高等数学

Sunday 3rd November, 2024

目录

Part I

极限

1 基础

1.1 常用极限

$$\begin{aligned}
\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x &= 1\\
\lim_{x\to \infty}\left(1+\frac{1}{x}\right)^x &= \mathrm{e}\\
\lim_{n\to \infty}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) &= \int_0^1 f\left(x\right)\mathrm{d}x\left(n\in\mathbb{N}^+\right)
\end{aligned}$$

1.2 常用等价无穷小

x 为函数，$\lim\limits_{x\to 0}$ 时，可对乘除因子替换

$$x\sim\sin x\sim\tan x\sim\arcsin x\sim\arctan x$$

$$x\sim\left(\mathrm{e}^x-1\right)\sim\ln\left(x+1\right)\sim\ln\left(x+\sqrt{1+x^2}\right)$$

$$x^3\sim 6\left(x-\sin x\right)\sim 6\left(\arcsin x-x\right)\sim 3\left(\tan x-x\right)$$

$$x^3\sim 3\left(x-\arctan x\right)\sim 2\left(\tan x-\sin x\right)$$

$$\begin{aligned}
1-\cos x &\sim \frac{x^2}{2}\\
\log_a\left(1+x\right) &\sim \frac{x}{\ln a}\\
\left(1+x\right)^a &\sim ax+1\\
a^x-1 &\sim x\ln a\left(0<a\neq 1\right)\\
\left(1+ax\right)^{\frac{1}{bx}} &\sim \mathrm{e}^{\frac{a}{b}}(1-\frac{a^2}{2b}x)
\end{aligned}$$

2 间断点

2.1 第一类间断点

$$\exists\lim_{x\to x_0^-}\text{且}\exists\lim_{x\to x_0^+}$$

2.1.1 可去间断点

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = A \left(\iff \lim_{x\to x_0} f(x) = A \right)$$

2.1.2 跳跃间断点

$$\lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0^+} f(x)$$

2.2 第二类间断点

$$\lim_{x\to x_0^-}, \lim_{x\to x_0^+} \text{至少满足有一个}\nexists$$

2.2.1 振荡间断点

左、右极限至少一个为振荡不存在

2.2.2 无穷间断点

左、右极限至少一个为 ∞

3 洛必达法则

3.1 使用条件

定义存在

$$x \in \mathring{U}(x_0)\text{（}x_0\text{可取}\infty\text{）}, \exists f'(x_0), \exists g'(x_0)$$

极限存在或为无穷

$$g'(x_0) \neq 0, \exists \lim_{x\to x_0} \frac{f'(x)}{g'(x)} \text{或} = \infty$$

符合 $\frac{0}{0}$ 或 $\frac{\text{任意}}{\infty}$

3.2 结论

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = A \implies \lim_{x\to x_0} \frac{f(x)}{g(x)} = A$$
$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \infty \implies \lim_{x\to x_0} \frac{f(x)}{g(x)} = \infty$$
$$\nexists \lim_{x\to x_0} \frac{f'(x)}{g'(x)} \not\Longrightarrow \nexists \lim_{x\to x_0} \frac{f(x)}{g(x)}$$

4 极限审敛

$$\lim_{\substack{x\to 0^+\\ y\to 0^+}} \frac{x^p y^q}{x^m + y^n}$$

m、n 全为偶数且 $\dfrac{p}{m} + \dfrac{q}{n} > 1$ 时 $\displaystyle\lim_{\substack{x\to 0^+\\ y\to 0^+}} \frac{x^p y^q}{x^m + y^n} = 0$，否则不存在

$\dfrac{p}{m} + \dfrac{q}{n} \leqslant 1$ 时，路径 $y = kx^{\frac{m-p}{q}}$ 可说明极限不存在

Part II

导数

5 基础

5.1 求导法则

$$\begin{aligned}
\left(f\left(x\right) + g\left(x\right)\right)' &= f'\left(x\right) + g'\left(x\right) \\
\left(f\left(x\right) g\left(x\right)\right)' &= f\left(x\right) g'\left(x\right) + f'\left(x\right) g\left(x\right) \\
\left(f\left(g\left(x\right)\right)\right)' &= f'\left(g\left(x\right)\right) g'\left(x\right) \\
\left(\int_{v(x)}^{u(x)} f\left(t\right) \mathrm{d}t\right)' &= f\left[u\left(x\right)\right] u'\left(x\right) - f\left[v\left(x\right)\right] v'\left(x\right) \\
\left(\int_{v(x)}^{u(x)} f\left(x,t\right) \mathrm{d}t\right)' &= \int_{v(x)}^{u(x)} f'_x\left(x,t\right) \mathrm{d}t + f\left[x, u\left(x\right)\right] u'\left(x\right) - f\left[x, v\left(x\right)\right] v'\left(x\right)
\end{aligned}$$

5.2 常用高阶导数

$$\begin{aligned}
&\sin^{(n)} \omega x && = \omega^n \sin\left(\omega x + \frac{n\pi}{2}\right) && (n \in \mathbb{N}) \\
&\cos^{(n)} \omega x && = \omega^n \cos\left(\omega x + \frac{n\pi}{2}\right) && (n \in \mathbb{N}) \\
&\ln^{(n)}\left(1+x\right) && = \left(-1\right)^{n-1} \frac{(n-1)!}{\left(1+x\right)^n} && \left(n \in \mathbb{N}^+\right) \\
&\ln^{(n)}\left(1-x\right) && = -\frac{(n-1)!}{\left(1-x\right)^n} && \left(n \in \mathbb{N}^+\right)
\end{aligned}$$

5.3 莱布尼茨公式

$$\left(uv\right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

5.4 中值定理

定理	公式	约束
积分中值定理	$f(\xi)=\dfrac{\int_a^b f(x)\,\mathrm{d}x}{x\vert_a^b}$	$\xi\in[a,b]$
罗尔中值定理	$f'(\xi)=0$	$\xi\in(a,b)$
拉格朗日中值定理	$f'(\xi)=\dfrac{f(x)\vert_a^b}{x\vert_a^b}$	$\xi\in(a,b)$
柯西中值定理	$\dfrac{f'(\xi)}{g'(\xi)}=\dfrac{f(x)\vert_a^b}{g(x)\vert_a^b}$	$\xi\in(a,b)$

5.5 泰勒中值定理

$R_n(x)$ 为余项

$$\begin{aligned}
P_n(x) \quad &= \sum_{i=0}^{n}\left[(x-x_0)\frac{\mathrm{d}}{\mathrm{d}x}\right]^i\frac{f(x_0)}{i!}+R_n(x)\\
P_n(x,y) \quad &= \sum_{i=0}^{n}\left[(x-x_0)\partial_x+(y-y_0)\partial_y\right]^i\frac{f(x_0,y_0)}{i!}+R_n(x,y)
\end{aligned}$$

5.5.1 拉格朗日型余项

$\theta\in(0,1)$

$$\begin{aligned}
R_n(x) &= \left[(x-x_0)\frac{\mathrm{d}}{\mathrm{d}x}\right]^{n+1}\frac{f(x_0+\theta(x-x_0))}{(n+1)!}\\
R_n(x,y) &= \left[(x-x_0)\partial_x+(y-y_0)\partial_y\right]^i\frac{f(x_0+\theta(x-x_0),y_0+\theta(y-y_0))}{(n+1)!}
\end{aligned}$$

5.5.2 佩亚诺型余项

$$\begin{aligned}
R_n(x) &= o\left[(x-x_0)^n\right]\\
R_n(x,y) &= o\left[\sqrt{(x-x_0)^2+(y-y_0)^2}^{\,n}\right]
\end{aligned}$$

5.5.3 误差估计式

$$n\in\mathbb{N};\exists M>0\forall x\in D\to M\geqslant\left|f^{(n+1)}(\xi)\right|$$

$$\implies |R_n(x)|\leqslant M\cdot\frac{|x-x_0|^{n+1}}{(n+1)!}$$

5.5.4 特别的：麦克劳林公式

$$\left.\begin{array}{c}(5.5.1)\\ x_0=y_0=0\end{array}\right\}\Longrightarrow\begin{cases}P_n\left(x\right)=\sum\limits_{i=0}^{n}\left(x\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^i\dfrac{f\left(0\right)}{i!}+R_n\left(x\right)\\ P_n\left(x,y\right)=\sum\limits_{i=0}^{n}\left(x\partial_x+y\partial_y\right)^i\dfrac{f\left(0,0\right)}{i!}+R_n\left(x,y\right)\end{cases}$$

5.5.5 常用麦克劳林公式

$\cos x$ 的 $2k$ 和 $2k+1$ 阶

$$\cos x=\sum_{i=0}^{k}\left(-1\right)^i\frac{x^{2i}}{2i!}+\left(-1\right)^{k+1}\cos\theta x\frac{x^{2k+2}}{(2k+2)!}$$

$\sin x$ 的 $2k-1$ 和 $2k$ 阶

$$\sin x=\sum_{i=1}^{k}\left(-1\right)^{i-1}\frac{x^{2i-1}}{(2i-1)!}+\left(-1\right)^{k}\cos\theta x\frac{x^{2k+1}}{(2k+1)!}$$

其他函数的 n 阶

$$\begin{aligned}\mathrm{e}^x&=\sum_{i=0}^{n}\frac{x^i}{i!}+\mathrm{e}^{\theta x}\frac{x^{n+1}}{(n+1)!}\\ \ln\left(1+x\right)&=\sum_{i=0}^{n}\left(-1\right)^{i-1}\frac{x^i}{i}+\frac{\left(-1\right)^n}{\left(1+\theta x\right)^{n+1}}\cdot\frac{x^{n+1}}{(n+1)}\left(x>-1\right)\\ \left(1+x\right)^\alpha&=\sum_{i=0}^{n}\binom{\alpha}{i}x^i+\binom{\alpha}{n+1}\frac{x^{n+1}}{\left(1+\theta x\right)^{n+1-\alpha}}\end{aligned}$$

5.6 极值（拉格朗日乘数法）

二元情况

$$\left.\begin{array}{l}\begin{cases}\text{约束条件：}\varphi\left(x,y\right)=0\\ \text{目标函数：}f\left(x,y\right)\end{cases}\\ \begin{cases}\nabla f=\lambda\nabla\varphi\left(\text{即}\nabla f\parallel\nabla\varphi\right)\\ \varphi\left(x,y\right)=0\end{cases}\end{array}\right\}\Longrightarrow\begin{array}{l}\text{解得几组 }\left(x_i,y_i\right)\text{ 即为可能的极值点}\\ \text{（若无约束条件 }\varphi\left(x,y\right)=0\text{，}\\ \text{可设约束为 }0=0\text{，即}\nabla\varphi=(0,0)\\ \text{则 }\nabla f=(0,0)\text{）}\end{array}$$

检验可能的极值点 (x_0,y_0)

$$\begin{aligned}&\left.\begin{array}{r}f''^2_{xy}\left(x_0,y_0\right)<f''_{xx}\left(x_0,y_0\right)f''_{yy}\left(x_0,y_0\right)\\ f''_{xx}\left(x_0,y_0\right)>0\end{array}\right\}\Longrightarrow f(x_0,y_0)\text{为极小值点}\\ &\left.\begin{array}{r}f''^2_{xy}\left(x_0,y_0\right)<f''_{xx}\left(x_0,y_0\right)f''_{yy}\left(x_0,y_0\right)\\ f''_{xx}\left(x_0,y_0\right)<0\end{array}\right\}\Longrightarrow f(x_0,y_0)\text{为极大值点}\\ &f''^2_{xy}\left(x_0,y_0\right)>f''_{xx}\left(x_0,y_0\right)f''_{yy}\left(x_0,y_0\right)\Longrightarrow f(x_0,y_0)\text{不取极值}\\ &f''^2_{xy}\left(x_0,y_0\right)=f''_{xx}\left(x_0,y_0\right)f''_{yy}\left(x_0,y_0\right)\Longrightarrow\text{需进一步讨论}\end{aligned}$$

n 元情况

$$\left.\begin{array}{l}\left\{\begin{array}{rl}\text{约束条件}\Phi\text{：} & \varphi_1\left(x_1,x_2,\cdots,x_n\right)=0\\ & \varphi_2\left(x_1,x_2,\cdots,x_n\right)=0\\ & \vdots\\ & \varphi_{n-1}\left(x_1,x_2,\cdots,x_n\right)=0\\ \text{目标函数：} & f\left(x_1,x_2,\cdots,x_n\right)\end{array}\right.\\ \left\{\begin{array}{l}\nabla f=\sum\limits_i\lambda_i\nabla\varphi_i\text{（三元时共面）}\\ \text{约束条件}\Phi\end{array}\right.\end{array}\right\}\Longrightarrow \text{解得几组}\left(x_1,x_2,\cdots,x_n\right)\text{即为可能的极值点}$$

5.7 隐函数存在定理

$F\left(x,y\right)$（二元）

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x'}{F_y'}\left(F_y'\neq0\right)$$

$F\left(x,y,z\right)$（多元）

$$\frac{\partial y}{\partial x}=-\frac{F_x'}{F_y'}\left(F_y'\neq0\right)$$

5.8 雅可比行列式

$$\frac{\partial\left(u_1,u_2,\cdots,u_n\right)}{\partial\left(x_1,x_2,\cdots,x_n\right)}=\begin{vmatrix}\partial_{x_1}u_1 & \partial_{x_2}u_1 & \cdots & \partial_{x_n}u_1\\ \partial_{x_1}u_2 & \partial_{x_2}u_2 & \cdots & \partial_{x_n}u_2\\ \vdots & \vdots & \ddots & \vdots\\ \partial_{x_1}u_n & \partial_{x_2}u_n & \cdots & \partial_{x_n}u_n\end{vmatrix}$$

Part III
积分

6 基础

6.1 牛顿-莱布尼茨公式

$$\int_a^b f'\left(x\right)\mathrm{d}x=f\left(x\right)\Big|_a^b$$

6.2 第一类换元（凑微分）法

$$\int f\left(x\right)g\left(x\right)\mathrm{d}x=\int f\left(x\right)\mathrm{d}\left(\int g\left(x\right)\mathrm{d}x\right)$$

6.3 第二类换元法

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= \int f(t)\,\mathrm{d}t\Big|_{t=\varphi(x)}\\ \int_a^b f[\varphi(x)]\,\mathrm{d}x &= \int_{\varphi(a)}^{\varphi(b)} f(t)\,\frac{\mathrm{d}\varphi^{-1}(t)}{\mathrm{d}t}\mathrm{d}t\Big|_{t=\varphi(x)}\end{aligned}$$

6.4 分部积分

$$\begin{cases} u = u(x) \\ v = v(x) \end{cases}$$

$$uv = \int udv + \int vdu$$

$$uv|_a^b = \int_a^b udv + \int_a^b vdu$$

6.5 常用积分表

6.5.1 三角函数总表

$\int f(x)\,\mathrm{d}x + \mathrm{C}$	$f(x)$	$f'(x)$	$\int f(x)\,\mathrm{d}x + \mathrm{C}$	$f(x)$	$f'(x)$
$-\cos x$	$\sin x$	$\cos x$	$\sin x$	$\cos x$	$-\sin x$
$-\ln\lvert\cos x\rvert$	$\tan x$	$\sec^2 x$	$\ln\lvert\sin x\rvert$	$\cot x$	$-\csc^2 x$
$\ln\lvert\sec x + \tan x\rvert$	$\sec x$	$\sec x \tan x$	$-\ln\lvert\csc x + \cot x\rvert$	$\csc x$	$-\csc x \cot x$
	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$		$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
	$\arctan x$	$\frac{1}{1+x^2}$		$\mathrm{arccot} x$	$-\frac{1}{1+x^2}$
	$\mathrm{arcsec} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$		$\mathrm{arccsc} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\cosh x$	$\sinh x$	$\cosh x$	$\sinh x$	$\cosh x$	$\sinh x$
$\ln\lvert\cosh x\rvert$	$\tanh x$	$\mathrm{sech}^2 x$	$\ln\lvert\sinh x\rvert$	$\coth x$	$-\mathrm{csch}^2 x$
$\arctan(\mathrm{e}^x)$	$\mathrm{sech} x$	$-\mathrm{sech} x \tanh x$	$-\ln\lvert\mathrm{csch} x + \coth x\rvert$	$\mathrm{csch} x$	$-\mathrm{csch} x \coth x$
	$\mathrm{arsinh} x$	$\frac{1}{\sqrt{x^2+1}}$		$\mathrm{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
	$\mathrm{artanh} x$	$\frac{1}{1-x^2}$		$\mathrm{arcoth} x$	$\frac{1}{1-x^2}$
	$\mathrm{arsech} x$	$-\frac{1}{\lvert x\rvert\sqrt{1-x^2}}$		$\mathrm{arcsch} x$	$-\frac{1}{\lvert x\rvert\sqrt{1+x^2}}$

6.5.2 其他

$$\begin{aligned}
\int a^x \mathrm{d}x &= \frac{a^x}{\ln a} && + \mathrm{C} \\
\int \frac{\mathrm{d}x}{x^2 - a^2} &= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| && + \mathrm{C} \\
\int \frac{\mathrm{d}x}{a^2 + x^2} &= \frac{1}{a} \arctan \frac{x}{a} && + \mathrm{C} \\
\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} &= \ln\left|x + \sqrt{x^2 \pm a^2}\right| && + \mathrm{C} \\
\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} &= \arcsin \frac{x}{a} && + \mathrm{C}
\end{aligned}$$

6.5.3 华里士公式

$$\int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \mathrm{d}x = \begin{cases} \dfrac{(n-1)!!}{n!!} \cdot \dfrac{\pi}{2} & n\text{为正偶数} \\ \dfrac{(n-1)!!}{n!!} & n\text{为正奇数} \end{cases}$$

6.6 * 有理函数积分通解（递推）

$$\int \frac{x+N}{\left(x^2+px+q\right)^{\lambda}} \mathrm{d}x \begin{cases} 0 > p^2 - 4q \\ a = \sqrt{q - \dfrac{p^2}{4}} \\ b = N - \dfrac{p}{2} \end{cases}$$

$$= \begin{cases} \dfrac{2bx + bp - 2a^2}{4(\lambda-1)a^2\left(x^2+px+q\right)^{\lambda-1}} + \dfrac{b(2\lambda-3)}{2(\lambda-1)a^2} \displaystyle\int \frac{\mathrm{d}x}{\left(x^2+px+q\right)^{\lambda-1}} & (\lambda > 1) \\ \dfrac{\ln\left(x^2+px+q\right)}{2} + \dfrac{b}{a} \arctan \dfrac{x+2p}{2a} + \mathrm{C} & (\lambda = 1) \end{cases}$$

6.7 万能代换

$$x = 2\arctan u \implies \begin{cases} \sin x = \dfrac{2u}{1+u^2} \\ \cos x = \dfrac{1-u^2}{1+u^2} \\ \mathrm{d}x = \dfrac{2}{1+u^2} \mathrm{d}u \end{cases}$$

6.8 区间再现

$$\int_a^b f(x) \mathrm{d}x = \int_a^b f(a+b-x) \mathrm{d}x$$

6.8.1 对称区间

$$\int_{-a}^{a} f\left(x\right) \mathrm{d}x = \int_{0}^{a} \left[f\left(x\right) + f\left(-x\right)\right] \mathrm{d}x$$

6.9 极坐标图形面积

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2\left(\theta\right) \mathrm{d}\theta$$

Definition 6.9.1 (以下参数方程中都有，且都可轮换)**.**

$$\begin{cases} x = x\left(t\right) \\ y = y\left(t\right) \end{cases}$$

6.10 旋转体体积（参数方程）

绕 x 轴

圆盘法

$$V = \pi \int_{a}^{b} x' y^2 dt$$

柱壳法

$$V = 2\pi \int_{a}^{b} x y' y dt$$

6.11 旋转体侧面积（参数方程）

绕 x 轴

$$S = 2\pi \int_{a}^{b} y \sqrt{x'^2 + y'^2} \mathrm{d}t$$

6.12 平面曲线弧长（参数方程）

$$s = \int_{a}^{b} \sqrt{{x'}^2 + {y'}^2} \mathrm{d}t = \int_{\alpha}^{\beta} \sqrt{r^2\left(\theta\right) + r'^2\left(\theta\right)} \mathrm{d}\theta$$

6.13 平面曲线曲率（参数方程）

曲率半径：K^{-1}

$$K = \frac{\left|x' y'' - x'' y'\right|}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}$$

7 重积分

7.1 二重积分

Definition 7.1.1 ($\mathrm{d}\sigma = \mathrm{d}x\mathrm{d}y$)**.**

$$\iint\limits_{D} f\left(x, y\right) \mathrm{d}\sigma$$

7.1.1 换元

$$\left.\begin{array}{l} \begin{cases} x = x\left(u, v\right) \\ y = y\left(u, v\right) \end{cases} \\ \left.\dfrac{\partial\left(x, y\right)}{\partial\left(u, v\right)}\right|_{D'} \neq 0 \end{array}\right\} \Longrightarrow \iint\limits_{D} f\left(x, y\right) \mathrm{d}x\mathrm{d}y = \iint\limits_{D'} f\left(x, y\right) \left|\frac{\partial\left(x, y\right)}{\partial\left(u, v\right)}\right| \mathrm{d}u\mathrm{d}v$$

7.1.2 广义极坐标变换

$$\begin{cases} x\left(r, \theta\right) = x_0 + ar\cos\theta \\ y\left(r, \theta\right) = y_0 + br\sin\theta \end{cases} \Longrightarrow \iint\limits_{D} f\left(x, y\right) \mathrm{d}x\mathrm{d}y = ab \iint\limits_{D} f\left(x, y\right) r\mathrm{d}r\mathrm{d}\theta$$

7.2 三重积分

Definition 7.2.1 ($\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$)**.**

$$\iiint\limits_{\Omega} f\left(x, y, z\right) \mathrm{d}V$$

7.2.1 换元

$$\left.\begin{array}{l} \begin{cases} x = x\left(u, v, w\right) \\ y = y\left(u, v, w\right) \\ z = z\left(u, v, w\right) \end{cases} \\ \left.\dfrac{\partial\left(x, y, z\right)}{\partial\left(u, v, w\right)}\right|_{\Omega'} \neq 0 \end{array}\right\} \Longrightarrow \iiint\limits_{\Omega} f\left(x, y, z\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega'} f\left(x, y, z\right) \left|\frac{\partial\left(x, y, z\right)}{\partial\left(u, v, w\right)}\right| \mathrm{d}u\mathrm{d}v\mathrm{d}w$$

7.2.2 柱面坐标

$$\begin{cases} x = x\left(r, \theta, z\right) = x_0 + ar\cos\theta \\ y = y\left(r, \theta, z\right) = y_0 + br\sin\theta \\ z = z\left(r, \theta, z\right) = z \end{cases}$$

$$\iiint\limits_{\Omega} f\left(x, y, z\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega} f\left(x, y, z\right) r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$

7.2.3 球面坐标

$$\begin{cases} x = x\left(r,\varphi,\theta\right) = \rho\sin\varphi\cos\theta \\ y = y\left(r,\varphi,\theta\right) = \rho\sin\varphi\sin\theta \\ z = z\left(r,\varphi,\theta\right) = \rho\cos\varphi \end{cases}$$

$$\iiint\limits_{\Omega} f\left(x,y,z\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega} f\left(x,y,z\right)\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$$

7.2.4 曲面面积（可轮换）

$$\left.\begin{aligned} z = z\left(x,y\right) \\ \left(x,y\right)\in D_{xy} \end{aligned}\right\} \implies S = \iint\limits_{D_{xy}} \sqrt{1+z_x'^2+z_y'^2}\mathrm{d}x\mathrm{d}y$$

7.3 积分应用

密度为 $\rho\left(x,y\right)$ 或 $\rho\left(x,y,z\right)$

7.3.1 质量

$$M = \iint\limits_{D} \rho\left(x,y\right)\mathrm{d}\sigma, M = \iiint\limits_{\Omega} \rho\left(x,y,z\right)\mathrm{d}\sigma$$

7.3.2 质心

质心的 x 坐标为

$$\bar{x} = \frac{\iint\limits_{D} x\rho\left(x,y\right)\mathrm{d}\sigma}{M}, \bar{x} = \frac{\iiint\limits_{\Omega} x\rho\left(x,y,z\right)\mathrm{d}\sigma}{M}$$

$\rho\left(\cdots\right)\equiv 1$ 时，质心相当于形心

7.3.3 转动惯量

绕 x 轴时

$$I_x = \iint\limits_{D} y^2\rho\left(x,y\right)\mathrm{d}\sigma, I_x = \iiint\limits_{\Omega} \left(y^2+z^2\right)\rho\left(x,y,z\right)\mathrm{d}\sigma$$

7.3.4 古尔丁定理

旋转体体积（平面图形 D 绕直线 $l: Ax+By+C=0$ 旋转）

$$V = \iint\limits_{D} 2\pi d_l\left(x,y\right)\mathrm{d}x\mathrm{d}y = 2\pi\iint\limits_{D} \frac{\left|Ax+By+C\right|}{\sqrt{A^2+B^2}}\mathrm{d}x\mathrm{d}y$$

若 D 形心为 (x_0,y_0)

$$V=2\pi d_l\left(x_0,y_0\right)S_D=2\pi\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}\iint\limits_D \mathrm{d}x\mathrm{d}y$$

8 曲线与曲面积分

8.1 曲线积分

Definition 8.1.1.

$$\begin{cases}x=x\left(t\right)\\ y=y\left(t\right)\\ z=z\left(t\right)\end{cases}\begin{cases}P=P\left(x,y,z\right)\\ Q=Q\left(x,y,z\right)\\ R=R\left(x,y,z\right)\end{cases}$$

Definition 8.1.2 (第一类)**.** $t\in[\alpha,\beta]$ $\theta\in[\theta_1,\theta_2]$

$$\int_L f\left(x,y\right)\mathrm{d}s=\int_\alpha^\beta f\left(x,y\right)\sqrt{x'^2+y'^2}\mathrm{d}t=\int_{\theta_2}^{\theta_1}\sqrt{r^2\left(\theta\right)+r'^2\left(\theta\right)}\mathrm{d}\theta$$

$$\int_\Gamma f\left(x,y,z\right)\mathrm{d}s=\int_\alpha^\beta f\left(x,y,z\right)\sqrt{x'^2+y'^2+z'^2}\mathrm{d}t$$

Definition 8.1.3 (第二类（坐标积分）)**.** $t:\alpha\to\beta$

$$\int_\Gamma P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z=\int_\alpha^\beta\left(Px'+Qy'+Rz'\right)\mathrm{d}t$$

8.1.1 格林公式

L 围成 D

$$\oint\limits_L P\mathrm{d}x+Q\mathrm{d}y=\iint\limits_D\begin{vmatrix}\partial_x & \partial_y\\ P & Q\end{vmatrix}\mathrm{d}x\mathrm{d}y=\iint\limits_D\left(Q'_x-P'_y\right)\mathrm{d}x\mathrm{d}y$$

8.1.2 平面曲线积分与积分路径无关条件

$\int_L P\mathrm{d}x+Q\mathrm{d}y$ 与积分路径无关

$$\begin{aligned}&\int_L P\mathrm{d}x+Q\mathrm{d}y=\int_A^B P\mathrm{d}x+Q\mathrm{d}y\\ \Longleftrightarrow&\oint\limits_L P\mathrm{d}x+Q\mathrm{d}y=0\\ \Longleftrightarrow&\exists u=u\left(x,y\right),\mathrm{d}u=P\mathrm{d}x+Q\mathrm{d}y\\ \Longleftrightarrow&D\text{内},Q'_x=P'_y\end{aligned}$$

8.1.3 曲线积分路径无关

$$\int_\Gamma P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y \to \begin{cases} R'_y = Q'_z \\ P'_z = R'_x \\ Q'_x = P'_y \end{cases}$$

8.2 曲面积分

Definition 8.2.1.

$$z = z\left(x, y\right) \begin{cases} P = P\left(x, y, z\right) \\ Q = Q\left(x, y, z\right) \\ R = R\left(x, y, z\right) \end{cases}$$

Definition 8.2.2 (第一类（可轮换）).

$$\iint\limits_{\Sigma} f\left(x, y, z\right) \mathrm{d}S = \iint\limits_{D_{xy}} f\left(x, y, z\right) \sqrt{z'^2_x + z'^2_y + 1} \mathrm{d}x\mathrm{d}y$$

Definition 8.2.3 (第二类（坐标积分）（外（远离原点）侧取正，内（指向原点）侧取负）).

$$\iint\limits_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \pm \iint\limits_{\Sigma} \frac{-Pz'_x - Qz'_y + R}{\sqrt{z'^2_x + z'^2_y + 1}} \mathrm{d}S$$

8.2.1 三合一投影法（外侧取正，内侧取负）

$$\iint\limits_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \pm \iint\limits_{D_{xy}} \left(-Pz'_x - Qz'_y + R\right) \mathrm{d}x\mathrm{d}y$$

8.2.2 高斯公式

$$\oint\!\!\!\oint\limits_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iiint\limits_{\Omega} \left(P'_x + Q'_y + R'_z\right) \mathrm{d}V$$

8.2.3 曲面积分路径无关

$$\iint\limits_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y \to P'_x + Q'_y + R'_z = 0$$

8.3 斯托克斯公式

$$\oint\limits_{\Gamma} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint\limits_{\Sigma} \begin{vmatrix} \mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x & \mathrm{d}x\mathrm{d}y \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$
$$= \iint\limits_{\Sigma} \left(R'_y - Q'_z\right) \mathrm{d}y\mathrm{d}z + \left(P'_z - R'_x\right) \mathrm{d}z\mathrm{d}x + \left(Q'_x - P'_y\right) \mathrm{d}x\mathrm{d}y$$

9 向量分析

Definition 9.0.1 (向量场)**.**

$$\vec{\psi} = \{P, Q, R\}$$

9.1 梯度

Definition 9.1.1 (梯度)**.**

$$\nabla = \{\partial_x, \partial_y, \partial_z\}$$

9.2 散度

Definition 9.2.1 (通过 Σ 流向指定侧的通量)**.**

$$\Phi = \iint\limits_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y$$

Theorem 9.2.1 (散度)**.**

$$\mathrm{div}\vec{\psi} = \nabla \cdot \vec{\psi} = P'_x + Q'_y + R'_z$$

9.3 旋度

Definition 9.3.1 (沿封闭曲线 Γ 的环流量)**.**

$$\oint\limits_{\Gamma} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$$

Theorem 9.3.1 (旋度)**.**

$$\mathrm{rot}\vec{\psi} = \nabla \times \vec{\psi} = \left\{R'_y - Q'_z, P'_z - R'_x, Q'_x - P'_y\right\}$$

Part IV

微分方程

10 n 阶线性微分方程

Definition 10.0.1.

$$y^{(n)} + \sum_{i=0}^{n-1} p_i(x) y^{(i)} = f(x)$$

10.1 线性相关

$$\frac{f(x)}{g(x)} = \mathrm{C}(\mathrm{C} \in \mathbb{C})$$

10.2 伯努利方程

$$y' + P(x) y = Q(x) y^{\alpha} \xrightarrow{z=y^{1-\alpha}} z' + (1-\alpha) P(x) z = (1-\alpha) Q(x)$$

11 一阶线性微分方程

Definition 11.0.1 ($f(x) \equiv 0$ 时，为齐次)**.**

$$\left.\begin{aligned} (10.0.1) \\ n = 1 \end{aligned}\right\} \implies y' + P(x) y = f(x)$$

Theorem 11.0.1 (通解)**.**

$$y = \frac{\int f(x) \exp\left(\int P(x)\,\mathrm{d}x\right) \mathrm{d}x + \mathrm{C}}{\exp\left(\int P(x)\,\mathrm{d}x\right)}$$

12 二阶线性微分方程

Definition 12.0.1.

$$y'' + P(x) y' + Q(x) y = f(x)$$

12.1 齐次、非齐次、通解、特解关系

齐特 + 齐特（线性无关）= 齐通
齐通 + 非特 = 非通
齐特 + 非特 = 非特
非特 − 非特 = 齐特

13 n 阶常系数线性齐次微分方程

Definition 13.0.1.

$$y^{(n)}+\sum_{i=0}^{n-1}p_iy^{(i)}=0\,(p_i\in\mathbb{C})$$

13.1 特征方程

$$r^n+\sum_{i=0}^{n-1}p_ir^i=0$$

13.2 通解对应项

k 重实根 r 在通解中对应项

$$y_r=\sum_{i=1}^{k}\mathrm{C}_ix^{i-1}\cdot\mathrm{e}^{rx}$$

特别的：r 为共轭复根（$r=\alpha\pm\beta\mathrm{i}$）时，可改写为两个实根

$$y_r=(\mathrm{C}_1\cos\beta x+\mathrm{C}_2\sin\beta x)\mathrm{e}^{\alpha x}$$

14 二阶常系数线性齐次微分方程

Definition 14.0.1.

$$y''+py'+qy=0$$

14.1 特征方程

$$r^2+pr+q=0$$

14.2 通解

$r_1\neq r_2$

$$y=\mathrm{C}_1\mathrm{e}^{r_1x}+\mathrm{C}_2\mathrm{e}^{r_2x}$$

$r_1=r_2$

$$y=(\mathrm{C}_1+\mathrm{C}_2x)\,\mathrm{e}^{r_1x}$$

$r_{1,2}=\alpha\pm\beta\mathrm{i}$

$$y=(\mathrm{C}_1\cos\beta x+\mathrm{C}_2\sin\beta x)\mathrm{e}^{\alpha x}$$

15 二阶常系数非齐次线性微分方程

Definition 15.0.1.

$$y'' + py' + qy = f(x)$$

15.1 特解

Theorem 15.1.1 (特解)**.** $\mathcal{P}_n$ 表示 n 次多项式

$$\left.\begin{aligned} &(15.0.1) \\ &f(x) = \left[\mathcal{P}_{n_1}(x)\cos\omega x + \mathcal{P}_{n_2}(x)\sin\omega x\right]\mathrm{e}^{\lambda x} \\ &m = \max\{n_1, n_2\} \end{aligned}\right\} \Longrightarrow$$

$$y^* = x^k\left[\mathcal{U}_m(x)\cos\omega x + \mathcal{V}_m(x)\sin\omega x\right]\mathrm{e}^{\lambda x}\begin{cases} k=0 & \left(\lambda \pm \omega\mathrm{i}\text{不是特征方程根}\right) \\ k=1 & \left(\lambda \pm \omega\mathrm{i}\text{是特征方程根}\right) \end{cases}$$

Theorem 15.1.2 (特解的特例)**.** $\omega = 0$ 时，$m = n_1$

$$\left.\begin{aligned} &(15.0.1) \\ &f(x) = \mathcal{P}_m(x)\mathrm{e}^{\lambda x} \end{aligned}\right\} \Longrightarrow y^* = x^k\mathcal{Q}_m(x)\mathrm{e}^{\lambda x}\begin{cases} k=0 & \left(\lambda\text{不是特征方程根}\right) \\ k=1 & \left(\lambda\text{是特征方程单根}\right) \\ k=2 & \left(\lambda\text{是特征方程重根}\right) \end{cases}$$

15.1.1 常系数非齐次通解的大致形式

$$\underbrace{\overbrace{\mathrm{C}_1\overbrace{a(x)\mathrm{e}^{r_1x}}^{\text{齐次特解}} + \mathrm{C}_2\underbrace{\overbrace{b(x)\mathrm{e}^{r_2x}}^{\text{齐次特解}} + \overbrace{x^kc(x)\mathrm{e}^{\lambda x}}^{\text{非齐次特解}}}_{\text{非齐次特解}}}^{\text{齐次通解}}}_{\text{非齐次通解}}$$

15.1.2 算子法求特解

Definition 15.1.1 (D 算子)**.**

$$Df(x) = f'(x), \frac{1}{D}f(x) = \int f(x)\,\mathrm{d}x$$

对于 (15.0.1)：

$$y^* = \frac{1}{D^2 + pD + q}f(x) = \frac{1}{\mathcal{F}(D)}f(x)$$

若代入 D 后分母 $\mathcal{P}(D)$ 出现为 0 的状况，则（可多次使用，D 算子只对右侧 $f(x)$ 有效）：

$$y^* = x^n\frac{1}{\mathcal{P}(D)}f(x) \longrightarrow y^* = x^{n+1}\frac{1}{\mathcal{P}'(D)}f(x)$$

$f(x) = \mathrm{Ce}^{kx}$：D 换为 k

$f(x)=\mathrm{C}\sin ax$ 或 $\mathrm{C}\sin ax$：D^2 换为 $-a^2$　若代入 D^2 后分母有 $mD+n\,(mn>0)$ 一次多项式，可以配平方将一次多项式化到分子，再代入 D^2 后直接使用 D 算子求导

$f(x)=\mathcal{P}_n(x)$：　使用 $\dfrac{1}{1+x}=\sum\limits_{n\in\mathbb{N}}(-x)^n$ 泰勒展开 $\dfrac{1}{\mathcal{F}(D)}$（$\mathcal{F}(D)-1$ 当作 x，不考虑收敛域），使得展开后 D 的最高次幂与 $\mathcal{P}_n(x)$ 相同即可

$f(x)=\mathrm{e}^{kx}y(x)$：移位定理

$$y^*=\frac{1}{\mathcal{F}(D)}\mathrm{e}^{kx}y(x)=\mathrm{e}^{kx}\frac{1}{\mathcal{F}(D+k)}y(x)$$

$f(x)=\mathcal{P}_n(x)\,\mathrm{C}\sin ax$：

$$y^*=\frac{1}{\mathcal{F}(D)}\mathcal{P}_n(x)\,\mathrm{C}\sin ax=\mathrm{Im}\left[\frac{1}{\mathcal{F}(D)}\mathcal{P}_n\mathrm{e}^{\mathrm{i}ax}\right]$$

$f(x)=\mathcal{P}_n(x)\,\mathrm{C}\cos ax$：

$$y^*=\frac{1}{\mathcal{F}(D)}\mathcal{P}_n(x)\,\mathrm{C}\cos ax=\mathrm{Re}\left[\frac{1}{\mathcal{F}(D)}\mathcal{P}_n\mathrm{e}^{\mathrm{i}ax}\right]$$

16　全微分方程

16.1　条件（微分换序）

$$P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y=0\text{是全微分方程}\iff P'_y=Q'_x$$

Part V
空间解析几何

17　基础

17.1　向量的方向余弦

$$\vec{\mathrm{v}}^0=\begin{bmatrix}\cos\alpha\\\cos\beta\\\cos\gamma\end{bmatrix}=\frac{1}{\|\vec{\mathrm{v}}\|}\begin{bmatrix}\vec{\mathrm{v}}_x\\\vec{\mathrm{v}}_y\\\vec{\mathrm{v}}_z\end{bmatrix}$$

18　空间曲面

18.1　基础

Definition 18.1.1.

$$F(x,y,z)=0$$

18.1.1 法向量

$$\nabla F$$

18.1.2 方向导数

方向为 $\boldsymbol{l}$

$$\frac{\partial F}{\partial \boldsymbol{l}} = \partial_x F \cdot \cos\alpha + \partial_y F \cdot \cos\beta + \partial_z F \cdot \cos\gamma = \partial_x F \cdot \frac{\boldsymbol{l} \cdot \boldsymbol{i}}{|\boldsymbol{l}|} + \partial_y F \cdot \frac{\boldsymbol{l} \cdot \boldsymbol{j}}{|\boldsymbol{l}|} + \partial_z F \cdot \frac{\boldsymbol{l} \cdot \boldsymbol{k}}{|\boldsymbol{l}|}$$

$$\boldsymbol{l} = \nabla F \implies \frac{\partial F}{\partial \boldsymbol{l}} = |\nabla F|$$

18.2 平面

Definition 18.2.1.

$$Ax + By + Cz + D = 0$$

18.2.1 平面点法式

过 (x_0, y_0, z_0)，法向量 $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

18.2.2 平面截距式

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

18.2.3 点面距离公式

点 (x_0, y_0, z_0)

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

19 空间曲线

Definition 19.0.1.

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$$

19.1 参数方程

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

19.2 切向量

$$\vec{\tau} = \nabla F \times \nabla G$$

19.3 直线

Definition 19.3.1.

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

19.4 直线对称式（点向式）方程

过 (x_0, y_0, z_0)，方向向量 $\begin{bmatrix} m \\ n \\ p \end{bmatrix}$

$$\frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p} = t$$

19.5 直线参数方程

$$\begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases}$$

20 特殊曲面

Definition 20.0.1 (绕 z 轴旋转曲面：（原曲线为 $f(y_1, z) = 0$）)**.**

$$\left.\begin{matrix} f(y_1, z) = 0 \\ \sqrt{x^2 + y^2} = |y_1| \end{matrix}\right\} \implies f\left(\pm\sqrt{x^2 + y^2}, z\right) = 0$$

20.1 圆锥面

$$z^2 = \cot^2 \alpha \cdot \left(x^2 + y^2\right)$$

Definition 20.1.1 (以下二次曲面方程中都有)**.**

$$pq > 0$$

20.2 椭球面

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

20.3 椭圆抛物面

$$\frac{x^2}{2p}+\frac{y^2}{2q}=z$$

20.4 双曲抛物面（马鞍面）

$$-\frac{x^2}{2p}+\frac{y^2}{2q}=z$$

$$z=xy$$

20.5 单叶双曲面

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

20.6 双叶双曲面

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

Part VI

级数

21 收敛与发散

21.1 绝对收敛

$$\sum_{n\in\mathbb{N}^+}|u_n|=s,s\in\mathbb{C}$$

21.2 条件收敛

$$\sum_{n\in\mathbb{N}^+}u_n=s,\ \sum_{n\in\mathbb{N}^+}|u_n|\,\text{发散},s\in\mathbb{C}$$

$$\text{绝对收敛}\cap\text{条件收敛}=\varnothing$$

21.3 无穷大比较

$$n \to +\infty$$

$$n^n \gg n! \gg a^n\,(a>1) \gg n^p\,(p>1) \gg | \gg n^p\,(1 \geqslant p > 0) \gg (\ln n)^q\,(q>0)$$

$\sum\limits_{n\in\mathbb{N}+} \dfrac{1}{u_n}$ 中，若 u_n 在 $|$ 记号左侧则收敛，在 $|$ 记号右侧则发散

$\sum\limits_{n\in\mathbb{N}+} \dfrac{v_n}{u_n}$ 中，若 u_n 在 $|$ 记号左侧，且 v_n 在 u_n 右侧时收敛，否则发散（$\dfrac{n^p\,(1 \geqslant p > 0)}{n^q\,(q>1)}$ 除外，需进一步 $p > q+1$ 才收敛）

22 正项级数

22.1 积分审敛法

$$\sum_{n\in\mathbb{N}+} f(n) \text{与} \int_1^{+\infty} f(x)\,\mathrm{d}x \text{敛散同}$$

22.2 比较审敛法

$$\lim_{n\to\infty} \frac{u_n}{v_n} \begin{cases} =0 & \Longrightarrow \sum v_n \text{收敛则} \sum u_n \text{收敛} \\ \in (0,+\infty) & \Longrightarrow \sum v_n, \sum u_n \text{敛散同} \\ =+\infty & \Longrightarrow \sum v_n \text{发散则} \sum u_n \text{发散} \end{cases}$$

22.3 比值审敛法（达朗贝尔判别法）

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} \begin{cases} <1 & \Longrightarrow \sum u_n \text{收敛} \\ =1 & \Longrightarrow \sum u_n \text{可能收敛可能发散} \\ >1 & \Longrightarrow \sum u_n \text{发散} \end{cases}$$

22.4 根值审敛法（柯西判别法）

$$\lim_{n\to\infty} \sqrt[n]{u_n} \begin{cases} <1 & \Longrightarrow \sum u_n \text{收敛} \\ =1 & \Longrightarrow \sum u_n \text{可能收敛可能发散} \\ >1 & \Longrightarrow \sum u_n \text{发散} \end{cases}$$

23 交错级数

23.1 莱布尼兹判别法

$$\left.\begin{array}{r}\text{正项级数}u_n\searrow\\ \lim\limits_{n\to\infty}u_n=0\end{array}\right\}\Longrightarrow\text{交错级数}\sum(-1)^{n(\text{或}n-1)}u_n\text{收敛}$$

24 幂（泰勒）级数

Definition 24.0.1 (以下默认幂级数形式)**.**

$$\sum_{n\in\mathbb{N}}=a_nx^n,\text{收敛半径为}R$$

24.1 阿贝尔定理

$$\begin{cases}x\in(-R,R) & \text{绝对收敛}\\ x\in\{-R,R\} & \text{单独讨论}\\ x\in(-\infty,-R)\cup(R,+\infty) & \text{发散}\end{cases}$$

收敛区间为 $(-R,R)$，收敛域需要讨论端点 $\pm R$ 处的值

24.2 系数模比值法

$$R^{-1}=\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

24.3 系数模根值法

$$R^{-1}=\rho=\lim_{n\to\infty}\sqrt[n]{|a_n|}$$

24.4 加减运算

$$\left.\begin{array}{r}\sum a_nx^n\text{收敛域为}I_a\\ \sum b_nx^n\text{收敛域为}I_b\end{array}\right\}\Longrightarrow\sum a_nx^n\pm\sum b_nx^n=\sum(a_n\pm b_n)x^n,x\in I_a\cap I_b$$

24.5 泰勒级数

$$f(x)\sim\sum_{n\in\mathbb{N}}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

24.6 常用泰勒级数

$$\begin{aligned}
\mathrm{e}^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+o\left(x^3\right) && =\sum_{n\in\mathbb{N}}\frac{x^n}{n!} \\
\sin x &= x-\frac{x^3}{3!}+\frac{x^5}{5!}+o\left(x^5\right) && =\sum_{n\in\mathbb{N}}\left(-1\right)^n\frac{x^{2n+1}}{(2n+1)!} \\
\cos x &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}+o\left(x^4\right) && =\sum_{n\in\mathbb{N}}\left(-1\right)^n\frac{x^{2n}}{(2n)!} \\
\tan x &= x+\frac{1}{3}x^3+\frac{2}{15}x^5+o\left(x^5\right) && =\sum_{n\in\mathbb{N}^+}\frac{B_{2n}\left(-4\right)^n\left(1-4^n\right)^{2n-1}}{(2n)!} && x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \\
\arctan x &= x-\frac{x^3}{3}+\frac{x^5}{5}+o\left(x^5\right) && =\sum_{n\in\mathbb{N}}\frac{\left(-1\right)^n}{2n+1}x^{2n+1} && x\in[-1,1] \\
\arcsin x &= x+\frac{1}{6}x^3+\frac{3}{40}x^5+o\left(x^5\right) && =\sum_{n\in\mathbb{N}}\frac{(2n)!}{4^n\left(n!\right)^2\left(2n+1\right)}x^{2n+1} && x\in(-1,1) \\
\ln\left(1+x\right) &= x-\frac{x^2}{2}+\frac{x^3}{3}+o\left(x^3\right) && =\sum_{n\in\mathbb{N}^+}\left(-1\right)^{n-1}\frac{x^n}{n} && x\in(-1,1] \\
\ln\left(1-x\right) &= -x-\frac{x^2}{2}-\frac{x^3}{3}+o\left(x^3\right) && =\sum_{n\in\mathbb{N}^+}-\frac{x^n}{n} && x\in(-1,1] \\
\frac{1}{1+x} &= 1-x+x^2-x^3+o\left(x^3\right) && =\sum_{n\in\mathbb{N}}\left(-x\right)^n && x\in(-1,1) \\
\frac{1}{1-x} &= 1+x+x^2+x^3+o\left(x^3\right) && =\sum_{n\in\mathbb{N}}x^n && x\in(-1,1) \\
\left(1+x\right)^\alpha &= 1+\alpha x+\frac{\alpha\left(\alpha-1\right)}{2!}x^2+o\left(x^2\right) && =\sum_{n\in\mathbb{N}}\binom{\alpha}{n}x^n && x\in(-1,1)
\end{aligned}$$

25 三角（傅里叶）级数

25.1 傅里叶级数

$$f\left(x\right)\sim\frac{a_0}{2}+\sum_{n\in\mathbb{N}^+}\left(a_n\cos nx+b_n\sin nx\right)$$

$f\left(x\right)$ 周期为 $T=2l$ 时（l 常取 π），有傅里叶系数：

$$\begin{cases}
a_n=\dfrac{1}{l}\displaystyle\int_{-l}^{l}f\left(x\right)\cos\frac{n\pi x}{l}\mathrm{d}x, n\in\mathbb{N} & =\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}f\left(x\right)\cos n\omega x\mathrm{d}x \\
b_n=\dfrac{1}{l}\displaystyle\int_{-l}^{l}f\left(x\right)\sin\frac{n\pi x}{l}\mathrm{d}x, n\in\mathbb{N}^+ & =\dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}f\left(x\right)\sin n\omega x\mathrm{d}x
\end{cases}$$

$$\omega=\frac{2\pi}{T}=\frac{\pi}{l}$$

25.2 狄利克雷收敛定理

$f(x)$ 在一个周期内有：

1. 连续或只有有限个第一类间断点
2. 只有有限个极值点

即 $f(x)$ 的傅里叶级数在 $\mathbb{R}$ 连续，且

1. x_0 连续时，级数收敛于 $f(x_0)$
2. x_0 是第一类间断点时，级数收敛于 $\dfrac{f(x_0^-)+f(x_0^+)}{2}$