, combing Lemma 3.3 and (20), (21), we deduce

$$\|u^h\|_{1,h} \le N_0 \cdot \left(\frac{c_1^2\lambda}{\nu}\varepsilon\right) + c_1^2\lambda\varepsilon \le \frac{7c_1^2\lambda\varepsilon}{6\nu}.$$

In conclusion, we complete the proof.

Theorem 4.2. *If the assumption (**A**) holds,* (u,p,T) *and* (u^h, p^h, T^h) *are the solution of problem (6) and (19), respectively. Then, there holds that*

$$\|u - u^h\|_{1,h} + \|\nabla(T - T^h)\|_0 + \|p - p^h\|_0 \le ch.$$

Proof. Multiplying (1) by $(v, q) \in (NC_\mu, M_\mu)$ and integrateing it over Ω, we get

$$\begin{aligned} &\nu a_h(u, v) - d_h(v, q) + d_h(u, q) + b_h(u; u, v) - \nu\sum_j\left\langle\frac{\partial u}{\partial n_j}, v\right\rangle_j \\ &+ \sum_j\langle p, v\cdot n_j\rangle_j + \frac{1}{2}\sum_j\langle(u\cdot n_j)u, v\rangle_j = \lambda\sum_j(\mathbf{j}T, v)_j. \end{aligned} \tag{22}$$

Next, subtracting the second equation of (19) from the second equation of (4), we get

$$\bar{a}(T - T^h, s) = -\lambda\bar{b}(u - u_H; T, s) - \lambda\bar{b}(u_H; T - T_H, s). \tag{23}$$

Then, letting $s = T - T^h$ in (23) and with Lemma 3.6, Theorem 3.3, and (2), we have

$$\|\nabla(T - T^h)\|_0 \le \lambda\overline{N}_0\|\nabla T\|_0\|u - u_H\|_0 + \lambda\overline{N}_0\|u_H\|_0\|\nabla(T - T_H)\|_0 \le ch. \tag{24}$$

Besides, subtracting the first equation of (19) from (22), we obtain

$$\begin{aligned} &\nu a_h(u - u^h, v) - d_h(v, p - p^h) + d_h(u - u^h, q) + b_h(u_H; u - u_H, v) + b_h(u - u_H; u, v) \\ &- G_h(p^h, q) - \nu\sum_j\langle\frac{\partial u}{\partial n_j}, v\rangle_j + \sum_j\langle p, v\cdot n_j\rangle_j + \frac{1}{2}\sum_j\langle(u\cdot n_j)u, v\rangle_j = \lambda\sum_j(\mathbf{j}(T - T^h), v)_j. \end{aligned} \tag{25}$$

Next, setting $(e^h, \eta^h) = (R_h(u,p) - u^h, Q_h(u,p) - p^h)$ and taking $(v, q) = (e^h, \eta^h)$ in (25), we have

$$\begin{aligned} &\nu a_h(e^h, e^h) + b_h(u_H; u - u_H, e^h) + b_h(u - u_H; u, e^h) + G_h(\eta^h, \eta^h) \\ &- \nu \sum_j \langle \frac{\partial u}{\partial n_j}, e^h \rangle_j + \sum_j \langle p, e^h \cdot n_j \rangle_j + \frac{1}{2} \sum_j \langle (u \cdot n_j) u, e^h \rangle_j = \lambda \sum_j (\mathbf{j}(T - T^h), e^h)_j. \end{aligned} \tag{26}$$

It follows from Theorem 3.3 and (24), (26), we get

$$\nu \|e^h\|_{1,h} \leq \|u_H\|_0 \|u - u_H\|_0 + \|u\|_0 \|u - u_H\|_0 + ch(\|u\|_2 + \|p\|_1) + ch \leq ch. \tag{27}$$

Combing the triangle inequality and (14), we have

$$\|u - u_h\|_{1,h} \leq \|u - R_h(u,p)\|_{1,h} + \|e^h\|_{1,h} \leq ch. \tag{28}$$

Then, with (12), (24), (26), and (28), we obtain

$$\begin{aligned} \beta(\|e^h\|_{1,h} + \|\eta^h\|_0) &\leq \sup_{(v,q) \in (NC_h, M_h)} \frac{|\nu a_h(e^h, v) - d_h(v, \eta^h) + d_h(e^h, q) + G_h(\eta^h, q)|}{\|v\|_{1,h} + \|q\|_0} \\ &\leq \sup_{(v,q) \in (NC_h, M_h)} \frac{1}{\|v\|_{1,h} + \|q\|_0} \times \{ b_h(u_H; u - u_H, v) + b_h(u - u_H; u, v) \\ &\quad - \nu \sum_j \langle \frac{\partial u}{\partial n_j}, v \rangle_j + \sum_j \langle p, v \cdot n_j \rangle_j + \frac{1}{2} \sum_j \langle (u \cdot n_j) u, v \rangle_j \}. \end{aligned} \tag{29}$$

Combining (17) and (29) shows that

$$\|p - p^h\|_0 \leq ch.$$

Hence, we finish the proof.

Algorithm 4.2 (Oseen correction).

Step I. Solve a conduction–convection problem on coarse mesh: Find $(u_H, p_H, T_H) \in (NC_H, M_H, W_H)$ by (18).

Step II. Solve a linearized conduction–convection problem on fine mesh: Find $(u^h, p^h, T^h) \in (NC_h, M_h, W_h)$ such that for all $(v, q) \in (NC_h, M_h, W_{0,h})$,

$$\begin{cases} B_h((u^h, p^h); (v, q)) + b_h(u_H; u^h, v) = \lambda \sum_j (\mathbf{j} T^h, v), \\ \bar{a}(T^h, s) + \lambda \bar{b}(u_H; T^h, s) = 0. \end{cases} \tag{30}$$

Theorem 4.3. *Under the assumptions of Lemma 3.3 with $\mu = h$, (u^h, p^h, T^h) defined by scheme (30), satisfies*

$$\|u^h\|_{1,h} \leq \frac{5c_1^2 \lambda \varepsilon}{8\nu}, \quad \|\nabla T^h\|_0 \leq \frac{5}{8} \varepsilon.$$

Proof. Taking $(s, T^h) = (\omega^h, \omega^h + T_0)$ in the second equation of (30), we get

$$\bar{a}(\omega^h, \omega^h) + \bar{a}(T_0, \omega^h) + \lambda \bar{b}(u_H; \omega^h + T_0, \omega^h) = 0.$$

Then, with Lemma 3.3 and (2), (8), yields

$$\|\nabla\omega^h\|_0 \leq \lambda\overline{N}_0\|u_H\|_{1,h}\|\nabla T_0\|_0 + \|\nabla T_0\|_0 \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{8} \leq \frac{3}{8}\varepsilon. \tag{31}$$

Next, combing Lemma 3.3 and (31), we know that

$$\|\nabla T^h\|_0 \leq \|\nabla\omega^h\|_0 + \|\nabla T_0\|_0 \leq \frac{\varepsilon}{4} + \frac{3}{8}\varepsilon \leq \frac{5}{8}\varepsilon.$$

Besides, taking $(v, q, T^h) = (u^h, p^h, \omega^h + T_0)$ in the first equation of (30), we obtain

$$\nu a_h(u^h, u^h) \leq \lambda \sum_j \big(\mathbf{j}(\omega^h + T_0), u^h\big).$$

Then, combing (2) and (31), we deduce

$$\|u^h\|_{1,h} \leq \frac{c_1^2\lambda}{\nu}\left(\|\nabla\omega^h\|_0 + \|\nabla T_0\|_0\right) \leq \frac{5c_1^2\lambda\varepsilon}{8\nu}.$$

In conclusion, we complete the proof.

Theorem 4.4. *If the assumption* (A) *holds and* (u,p,T) *and* (u^h, p^h, T^h) *are the solution of problem (4) and (30), respectively. Then, there holds that*

$$\|u - u^h\|_{1,h} + \|\nabla(T - T^h)\|_0 + \|p - p^h\|_0 \leq ch.$$

Proof. Subtracting the second equation of (30) from the second equation of (4), we can get

$$\bar{a}(T - T^h, s) + \lambda\bar{b}(u - u_H; T, s) + \lambda\bar{b}(u_H; T - T^h, s) = 0. \tag{32}$$

Then, letting $s = T - T^h$ in (32),with Theorem 3.3, we have

$$\|\nabla(T - T^h)\|_0 \leq \lambda\overline{N}_0\|\nabla T\|_0\|u - u_H\|_0 \leq ch. \tag{33}$$

Besides, subtracting the first equation of (30) from (22), we can also get

$$\begin{aligned} &\nu a_h(u - u^h, v) - d_h(v, p - p^h) + d_h(u - u^h, q) + b_h(u_H; u - u^h, v) + b_h(u - u_H; u, v) \\ &\quad - G_h(p^h, q) - \nu\sum_j\left\langle\frac{\partial u}{\partial n_j}, v\right\rangle_j + \sum_j\langle p, v\cdot n_j\rangle_j + \frac{1}{2}\sum_j\langle(u\cdot n_j)u, v\rangle_j = \lambda\sum_j(\mathbf{j}(T - T^h), v)_j \end{aligned}. \tag{34}$$

Then, setting $(e^h, \eta^h) = (R_h(u,p) - u^h, Q_h(u,p) - p^h)$ and taking $(v, q) = (e^h, \eta^h)$ in (34), we have

$$\begin{aligned} &\nu a_h(e^h, e^h) + b_h(u_H; u - u^h, e^h) + b_h(u - u_H; u, e^h) + G_h(\eta^h, \eta^h) \\ &\quad - \nu\sum_j\left\langle\frac{\partial u}{\partial n_j}, e^h\right\rangle_j + \sum_j\langle p, e^h\cdot n_j\rangle_j + \frac{1}{2}\sum_j\langle(u\cdot n_j)u, e^h\rangle_j = \lambda\sum_j(\mathbf{j}(T - T^h), e^h)_j. \end{aligned} \tag{35}$$

It follows from (33), (35), and (14), we get

$$\begin{aligned} \nu\|e^h\|_{1,h} &\leq \overline{N}_0\|u_H\|_{1,h}\|u - R_h(u,p)\|_0 + \overline{N}_0\|u\|_{1,h}\|u - u_H\|_0 + ch(\|u\|_2 + \|p\|_1) + ch \\ &\leq ch. \end{aligned}$$

Combing the triangle inequality and (14), we have

$$\|u-u_h\|_{1,h} \leq \|u-R_h(u,p)\|_{1,h} + \|e^h\|_{1,h} \leq ch. \tag{36}$$

Then, with (12), (33), (34), and (36), we obtain

$$\begin{aligned}\beta(\|e^h\|_{1,h}+\|\eta^h\|_0) &\leq \sup_{(v,q)\in(NC_h,M_h)} \frac{|\nu a_h(e^h,v)-d_h(v,\eta^h)+d_h(e^h,q)+G_h(\eta^h,q)|}{\|v\|_{1,h}+\|q\|_0}\\ &\leq \sup_{(v,q)\in(NC_h,M_h)} \frac{1}{\|v\|_{1,h}+\|q\|_0}\times\{b_h(u_H;u-u^h,v)+b_h(u-u_H;u,v)\\ &\quad -\nu\sum_j\langle\frac{\partial u}{\partial n_j},v\rangle_j+\sum_j\langle p,v\cdot n_j\rangle_j+\frac{1}{2}\sum_j\langle(u\cdot n_j)u,v\rangle_j\}.\end{aligned} \tag{37}$$

Combining (17) and (37) shows that

$$\|p-p^h\|_0 \leq ch.$$

Hence, we finish the proof.

Algorithm 4.3 (Newton correction).

Step I. Solve a conduction–convection problem on coarse mesh: Find $(u_H^m, p_H^m, T_H^m) \in (NC_H, M_H, W_H)$ by (18).

Step II. Solve a conduction–convection problem on fine mesh: Find $(u_{mh}, p_{mh}, T_{mh}) \in (NC_h, M_h, W_h)$ such that for all $(v,q) \in (NC_h, M_h, W_{0,h})$,

$$\begin{cases} B_h((u^h,p^h);(v,q)) + b(u^h;u_H,v) + b(u_H;u^h,v) = \lambda\sum_j(\mathbf{j}T^h,v) + b(u_H;u_H,v),\\ \bar{a}(T^h,s) + \lambda\bar{b}(u_H;T^h,s) + \lambda\bar{b}(u^h;T_H,s) = \lambda\bar{b}(u_H;T_H,s). \end{cases} \tag{38}$$

Theorem 4.5. *Under the assumptions of Lemma 3.3 with* $\mu = h$, (u^h, p^h, T^h) *defined by scheme (38), satisfies*

$$\|u^h\|_{1,h} \leq \frac{8c_1^2\lambda\varepsilon}{\nu}, \quad \|\nabla T^h\|_0 \leq 4\varepsilon.$$

Proof. Taking $(s, T^h) = (\omega^h + T_0, \omega^h)$ in the second equation of (38), we obtain

$$\bar{a}(\omega^h+T_0,\omega^h) + \lambda\bar{b}(u_H;T^h,\omega^h)\lambda\bar{b}(u^h;T_H,\omega^h) = \lambda\bar{b}(u_H;T_H,\omega^h).$$

Then, with (2), (8), and Lemma 3.3, yields

$$\begin{aligned}\|\nabla\omega^h\|_0 &\leq \|\nabla T_0\|_0 + \lambda\overline{N}_0\|u_H\|_{1,h}\|\nabla T_H\|_0 + 2\lambda\overline{N}_0\|u^h|_{1,h}\|\nabla T_H\|_0\\ &\leq \frac{7}{8}\varepsilon + \lambda\overline{N}_0\varepsilon\|u^h\|_{1,h}.\end{aligned} \tag{39}$$

Besides, taking $(v, q, T^h) = (u^h, p^h, \omega^h + T_0)$ in the first equation of (38), we get

$$\nu a_h(u^h, u^h) + G_h(p^h, p^h) + b_h(u^h; u_H, u^h) = \lambda \sum_j (\mathbf{j}(\omega^h + T_0), u^h)_j + b_h(u_H; u_H, u^h).$$

From (2), (7), and Lemma 3.3 yields

$$\|u^h\|_{1,h} \le \left(\frac{\nu}{2}\right)^{-1} \left(c_1^2 \lambda \|\nabla \omega^h\|_0 + c_1^2 \lambda \|\nabla T_0\|_0 + N \|u_H\|_{1,h}^2\right) \le \frac{8c_1^2 \lambda \varepsilon}{\nu}. \tag{40}$$

Next, together with (39) and (40), we deduce

$$\|\nabla \omega^h\|_0 \le \frac{23}{8} \varepsilon. \tag{41}$$

Then, combing the triangle inequality, assumption (A) and (41), we know that

$$\|\nabla T^h\|_0 \le \|\nabla \omega^h\|_0 + \|\nabla T_0\|_0 \le 4\varepsilon.$$

In conclusion, we complete the proof.

Theorem 4.6. *If the assumption* (A) *holds and* (u,p,T) *and* (u^h, p^h, T^h) *are the solution of problem (4) and (38), respectively. Then, there holds that*

$$\|u - u^h\|_{1,h} + \|\nabla(T - T^h)\|_0 + \|p - p^h\|_0 \le ch.$$

Proof. Subtracting the second equation of (38) from the second equation of (4), we get

$$\bar{a}(T - T^h, s) + \lambda \bar{b}(u_H; T - T^h, s) + \lambda \bar{b}(u - u^h; T_H, s) + \lambda \bar{b}(u - u_H; T - T_H, s) = 0. \tag{42}$$

Then,letting $s = T - T^h$ in (42), and with Lemma 3.2, Theorem 3.3, (2), and (8), we have

$$\begin{aligned} \|\nabla(T - T^h)\|_0 &\le \lambda \overline{N}_0 \|u - u^h\|_{1,h} \|\nabla T_H\|_0 + \lambda \overline{N}_0 \|u - u_H\|_{1,h} \|\nabla(T - T_H)\|_0 \\ &\le ch + \lambda \overline{N}_0 \|e^h\|_{1,h} \|\nabla T_H\|_0. \end{aligned} \tag{43}$$

Besides, subtracting the first equation of (38) from (22), we can also get

$$\begin{aligned} &\nu a_h(u - u^h, v) - d_h(v, p - p^h) + d_h(u - u^h, q) + b_h(u_H; u - u^h, v) \\ &\quad + b_h(u - u^h; u_H, v) + b_h(u - u_H; u - u_H, v) + G_h(p^h, q) \\ &\quad - \nu \sum_j \left\langle \frac{\partial u}{\partial n_j}, v \right\rangle_j + \sum_j \langle p, v \cdot n_j \rangle_j + \frac{1}{2} \sum_j \langle (u \cdot n_j) u, v \rangle_j = \lambda \sum_j (\mathbf{j}(T - T^h), v)_j. \end{aligned} \tag{44}$$

Then, setting $(e^h, \eta^h) = (R_h(u,p) - u^h, Q_h(u,p) - p^h)$ and taking $(v, q) = (e^h, \eta^h)$ in (43), we have

$$\begin{aligned} &\nu a_h(e^h, e^h) + b_h(u - R_h(u,p); u, e^h) + b_h(R_h(u,p); u - R_h(u,p), e^h) \\ &\quad - b_h(u^h - R_h(u,p); u_H, e^h) - b_h(u_H - R_h(u,p); e^h, u_H - R_h(u,p)) + G_h(\eta^h, \eta^h) \\ &\quad - \nu \sum_j \left\langle \frac{\partial u}{\partial n_j}, e^h \right\rangle_j + \sum_j \langle p, e^h \cdot n_j \rangle_j + \frac{1}{2} \sum_j \langle (u \cdot n_j) u, e^h \rangle_j = \lambda \sum_j (\mathbf{j}(T - T^h), e^h)_j. \end{aligned} \tag{45}$$

It follows from Theorem 3.3 and Lemma 3.2 that

$$\frac{\nu}{2}\|e^h\|_{1,h} \le c(\|u - R_h(u,p)\|_{1,h}\|u\|_1 + \|R_h(u,p)\|_{1,h}\|u - R_h(u,p)\|_{1,h} + \|u_H - R_h(u,p)\|^2_{1,h})\|e^h\|_{1,h} + h(\|u\|_2 + \|p\|_1) + ch \le ch.$$

Combing the triangle inequality and (14), we have

$$\|u - u_h\|_{1,h} \le \|u - R_h(u,p)\|_{1,h} + \|e^h\|_{1,h} \le ch. \tag{46}$$

Then, with (46) we get

$$\|\nabla(T - T^h)\|_0 \le ch. \tag{47}$$

Then, with (12), (44), (46), and (47), we obtain

$$\begin{aligned}\beta(\|e^h\|_{1,h} + \|\eta^h\|_0) \le & \sup_{(v,q)\in(NC_h,M_h)} \frac{|\nu a_h(e^h,v) - d_h(v,\eta^h) + d_h(e^h,q) + G_h(\eta^h,q)|}{\|v\|_{1,h} + \|q\|_0} \\ \le & \sup_{(v,q)\in(NC_h,M_h)} \frac{1}{\|v\|_{1,h} + \|q\|_0} \times \{b_h(u - R_h(u,p); u, e^h) \\ & + b_h(R_h(u,p); u - R_h(u,p), e^h) - b_h(u^h - R_h(u,p); u_H, e^h) \\ & - b_h(u_H - R_h(u,p); e^h, u_H - R_h(u,p)) + \frac{1}{2}\sum_j \langle (u\cdot n_j)u, v\rangle_j \\ & - \nu \sum_j \left\langle \frac{\partial u}{\partial n_j}, v\right\rangle_j + \sum_j \langle p, v\cdot n_j\rangle_j\}.\end{aligned} \tag{48}$$

Combining (17) and (48) shows that

$$\|p - p^h\|_0 \le ch.$$

Hence, we finish the proof.

5. Numerical experiments

In this section, we present two numerical experiments to check the theoretical analysis in the previous sections and illustrate the effectiveness of the given algorithms. Our computational experiments center around the investigation of two main areas of interest associated with multiscale simulations: (i) verifying optimal finite element convergence estimates and (ii) investigating the computational cost.

In the numerical experiments, we select the viscosity $\nu = 1$, the Grashoff number $\lambda = 1$ and with a fixed tolerance 10^{-6}. And the nonlinear system on the coarse mesh is solved by the Stokes iteration.

5.1. Experiment 1

First, we consider the exact solution problem to investigate the first two respects, i.e., optimal convergence rate estimates and computational cost. we compare the numerical results of three Algorithms with the one-level nonconforming finite element methods. Then, let Ω be the unit square in $\mathbb{R}^2$. The uniform mesh is adopted; i.e., the mesh consists of triangular elements that are obtained by dividing Ω into sub-squares of equal size and then drawing the diagonal in each sub-square.

The chosen functions are added to the right-hand side of (1) such that the exact solution for the velocity $u = (u_1, u_2)$, the temperature T and the pressure p to the considered problem are given as follows:

$$\begin{cases} u_1(x,y) = 10x^2(x-1)^2 y(y-1)(2y-1), \\ u_2(x,y) = -10x(x-1)(2x-1)y^2(y-1)^2, \\ p(x,y) \;= 10(2x-1)(2y-1), \\ T(x,y) \;= u_1(x,y) + u_2(x,y). \end{cases}$$

5.1.1. Rates of convergence study

The experimental rates of convergence with respect to the mesh size h are calculated by the formula $log(E_i/E_{i+1})/log(h_i/h_{i+1})$, where E_i and E_{i+1} are the relative errors corresponding to the meshes of size h_i and h_{i+1}, respectively.

From Figure 2, we can see the result that is consistent with theory analysis, and the one-level methods and two-level methods keep the convergence rates. Moreover, from Figure 2(a), it can be seen that the L^2-convergence for the pressure is clearly faster than the indicated convergence of order 1. Because of we use the Gauss integration to deal with the stability term $G_h(\cdot, \cdot)$.

5.1.2. Computational cost

In Table 1, we compare the CPU time of three Algorithms and the one-level methods for the four values of h and H based on Section 5.1.1. As expected, the two-level methods spend much less time than one-level methods.

Furthermore, the results in Table 1 show that Algorithm 4.1 is the most efficient and Algorithm 4.3 spend the most computer time. We can see this phenomenon, because of the different nonlinear term $b(\cdot; \cdot, \cdot)$ and $\bar{b}(\cdot; \cdot, \cdot)$: trilinear form of Algorithm 4.1 is the most simple one, but the trilinear form of Algorithm 4.3 is the most complex one.

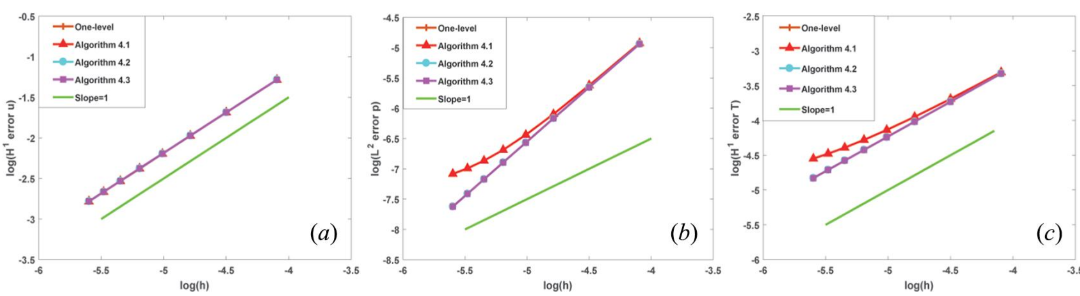

Figure 2. Convergence analysis using one-level and two-level nonconforming methods. (*a*) H^1 error for the velocity, (*b*) L^2 error for the pressure, and (*c*) H^1 error for the emperature.

Table 1. Comparisons of the one-level nonconforming method with the two-level nonconforming methods.

Methods	1/H	1/h	CPU-time(s)	Methods	1/H	1/h	CPU-time(s)
One-level	–	60	4.63e0	One-level	–	210	7.11e1
Algorithm 4.1	20	60	2.29e0	Algorithm 4.1	70	210	3.36e1
Algorithm 4.2	20	60	2.56e0	Algorithm 4.2	70	210	3.71e1
Algorithm 4.3	20	60	3.09e0	Algorithm 4.3	70	210	4.34e1
One-level	–	120	2.07e1	One-level	–	270	1.29e2
Algorithm 4.1	40	120	1.00e1	Algorithm 4.1	90	270	5.95e1
Algorithm 4.2	40	120	1.11e1	Algorithm 4.2	90	270	6.47e1
Algorithm 4.3	40	120	1.32e1	Algorithm 4.3	90	270	7.65e1

Table 2. Comparisons of the nonconforming two-level methods with the conforming two-level methods.

Methods	1/H	1/h	$\frac{\|u-u^h\|_{1,h}}{\|\nabla u\|_0}$	$\frac{\|p-p^h\|_0}{\|p\|_0}$	$\frac{\|\nabla(T-T^h)\|_0}{\|\nabla T\|_0}$	Methods	1/H	1/h	$\frac{\|u-u^h\|_{1,h}}{\|\nabla u\|_0}$	$\frac{\|p-p^h\|_0}{\|p\|_0}$	$\frac{\|\nabla(T-T^h)\|_0}{\|\nabla T\|_0}$
Algorithm 4.1	20	60	3.70e-2	3.89e-2	3.02e-2	Algorithm 4.1	70	210	1.06e-2	5.41e-3	8.64e-3
C-Stokes	20	60	4.28e-2	2.14e-1	3.06e-2	C-Stokes	70	210	1.22e-2	2.00e-2	8.65e-3
Algorithm 4.2	20	60	3.70e-2	3.94e-2	3.02e-2	Algorithm 4.2	70	210	1.06e-2	5.46e-3	8.64e-3
C-Oseen	20	60	4.28e-2	1.38e-1	3.04e-2	C-Oseen	70	210	1.22e-2	1.45e-2	8.64e-3
Algorithm 4.3	20	60	3.70e-2	3.44e-2	3.02e-2	Algorithm 4.3	70	210	1.06e-2	5.26e-3	8.65e-3
C-Newton	20	60	4.28e-2	8.28e-2	3.03e-2	C-Newton	70	210	1.22e-2	1.15e-2	8.65e-3
Algorithm 4.1	40	120	1.85e-2	1.29e-2	1.51e-2	Algorithm 4.1	90	270	8.23e-3	3.68e-3	6.72e-3
C-Stokes	40	120	2.14e-2	5.70e-2	1.52e-2	C-Stokes	90	270	9.50e-3	1.26e-2	6.72e-3
Algorithm 4.2	40	120	1.85e-2	1.31e-2	1.51e-2	Algorithm 4.2	90	270	8.23e-3	3.71e-3	6.72e-3
C-Oseen	40	120	2.14e-2	3.84e-2	1.51e-2	C-Oseen	90	270	9.50e-3	9.46e-3	6.72e-3
Algorithm 4.3	40	120	1.85e-2	1.21e-2	1.51e-2	Algorithm 4.3	90	270	8.23e-3	3.63e-3	6.73e-3
C-Newton	40	120	2.14e-2	2.74e-2	1.51e-2	C-Newton	90	270	9.50e-3	7.84e-3	6.74e-3

5.2. Experiment 2

Next, we further test the exact solution by solving the triangular function problem. The chosen functions are added to the right-hand side of (1) such that the exact solution for the velocity $u = (u_1, u_2)$, the temperature T and the pressure p to the considered problem are given as follows:

$$\begin{cases} u_1(x,y) = 2\pi\sin^2(\pi x)\sin(\pi y)\cos(\pi x), \\ u_2(x,y) = -2\pi\cos(\pi x)\sin^2(\pi y)\cos(\pi x), \\ p(x,y) \ = \cos(\pi x)\cos(\pi y), \\ T(x,y) \ = u_1(x,y) + u_2(x,y). \end{cases}$$

To validate our proposed algorithm advantages, we compare the numerical results of the methods that based on the three different corrections. For the sake of simplicity, we define the three correction methods of the two-level stabilized conforming finite element methods as: C-Stokes, C-Newton, and C-Oseen, respectively.

Then, we can see the result in Table 2. The error performance shows that the two-level nonconforming methods have smaller error than the two-level conforming methods, on account of the degrees of freedom of the nonconforming method is three times than that of the conforming method on the given mesh.

5.3. Experiment 3

The last example for testing the numerical experiment is the lid-driven flow, which has been analyzed in [25]. In this case, computations are performed in the domain $\Omega = [0,1]\times[0,1]$, and the boundary conditions are $T = 0$ on the left and bottom boundary, $\partial T/\partial n = 0$ on the tope side, $T = 4y(1-y)$ on

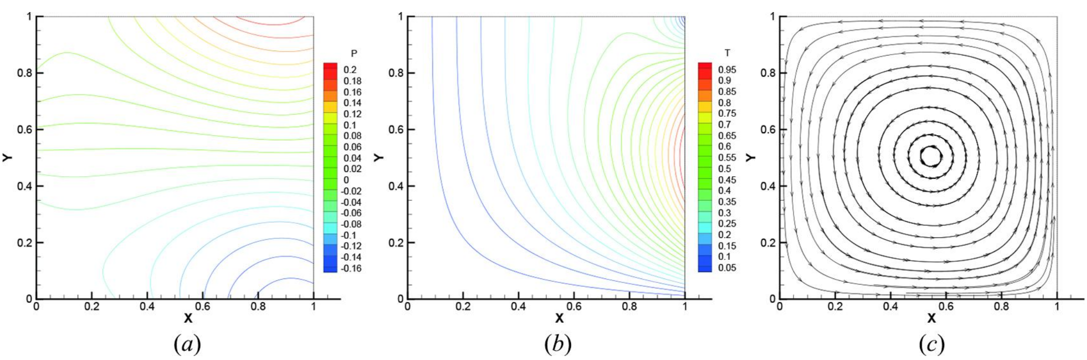

Figure 3. Physics models of the cavity for Algorithm 4.1 with $\nu = 0.1$; (a) isobars, (b) isotherms, and (c) the numerical streamlines.

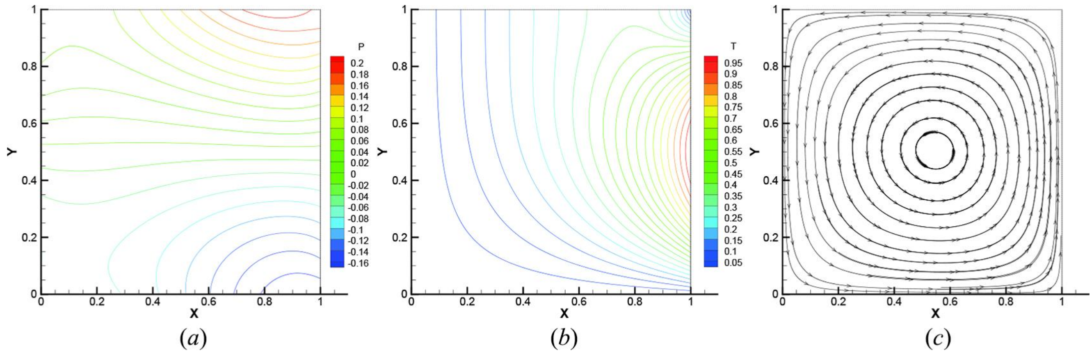

Figure 4. Physics models of the cavity for Algorithm 4.2 with $\nu = 0.1$; (*a*) isobars, (*b*) isotherms, and (*c*) the numerical streamlines.

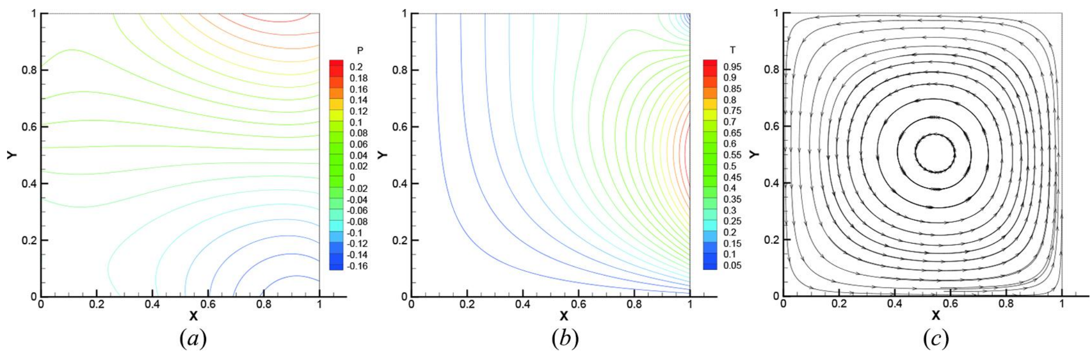

Figure 5. Physics models of the cavity for Algorithm 4.3 with $\nu = 0.1$; (*a*) isobars, (*b*) isotherms, and (*c*) the numerical streamlines.

the rest of the boundary, and zero Dirichlet conditions on velocity are imposed. The mesh consists of triangular element. We take the numerical solution by the standard Galerkin method (P_2, P_1, P_1) computed on a coarse mesh with mesh size $H = 50$ and a very fine mesh size $h = 150$. In Figures 3–5, the velocity streamlines, isobars and isotherms showed our methods. Seriously looking at the shape of the image and value, we can see that the three methods we proposed are reliable for the lid-driven flow and with a roughly similar feature.

6. Concluding remarks

Combining the nonconforming finite element pair and two-level scheme, we present three two-level iterative algorithms for the conduction–convection problem. The special mesh relationship $h = H/3$ we adopted in this paper is free of coarse-to-fine intergrid operator. Furthermore, the stability and error estimate are derived for the proposed methods. Besides, the numerical results verified the theoretical analysis and show that Algorithm 4.1 is the fastest method and Algorithm 4.3 has the best precision among the three algorithms, respectively. Naturally, this idea can be applied to other fluid dynamics equations.

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