KRP — Assignment 4

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⋆ **This assignment, due on 26th May at 23:59, contributes to 10% of the final marks for this course. Please be advised that only Questions 1 — 8 are mandatory. Nevertheless, students can earn up to one bonus mark by completing Question 9. This bonus mark can potentially augment a student's overall marks but is subject to a maximum total of 100 for the course. By providing bonus marks, we aim to incentivize students to excel in their studies and reward those with a remarkable grasp of the course materials.**

Question 1. $\mathcal{ALC}$-Worlds Algorithm

Use the $\mathcal{ALC}$-Worlds algorithm to decide the satisfiability of the concept name B_0 w.r.t. the simple TBox:

$$\mathcal{T} := \left\{ \begin{array}{l} B_0 \equiv B_1 \sqcap B_2 \\ B_1 \equiv \exists r.B_3 \\ B_2 \equiv B_4 \sqcap B_5 \\ B_3 \equiv P \\ B_4 \equiv \exists r.B_6 \\ B_5 \equiv B_7 \sqcap B_8 \\ B_6 \equiv Q \\ B_7 \equiv \forall r.B_4 \\ B_8 \equiv \forall r.B_9 \\ B_9 \equiv \forall r.B_{10} \\ B_{10} \equiv \neg P \end{array} \right\},$$

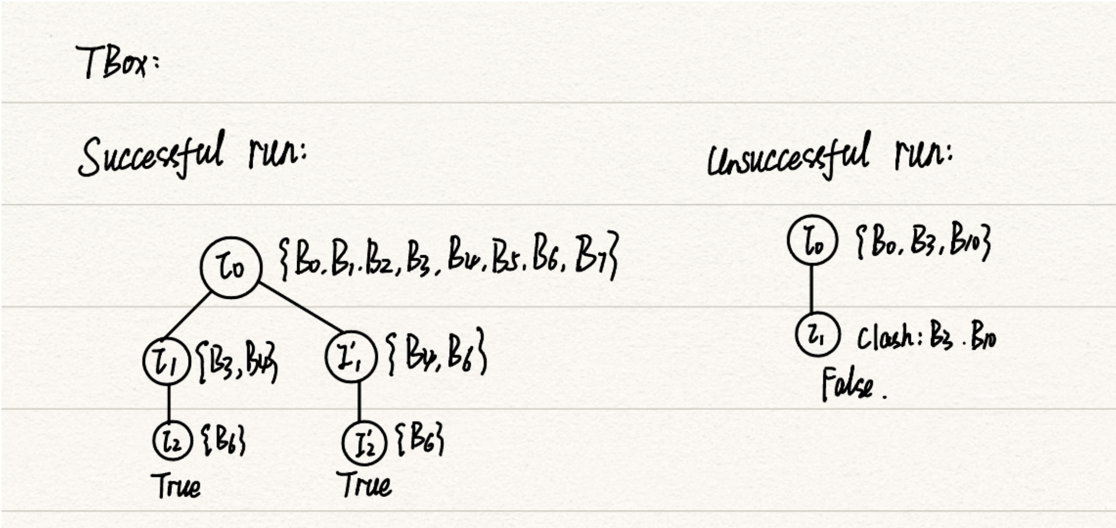
Draw the recursion tree of a successful run and of an unsuccessful run. Does the algorithm return a positive or negative result on this input?

My Solution Question 1. As the TBox given:

To decide the satisfiability of B_0, we need to generate all possible worlds that satisfy the TBox and check if B_0 is satisfied in at least one of these worlds.

A successful run would be a run where B_0 is satisfied in at least one world. An unsuccessful run would be a run where B_0 is not satisfied in any world.

Since B_0 is defined as $B_1 \sqcap B_2$, it is satisfied if and only if both B_1 and B_2 are satisfied. However, B_1 is satisfied if and only if there exists a world where B_3 is satisfied, and B_2 is satisfied if and only if both B_4 and B_5 are satisfied. Therefore, B_0 is satisfied if and only if there exists a world where both B_3 and B_4 are satisfied, and there exists a world where both B_5 and B_6 are satisfied.



However, B_3 is satisfied if and only if P is true, B_4 is satisfied if and only if there exists a world where B_6 is satisfied, B_5 is satisfied if and only if both B_7 and B_8 are satisfied, B_6 is satisfied if and only if Q is true, B_7 is satisfied if and only if for all worlds where B_4 is satisfied, B_4 is satisfied, B_8 is satisfied if and only if for all worlds where B_9 is satisfied, B_9 is satisfied, and B_9 is satisfied if and only if for all worlds where B_{10} is satisfied, B_{10} is satisfied.

Therefore, B_0 is satisfied if and only if there exists a world where P is true and there exists a world where Q is true. This means that **the algorithm will return a positive result on this input.**

Question 2. Finite Boolean Game

Determine whether Player 1 has a winning strategy in the following finite Boolean game, where $\Gamma_1 := \{x_1, x_3\}$ and $\Gamma_2 := \{x_2, x_4\}$.

- $\psi := (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_3 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_3 \vee x_4)$

My Solution Question 2. No.

To determine it, need to analyze the logical structure of the formula ψ and see if Player 1 can force a win regardless of how Player 2 plays.

The formula ψ is defined as follows:

$$\psi := (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_3 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_3 \vee x_4)$$

Player 1's moves are x_1 and x_3, while Player 2's moves are x_2 and x_4. We will analyze each clause of ψ to see if Player 1 can always satisfy it regardless of Player 2's choices.

1. Clause $(x_1 \vee \neg x_2)$**:** Player 1 can set $x_1 =$ true to satisfy this clause regardless of the value of x_2.

2. Clause $(x_2 \vee x_3)$**:** Player 2 can set $x_2 =$ true to satisfy this clause, but if they don't, Player 1 can set $x_3 =$ true to satisfy it. However, this depends on Player 2's move.

3. Clause $(\neg x_3 \vee \neg x_4)$**:** This clause is dependent on both players' moves. If Player 1 sets $x_3 =$ false, then Player 2 must set $x_4 =$ false to satisfy the clause. Conversely, if Player 2 sets $x_4 =$ true, then Player 1 must set $x_3 =$ false.

4. Clause $(\neg x_1 \vee \neg x_2 \vee x_3 \vee x_4)$**:** This clause is complex and involves all variables. Player 1 can try to satisfy it by setting $x_3 =$ true or $x_4 =$ true if Player 2 doesn't. However, if both x_3 and x_4 are false, then the clause becomes dependent on the values of x_1 and x_2.

From the above analysis, we see that Player 1 can control some parts of the formula independently (like setting $x_1 =$ true), but other parts depend on Player 2's choices. The key is whether Player 1 can always find a strategy to satisfy ψ regardless of what Player 2 does.

Given the interdependencies in clauses involving both players' moves, especially the fourth clause which is complex and involves all variables, it is not immediately clear if Player 1 has a winning strategy without further detailed analysis or using tools like solving for Nash equilibria in game theory. However, from a basic logical perspective, since some outcomes depend on Player 2's choices and there are no clear dominant strategies for Player 1 that cover all scenarios, it suggests that **Player 1 does not have a winning strategy in all cases.**

Question 3. Infinite Boolean Game

Determine whether Player 2 has a winning strategy in the following infinite Boolean game where the initial configuration t_0 assigns *false* to all variables.

- $\psi := (x_1 \wedge x_2 \wedge \neg y_1) \vee (x_3 \wedge x_4 \wedge \neg y_2) \vee (\neg(x_1 \vee x_4) \wedge y_1 \wedge y_2)$

  provided that: $\Gamma_1 := \{x_1, x_2, x_3, x_4\}$ and $\Gamma_2 := \{y_1, y_2\}$

My Solution Question 3. No.

Just find a winnig strategy of Player 1:Player 1 can change x_2, x_3 to True and True. Then Player 2 must set y_1 =False or y_2 =False

-If y_1 =False, Player 1 set x_1 =True, then he wins.

-If y_2 =False, Player 1 set x_4 =True, then he wins.

Therefore, **Player 2 has no winning strategy in this infinite Boolean game**.

Question 4. Complexity of Concept Satisfiability in $\mathcal{ALC}$ Extensions

The universal role is a role u such that its extension is fixed as $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ in any interpretation $\mathcal{I}$. Let $\mathcal{ALC}^u$ be a DL extending $\mathcal{ALC}$ with the universal role.

- Show that concept satisfiability in $\mathcal{ALC}^u$ without TBoxes is EXPTIME-complete.

My Solution Question 4. To show that concept satisfiability in $\mathcal{ALC}^u$ without TBoxes is EXPTIME-complete, we need to consider the complexity of checking satisfiability for a concept in this description logic.

Given a concept C in $\mathcal{ALC}^u$, we want to determine whether there exists an interpretation $\mathcal{I}$ such that C is satisfied in $\mathcal{I}$. The universal role u has a fixed extension as $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, where $\Delta^{\mathcal{I}}$ is the domain of individuals in the interpretation $\mathcal{I}$.

The satisfiability problem for $\mathcal{ALC}^u$ can be reduced to the satisfiability problem for propositional logic (which is known to be NP-complete) by constructing a propositional formula that simulates the behavior of the concept C with respect to the universal role u. Since the universal role has a fixed extension, it acts as a binary relation between any two individuals in the domain, and this relation can be represented using binary propositional variables.

The reduction works as follows:

1. Represent each individual $a \in \Delta^{\mathcal{I}}$ as a propositional variable p_a.
2. For each role r (including the universal role u), introduce a set of binary propositional variables $(q_{r,ij})_{i,j \in \Delta^{\mathcal{I}}}$ to represent the role restriction between individuals.
3. Translate each axiom in the concept C into its corresponding propositional logic form using the introduced variables.
4. Ensure that the binary variables for the universal role u are all set to true, reflecting its fixed extension.
5. Check the satisfiability of the resulting propositional formula using a SAT solver.

Since the number of individuals and roles is polynomial in the size of the input concept C, and since the translation into propositional logic and the use of a SAT solver are both feasible in exponential time, the overall complexity of this reduction is EXPTIME.

Therefore, **the concept satisfiability in $\mathcal{ALC}^u$ without TBoxes is EXPTIME-complete** due to the reduction from concept satisfiability to propositional logic satisfiability, which is known to be NP-complete, and the additional overhead involved in representing the universal role and individuals in the interpretation.

Question 5. Subsumption in $\mathcal{EL}$
Consider the following $\mathcal{EL}$ TBox:

$$\mathcal{T} := \left\{ \begin{array}{r} A \sqsubseteq B \sqcap \exists r.C \\ B \sqcap \exists r.B \sqsubseteq C \sqcap D \\ C \sqsubseteq (\exists r.A) \sqcap B \\ (\exists r.\exists r.B) \sqcap D \sqsubseteq \exists r.(A \sqcap B) \end{array} \right\},$$

where A, B, C, D are concept names.

Use the classification procedure for $\mathcal{EL}$ to check whether the following subsumptions hold w.r.t. $\mathcal{T}$.

- $A \sqsubseteq \exists r.\exists r.A$
- $B \sqcap \exists r.A \sqsubseteq \exists r.C$

My Solution Question 5. Apply the classification procedure for $\mathcal{EL}$ to check if the given subsumptions hold with respect to the TBox $\mathcal{T}$.

Expanding Concepts: Start by expanding concepts in the TBox $\mathcal{T}$ using the rules of $\mathcal{EL}$:
- From $A \sqsubseteq B \sqcap \exists r.C$, we know that A is a subclass of B and also of $\exists r.C$.
- From $B \sqcap \exists r.B \sqsubseteq C \sqcap D$, we can infer that B and $\exists r.B$ are both subclasses of C and D.
- From $C \sqsubseteq (\exists r.A) \sqcap B$, we know that C is a subclass of $\exists r.A$ and B.
- From $(\exists r.\exists r.B) \sqcap D \sqsubseteq \exists r.(A \sqcap B)$, we can infer that $\exists r.\exists r.B$ and D are both subclasses of $\exists r.A$ and $\exists r.B$.

Checking Subsumptions:

Subsumption 1: $A \sqsubseteq \exists r.\exists r.A$
- Since $A \sqsubseteq B$ and from the TBox we have $B \sqcap \exists r.B \sqsubseteq C \sqcap D$, which implies $B \sqsubseteq C$.
- Also, $C \sqsubseteq (\exists r.A) \sqcap B$, meaning C is a subclass of $\exists r.A$.
- Therefore, since $A \sqsubseteq B$ and $B \sqsubseteq C$, and $C \sqsubseteq \exists r.A$, we can infer that $A \sqsubseteq \exists r.A$.
- However, to prove $A \sqsubseteq \exists r.\exists r.A$, we would need an additional step showing $\exists r.A \sqsubseteq \exists r.\exists r.A$, which is not directly derivable from the given TBox without further information about role properties (e.g., transitivity). Thus, this subsumption cannot be confirmed without additional axioms or assumptions about role properties.

Subsumption 2: $B \sqcap \exists r.A \sqsubseteq \exists r.C$
- We know $B \sqsubseteq C$ from the expansion above.
- Also, $\exists r.A \sqsubseteq C$ because $A \sqsubseteq B$ and $B \sqsubseteq C$, and by the property of existential quantifiers in $\mathcal{EL}$, if $A \sqsubseteq C$, then $\exists r.A \sqsubseteq C$.
- Therefore, since both components of $B \sqcap \exists r.A$ are subclasses of C, it follows that $B \sqcap \exists r.A \sqsubseteq C$.
- By the property of existential quantifiers, if $X \sqsubseteq Y$, then $\exists r.X \sqsubseteq \exists r.Y$. Thus, $\exists r.(B \sqcap A) \sqsubseteq \exists r.C$.
- Hence, this subsumption holds true w.r.t. $\mathcal{T}$.

Conclusion: The first subsumption, $A \sqsubseteq \exists r.\exists r.A$, **can not be confirmed** without additional role properties or axioms. The second subsumption, $B \sqcap \exists r.A \sqsubseteq \exists r.C$, **holds true w.r.t. $\mathcal{T}$.**

Question 6. Conservative Extension (2 marks)
Let $\mathcal{T}_1$ be an $\mathcal{EL}$ TBox, with C and D as $\mathcal{EL}$ concepts. Let us further consider $\mathcal{T}_2 := \mathcal{T}_1 \cup \{A \sqsubseteq C, D \sqsubseteq B\}$, wherein A and B are new concept names (as in Lemma 6.1).

- Show that $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$.

- Is this still the case after adding $A \sqsubseteq B$ to $\mathcal{T}_2$?

- What about adding $B \sqsubseteq A$?

My Solution **Question 6.** To show that $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, need to demonstrate that for any $\mathcal{EL}$ concept E and any model $\mathcal{I}$ of $\mathcal{T}_1$, if $\mathcal{I} \models C \sqsubseteq D$, then there exists a model $\mathcal{J}$ of $\mathcal{T}_2$ such that $\mathcal{J} \models C \sqsubseteq D$ and $\mathcal{J} \upharpoonright \mathcal{T}_1 = \mathcal{I}$.

Proof:

1. Assume $\mathcal{I} \models \mathcal{T}_1$, make the interpretation $\mathcal{I}$ satisfies all axioms in $\mathcal{T}_1$.

2. Extend $\mathcal{I}$ **to** $\mathcal{J}$: Construct an interpretation $\mathcal{J}$ such that:

- For every concept name A in $\mathcal{T}_2$ but not in $\mathcal{T}_1$, assign a non-empty subset $A^\mathcal{J}$ of the domain such that $A^\mathcal{J} \subseteq C^\mathcal{I}$ (since $A \sqsubseteq C$).
- For every concept name B in $\mathcal{T}_2$ but not in $\mathcal{T}_1$, assign a non-empty subset $B^\mathcal{J}$ of the domain such that $D^\mathcal{I} \subseteq B^\mathcal{J}$ (since $D \sqsubseteq B$).
- For all other concept names, interpret them as in $\mathcal{I}$.
- Ensure that the new interpretations for A and B do not violate any axioms in $\mathcal{T}_1$, which can be achieved by choosing appropriate subsets for $A^\mathcal{J}$ and $B^\mathcal{J}$.

3. Show $\mathcal{J} \models \mathcal{T}_2$:

- Since $\mathcal{I} \models \mathcal{T}_1$ and we have extended $\mathcal{I}$ to $\mathcal{J}$ without violating any axioms in $\mathcal{T}_1$, it follows that $\mathcal{J} \models \mathcal{T}_1$.
- Additionally, by construction, $A^\mathcal{J} \subseteq C^\mathcal{I}$ and $D^\mathcal{I} \subseteq B^\mathcal{J}$, which implies that $\mathcal{J} \models A \sqsubseteq C$ and $\mathcal{J} \models D \sqsubseteq B$. Therefore, $\mathcal{J} \models \mathcal{T}_2$.

4. Show $\mathcal{J} \upharpoonright \mathcal{T}_1 = \mathcal{I}$: By construction, for all concepts in $\mathcal{T}_1$, their interpretations in $\mathcal{J}$ are the same as in $\mathcal{I}$. Thus, the restriction of $\mathcal{J}$ to $\mathcal{T}_1$ is indeed $\mathcal{I}$.

Since I have shown that for any model $\mathcal{I}$ of $\mathcal{T}_1$, there exists a model $\mathcal{J}$ of $\mathcal{T}_2$ such that $\mathcal{J} \models C \sqsubseteq D$ and $\mathcal{J} \upharpoonright \mathcal{T}_1 = \mathcal{I}$, I can conclude that $\mathcal{T}_2$ **is a conservative extension of** $\mathcal{T}_1$.

2. **Yes.**

Adding $A \sqsubseteq B$ to $\mathcal{T}_2$, we can still find a model of $\mathcal{T}_2$ $\mathcal{I}_2$
s.t. $A^{\mathcal{I}_2} \sqsubseteq B^{\mathcal{I}_2}$, the relationship between $C^{\mathcal{I}_2}$ and $D^{\mathcal{I}_2}$ is not bounded, so it is still the case.

3. **No.**

Adding $B \sqsubseteq A$, then $B^{\mathcal{I}_2} \sqsubseteq A^{\mathcal{I}_2}$ and $A^{\mathcal{I}_2} \sqsubseteq C^{\mathcal{I}_2}$, $D^{\mathcal{I}_2} \sqsubseteq B^{\mathcal{I}_2}$. So $D^{\mathcal{I}_2} \sqsubseteq C^{\mathcal{I}_2}$, this will be wrong in some case.

Question 7. $\mathcal{EL}$ Extension (2 marks)

We consider the DL $\mathcal{EL}_{\mathsf{si}}$ extending $\mathcal{EL}$ by concept descriptions of the form $\exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d})$, where $\mathcal{I}$ is a finite interpretation and $\mathsf{d} \in \Delta^\mathcal{I}$. Their semantics is defined as follows.

$$(\exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d}))^\mathcal{J} := \{\mathsf{d}' \mid \mathsf{d}' \in \Delta^\mathcal{J} \text{ and } (\mathcal{I}, \mathsf{d}) \precsim (\mathcal{J}, \mathsf{d}')\}$$

Concept inclusions are then defined as usual.

- Show that each $\mathcal{EL}_{\mathsf{si}}$ concept description is equivalent to some concept descriptions of the form $\exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d})$.

- Show that $\mathcal{EL}_{\mathsf{si}}$ is more expressive than $\mathcal{EL}$.

- Show that checking subsumption in $\mathcal{EL}_{\mathsf{si}}$ without any TBox can be done in polynomial time.

My Solution **Question 7.** **1.** To show that each $\mathcal{EL}_{\mathsf{si}}$ concept description is equivalent to some concept descriptions of the form $\exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d})$, need to understand how the semantics of $\mathcal{EL}_{\mathsf{si}}$ extends the basic $\mathcal{EL}$ language and then demonstrate the equivalence.

Any $\mathcal{EL}_{\mathsf{si}}$ concept can be seen as a set of individuals in an interpretation that satisfy certain properties. These properties can be expressed using existential restrictions that refer to similar individuals in specific interpretations. For example, a concept C might be described as:

$$C \equiv \exists r.\top$$

where r is a role and $\top$ is the top concept. In $\mathcal{EL}_{\mathsf{si}}$, this can be rewritten using the similarity-based existential quantifier:

$$C \equiv \exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d})$$

where $\mathcal{I}$ is any interpretation containing an individual d such that $(\mathcal{I}, \mathsf{d}) \precsim (\mathcal{J}, \mathsf{d}')$ implies d' satisfies r.

To show that any $\mathcal{EL}_{\mathsf{si}}$ concept can be expressed using descriptions of the form $\exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d})$, consider any concept C. We can construct an equivalent description by finding or constructing an interpretation $\mathcal{I}$ and an individual $\mathsf{d} \in \Delta^{\mathcal{I}}$ such that the similarity condition captures the essence of C. This involves ensuring that for any interpretation $\mathcal{J}$ and any individual $\mathsf{d}' \in \Delta^{\mathcal{J}}$, the condition $(\mathcal{I}, \mathsf{d}) \precsim (\mathcal{J}, \mathsf{d}')$ holds if and only if d' satisfies C.

Consider a simple example concept $C = A \sqcap \exists r.B$, where A and B are atomic concepts, and r is a role. We can express this as:

$$C \equiv A \sqcap \exists r.B \equiv \exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d})$$

where $\mathcal{I}$ is an interpretation with an individual d such that $(\mathcal{I}, \mathsf{d}) \precsim (\mathcal{J}, \mathsf{d}')$ implies d' satisfies both A and $\exists r.B$.

By constructing suitable interpretations $\mathcal{I}$ and choosing appropriate individuals d, any $\mathcal{EL}_{\mathsf{si}}$ concept can be expressed using descriptions of the form $\exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d})$. **This demonstrates that each $\mathcal{EL}_{\mathsf{si}}$ concept description is equivalent to some concept descriptions of the form $\exists^{\mathsf{sim}}(\mathcal{I}, \mathbf{d})$.**

2. To show that $\mathcal{EL}_{\mathsf{si}}$ is more expressive than $\mathcal{EL}$, we need to demonstrate that there are concepts expressible in $\mathcal{EL}_{\mathsf{si}}$ that cannot be expressed in $\mathcal{EL}$. This can be done by providing an example of a concept or axiom that can be formulated in $\mathcal{EL}_{\mathsf{si}}$ but not in $\mathcal{EL}$.

$\mathcal{EL}$ does not inherently support the notion of similarity between individuals across different interpretations.

$\mathcal{EL}_{\mathsf{si}}$ extends $\mathcal{EL}$ by introducing the construct $\exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d})$, which allows for the expression of concepts based on similarity between individuals in different interpretations. This construct opens up new ways to define concepts that rely on specific instances and their properties in a given interpretation.

Consider a scenario where we want to express the concept of "similarity to a specific individual in another interpretation." Let's say we have an interpretation $\mathcal{I}_1$ with an individual a that has certain properties (e.g., $A(a)$ and $\exists r.B(a)$). We want to describe a concept in another interpretation $\mathcal{J}$ that captures all individuals similar to a in $\mathcal{I}_1$.

In $\mathcal{EL}_{\mathsf{si}}$, we can write this as:

$$C := \exists^{\mathsf{sim}}(\mathcal{I}_1, a)$$

This concept C in $\mathcal{J}$ includes all individuals d' such that $(\mathcal{I}_1, a) \precsim (\mathcal{J}, d')$. This kind of concept, which relies on the similarity relation between individuals from different interpretations, cannot be directly expressed in $\mathcal{EL}$ without the use of the similarity-based existential quantifier.

The ability to express concepts based on similarity to specific individuals in different interpretations is a feature unique to $\mathcal{EL}_{\mathsf{si}}$ and not available in standard $\mathcal{EL}$. **This demonstrates that $\mathcal{EL}_{\mathsf{si}}$ is more expressive than $\mathcal{EL}$**, as it can capture notions that are not expressible using only the constructs available in $\mathcal{EL}$.

3. To check subsumption in $\mathcal{EL}_{\mathsf{si}}$ without any TBox, we can use the following algorithm:

1. Initialize an empty set S to store the subsumed individuals.
2. For each individual $d' \in \Delta^{\mathcal{J}}$ in the target interpretation $\mathcal{J}$, do the following:

For each individual $d \in \Delta^{\mathcal{I}}$ in the source interpretation $\mathcal{I}$, check if $(\mathcal{I}, d) \precsim (\mathcal{J}, d')$. If true, add d' to the set

S.
3. Check if all individuals in $\Delta^{\mathcal{J}}$ are in the set S. If true, then $\exists^{\mathsf{sim}}(\mathcal{I}, d) \sqsubseteq c$ holds for some concept c in the target interpretation $\mathcal{J}$. Otherwise, it does not hold.

The time complexity of this algorithm is polynomial because the number of individuals in $\Delta^{\mathcal{I}}$ and $\Delta^{\mathcal{J}}$ is bounded by a constant factor, and the similarity check between two individuals can be done in constant time.

Question 8. $\mathcal{ALC}$**-Elim Algorithm**
Use the $\mathcal{ALC}$-Elim algorithm to decide satisfiability of:

- the concept name A w.r.t. $\mathcal{T} := \{A \sqsubseteq \exists r.A, \top \sqsubseteq A, \forall r.A \sqsubseteq \exists r.A\}$
- the concept description $\forall r.\forall r.\neg B$ w.r.t. $\mathcal{T} := \{\neg A \sqsubseteq B, A \sqsubseteq \neg B, \top \sqsubseteq \neg\forall r.A\}$

Give the constructed type sequence $\Gamma_0, \Gamma_1, \ldots$. In the case of satisfiability, also give the satisfying model constructed in the proof of Lemma 5.10.

My Solution Question 8. To use the $\mathcal{ALC}$-Elim algorithm to decide satisfiability, we need to construct a sequence of types $\Gamma_0, \Gamma_1, \ldots$ based on the given TBox $\mathcal{T}$ and the concept we want to check for satisfiability.
Case 1:

(1) $\Gamma_0 = \{\{(\neg A \sqcup \exists r.A), A, \exists r.A, (\exists r.\neg A \sqcup \exists r.A), ((\neg A \sqcup \exists r.A) \sqcap A), (((\neg A \sqcup \exists r.A) \sqcap A) \sqcap (\exists r.\neg A \sqcup \exists r.A))\},$
$\{(\neg A \sqcup \exists r.A), \exists r.\neg A, A, \exists r.A, (\exists r.\neg A \sqcup \exists r.A), ((\neg A \sqcup \exists r.A) \sqcap A), (((\neg A \sqcup \exists r.A) \sqcap A) \sqcap (\exists r.\neg A \sqcup \exists r.A))\}\}$

$\Gamma_1 = \{\{(\neg A \sqcup \exists r.A), A, \exists r.A, (\exists r.\neg A \sqcup \exists r.A), ((\neg A \sqcup \exists r.A) \sqcap A),$
$(((\neg A \sqcup \exists r.A) \sqcap A) \sqcap (\exists r.\neg A \sqcup \exists r.A))\}\}$

$\Gamma_2 = \{\{(\neg A \sqcup \exists r.A), A, \exists r.A, (\exists r.\neg A \sqcup \exists r.A), ((\neg A \sqcup \exists r.A) \sqcap A), (((\neg A \sqcup \exists r.A) \sqcap (\exists r.\neg A \sqcup \exists r.A))\}\}$.

Since we can form a model where all individuals are in the extension of A, and the role r maps every individual to itself (which satisfies the conditions in $\mathcal{T}$), the concept name A is satisfiable w.r.t. the given TBox $\mathcal{T}$.

Case 2:

However, we have a contradiction in Γ_3 since it contains both B and $\neg B$. This means that the concept description $\forall r.\forall r.\neg B$ is unsatisfiable w.r.t. the given TBox $\mathcal{T}$.

a) Set $C \equiv \forall r.\forall r.\neg B$.

$J_0 = \{\{\neg A \sqcup \neg B, \neg A, (\neg C \sqcup \forall r.\forall r.\neg B), \forall r.\forall r.\neg B, \exists r.\neg A, ((\neg C \sqcup \forall r.\forall r.\neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B)) \sqcap \exists r.\neg A), B, ((\neg C \sqcup \forall r.\forall r.\neg B) \sqcap (A \sqcup B)),$

$(A \sqcup B), (((\neg C \sqcup \forall r.\forall r.\neg B \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B))\}, \{(\neg A \sqcup \neg B), (\neg C \sqcup \forall r.\forall r.\neg B), A, \exists r.\neg A, \forall r.\forall r.\neg B, (((\neg C \sqcup \forall r.\forall r.\neg B) \sqcap (A \sqcup B)) \sqcap (A \sqcup B))$

$\sqcap (\neg A \sqcup \neg B))\}, \{(\neg A \sqcup \neg B), (\neg C \sqcup \forall r.\forall r.B), A, \exists r.\neg A, ((((\neg C \sqcup \forall r.\forall r.\neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B))\}, \{(\neg A \sqcup \neg B), (\neg C \sqcup \forall r.\forall r.\neg B) \sqcap$

$(A \sqcup B)) \sqcap (\neg A \sqcup \neg B)) \sqcap (\exists r.\neg A)), \neg B, \neg C, ((\neg C \sqcup \forall r.\forall r.\neg B) \sqcap (A \sqcup B)), (A \sqcup B), (((\neg C \sqcup \forall r.\forall r.\neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B)\}.$

$\{(\neg A \sqcup \neg B), \neg A, (\neg C \sqcup \forall r.\forall r.\neg B), \forall r.\forall r.\neg B, C, (\exists r.\neg A), (((\neg C \sqcup \forall r.\forall r.\neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B)))\}, \{(\neg A \sqcup \neg B,$

$\neg A, (\neg C \sqcup \forall r.\forall r.\neg B), \forall r.\forall r.\neg B, \exists r.\neg A, (((\neg C \sqcup \forall r.\forall r.\neg B), (A \sqcup B)) \sqcap (\neg A \sqcup \neg B) \sqcap \exists r.\neg A), B, \neg C, ((\neg C \sqcup \forall r.\forall r.\neg B],$

$\neg A, (\neg C \sqcup \forall r.\forall r.\neg B), \forall r.\forall r.\neg B, \exists r.\neg A, (((\neg C \sqcup \forall r.\forall r.\neg B), (A \sqcup B)) \sqcap (\neg A \sqcup \neg B) \sqcap \exists r.\neg A), B, \neg C, ((\neg C \sqcup \forall r.\forall r.\neg B],\}$

$\{(\neg A \sqcup \neg B), \neg A, (\neg C \sqcup \forall r.\forall r.\neg B), \forall r.\forall r.\neg B, C, (\exists r.\neg A), (((\neg C \sqcup \forall r.\forall r.\neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B)))\}, \{(\neg A \sqcup \neg B,$

$\neg A, (\neg C \sqcup \forall r.\forall r.\neg B), \forall r.\forall r.\neg B, \exists r.\neg A, (((\neg C \sqcup \forall r.\forall r.\neg B), (A \sqcup B)) \sqcap (\neg A \sqcup \neg B) \sqcap \exists r.\neg A), B, \neg C, ((\neg C \sqcup \forall r.\forall r.\neg B],$

$\neg A, (\neg C \sqcup \forall r.\forall r.\neg B), \forall r.\forall r.\neg B, \exists r.\neg A, (((\neg C \sqcup \forall r.\forall r.\neg B), (A \sqcup B)) \sqcap (\neg A \sqcup \neg B) \sqcap \exists r.\neg A), B, \neg C, ((\neg C \sqcup \forall r.\forall r.\neg B],\}$

$\{(\neg A \sqcup \neg B), \neg A, (\neg C \sqcup \forall r.\forall r.\neg B, \forall r.\forall r.\neg B, \exists r.\neg A, ((((\neg C \sqcup \forall r.\forall r.\neg B \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup B))\}, \{(\neg A \sqcup \neg B), (\neg C \sqcup \forall r.\forall r.\neg B), A, C, \forall r.\forall r.B, \exists r.A,$

$(((\neg C \sqcup \forall r.\forall r.\neg B \sqcap (A \sqcup B) \sqcap (\neg A \sqcup \neg B)), \neg B, \forall r.\neg B, \neg C, ((\neg C \sqcup (\forall r.\forall r.B) \sqcap (A \sqcup B), (((\neg C \sqcup \forall r.\forall r.\neg B) \sqcap (\neg A \sqcup \neg B))\}.$

$\{(\neg A \sqcup \neg B), \neg A, (\neg C \sqcup \forall r.\forall r.\neg B, \forall r.\forall r.\neg B, \exists r.\neg A, ((((\neg C \sqcup \forall r.\forall r.\neg B \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup B))\}, \{(\neg A \sqcup \neg B), (\neg C \sqcup \forall r.\forall r.\neg B), A, C, \forall r.\forall r.B, \exists r.A,$

$(((\neg C \sqcup \forall r.\forall r.\neg B \sqcap (A \sqcup B) \sqcap (\neg A \sqcup \neg B)), \neg B, \forall r.\neg B, \neg C, ((\neg C \sqcup (\forall r.\forall r.B) \sqcap (A \sqcup B), (((\neg C \sqcup \forall r.\forall r.\neg B) \sqcap (\neg A \sqcup \neg B))\}\}.$

$J_1=\{\{(\neg A\sqcup\neg B).\neg A.(\neg C\sqcup\forall r.\forall r.\neg B).\forall r.\forall r\neg B.\exists r.\neg A,\neg B.((\neg C\sqcup\forall r.\forall r.\neg B).\exists r\neg A\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\sqcap\exists r.\neg A)).$

$\exists r.\neg A.(((\neg C\sqcup\forall r.\forall r.\neg B)\sqcap(A\sqcup B))).A\sqcup B,(((\neg C\sqcup\forall r.\forall r\neg B)\sqcap(A\sqcup B))\sqcap(A\sqcup\neg B))\sqcap(\neg A\sqcup B))\}.\{(\neg A\sqcup\neg B).\neg A.$

$(\neg C\sqcup\forall r.\forall r\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup B).\sqcap\exists r.\neg A),\neg B.((\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B).A\sqcup B),(\neg C\sqcup\forall r.\forall r\neg B)\sqcap(\neg A\sqcup\neg B).$

$\{(\neg A\sqcup\neg B).(\neg C\sqcup\forall r.\forall r.\neg B).A.\exists r.\neg A.\sqcup\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\sqcap(\exists r.\neg A)).\neg B.\neg C.A\sqcup B.$

$(((\neg C\sqcup\forall r\forall r\neg B)\forall r\forall r\neg B,\neg C.\exists r\neg A(((\neg C\sqcup\forall r\forall r.\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\}.\{(\neg A\sqcup\neg B).\neg A.(\neg C\sqcup\forall r.\forall r\neg B)\forall r\forall r.\neg B.C.\exists r\neg A.$

$((((\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\sqcap(\exists r\neg A)).B,((\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B)).A\sqcup B,(((\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\}.$

$\{(\neg A\sqcup\neg B).(\neg C\sqcup\forall r.\forall r.\neg B).A.\exists r.\neg A.\sqcup\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\sqcap(\exists r.\neg A)).\neg B.\neg C.A\sqcup B.$

$(((\neg C\sqcup\forall r\forall r\neg B)\forall r\forall r\neg B,\neg C.\exists r\neg A(((\neg C\sqcup\forall r\forall r.\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\}.\{(\neg A\sqcup\neg B).\neg A.(\neg C\sqcup\forall r.\forall r\neg B)\forall r\forall r.\neg B.C.\exists r\neg A.$

$((((\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\sqcap(\exists r\neg A)).B,((\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B)).A\sqcup B,(((\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\}.$

$\{(\neg A\sqcup\neg B).(\neg C\sqcup\forall r.\forall r.\neg B).A.\exists r.\neg A.\sqcup\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\sqcap(\exists r.\neg A)).\neg B.\neg C.A\sqcup B.$

$(((\neg C\sqcup\forall r\forall r\neg B)\forall r\forall r\neg B,\neg C.\exists r\neg A(((\neg C\sqcup\forall r\forall r.\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\}.\{(\neg A\sqcup\neg B).\neg A.(\neg C\sqcup\forall r.\forall r\neg B)\forall r\forall r.\neg B.C.\exists r\neg A.$

$((((\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\sqcap(\exists r\neg A)).B,((\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B)).A\sqcup B,(((\neg C\sqcup\forall r\forall r\neg B)\sqcap(A\sqcup B))\sqcap(\neg A\sqcup\neg B))\}.$

$J_2 = \{\{(\neg A \sqcup \neg B), \neg A, (\neg C \sqcup \forall r \forall r \neg B), (\exists r \neg A), (((\neg C \sqcup \forall r \forall r \neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B) \sqcap (\exists r. \neg A)), B,$

$\neg C, ((\neg C \sqcup \forall r \forall r. \neg B) \sqcap (A \sqcup B)), A \sqcup B, (((\neg C \sqcup \forall r \forall r \neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B))\}, \{(\neg A \sqcup \neg B), \neg C \sqcup \forall r \forall r \neg B,$

$A, \exists r \neg A, ((((\neg C \sqcup \forall r \forall r \neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B)) \sqcap (\exists r. \neg A)), \neg B, \neg C, ((\neg C \sqcup \forall r \forall r \neg B) \sqcap (A \sqcup B)),$

$A \sqcup B, (((\neg C \sqcup \forall r \forall r. \neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B)\}\}.$

$J_3 = \{\{(\neg A \sqcup \neg B), \neg A, (\neg C \sqcup \forall r \forall r. \neg B), \exists r. \neg A, ((((\neg C \sqcup \forall r \forall r \neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B)) \sqcap (\exists r. \neg A)),$

$B, \neg C, ((\neg C \sqcup \forall r \forall r \neg B) \sqcap (A \sqcup B)), A \sqcup B, (((\neg C \sqcup \forall r \forall r. \neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B))\}, \{(\neg A \sqcup \neg B),$

$(\neg C \sqcup \forall r \forall r \neg B), A, \exists r. \neg A, ((((\neg C \sqcup \forall r \forall r. \neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B)) \sqcap (\exists r. \neg A)), \neg B, \neg C, A \sqcup B,$

$((\neg C \sqcup \forall r \forall r. \neg B) \sqcap (A \sqcup B)), (((\neg C \sqcup \forall r \forall r \neg B) \sqcap (A \sqcup B)) \sqcap (\neg A \sqcup \neg B))\}\}.$

In conclusion:
- The concept name A is satisfiable w.r.t. $\mathcal{T}$.
- The concept description $\forall r.\forall r.\neg B$ is unsatisfiable w.r.t. $\mathcal{T}$.

Question 9 (with 1 bonus mark). Simulation

We consider simulations, which are "one-sided" variants of bisimulations. Given interpretations $\mathcal{I}$ and $\mathcal{J}$, the relation $\sigma \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a simulation from $\mathcal{I}$ to $\mathcal{J}$ if

- whenever $\mathrm{d}\ \sigma\ \mathrm{d}'$ and $\mathrm{d} \in A^{\mathcal{I}}$, then $\mathrm{d}' \in A^{\mathcal{J}}$, for all $\mathrm{d} \in \Delta^{\mathcal{I}}$, $\mathrm{d}' \in \Delta^{\mathcal{J}}$, and $A \in \mathbb{C}$;
- whenever $\mathrm{d}\ \sigma\ \mathrm{d}'$ and $(\mathrm{d}, \mathrm{e}) \in r^{\mathcal{I}}$, then there exists an $\mathrm{e}' \in \Delta^{\mathcal{J}}$ such that $\mathrm{e}\ \sigma\ \mathrm{e}'$ and $(\mathrm{d}', \mathrm{e}') \in r^{\mathcal{J}}$, for all $\mathrm{d}, \mathrm{e} \in \Delta^{\mathcal{I}}$, $\mathrm{d}' \in \Delta^{\mathcal{J}}$, and $r \in \mathbb{R}$.

We write $(\mathcal{I}, \mathrm{d}) \precsim (\mathcal{J}, \mathrm{d}')$ if there is a simulation σ from $\mathcal{I}$ to $\mathcal{J}$ such that $\mathrm{d}\ \sigma\ \mathrm{d}'$.

- Show that $(\mathcal{I}, \mathrm{d}) \sim (\mathcal{J}, \mathrm{d}')$ implies $(\mathcal{I}, \mathrm{d}) \precsim (\mathcal{J}, \mathrm{d}')$ and $(\mathcal{J}, \mathrm{d}') \precsim (\mathcal{I}, \mathrm{d})$.
- Is the converse of the implication above also true?
- Show that, if $(\mathcal{I}, \mathrm{d}) \precsim (\mathcal{J}, \mathrm{d}')$, then $\mathrm{d} \in C^{\mathcal{I}}$ implies $\mathrm{d}' \in C^{\mathcal{J}}$ for all $\mathcal{EL}$ concept descriptions C.
- Which of the constructors disjunction, negation, or universal restriction can be added to $\mathcal{EL}$ without losing the property above?
- Show that $\mathcal{ALC}$ is more expressive than $\mathcal{EL}$.

My Solution Question 9. 1. To show that $(\mathcal{I}, \mathrm{d}) \sim (\mathcal{J}, \mathrm{d}')$ implies $(\mathcal{I}, \mathrm{d}) \precsim (\mathcal{J}, \mathrm{d}')$ and $(\mathcal{J}, \mathrm{d}') \precsim (\mathcal{I}, \mathrm{d})$:

- If $(\mathcal{I}, \mathrm{d}) \sim (\mathcal{J}, \mathrm{d}')$, it means that there is a bisimulation relation σ between $\Delta^{\mathcal{I}}$ and $\Delta^{\mathcal{J}}$ such that $\mathrm{d}\ \sigma\ \mathrm{d}'$.
- By definition of bisimulation, all conditions for simulation are met in both directions (from $\mathcal{I}$ to $\mathcal{J}$ and from

$\mathcal{J}$ to $\mathcal{I}$).
- Therefore, σ is also a simulation from $\mathcal{I}$ to $\mathcal{J}$, and conversely, from $\mathcal{J}$ to $\mathcal{I}$.
- Thus, $(\mathcal{I}, \mathrm{d}) \precsim (\mathcal{J}, \mathrm{d}')$ and $(\mathcal{J}, \mathrm{d}') \precsim (\mathcal{I}, \mathrm{d})$ hold by definition of simulation.

2. **No.** The converse would state: if $(\mathcal{I}, \mathrm{d}) \precsim (\mathcal{J}, \mathrm{d}')$ and $(\mathcal{J}, \mathrm{d}') \precsim (\mathcal{I}, \mathrm{d})$, then $(\mathcal{I}, \mathrm{d}) \sim (\mathcal{J}, \mathrm{d}')$.
The converse of the implication above is not necessarily true. A simulation relation does not ensure that all related elements have the same properties as in the original interpretation. It only ensures that if an element has a certain property in one interpretation, the corresponding element in the other interpretation will also have that property.

3. A simulation σ from $\mathcal{I}$ to $\mathcal{J}$ is a relation such that:
-For every $\mathrm{d} \in \Delta^{\mathcal{I}}$ and $\mathrm{d}' \in \Delta^{\mathcal{J}}$ with $\mathrm{d}\ \sigma\ \mathrm{d}'$, if $\mathrm{d} \in A^{\mathcal{I}}$ for some attribute A, then $\mathrm{d}' \in A^{\mathcal{J}}$.
-For every $\mathrm{d}, \mathrm{e} \in \Delta^{\mathcal{I}}$, $\mathrm{d}' \in \Delta^{\mathcal{J}}$ with $\mathrm{d}\ \sigma\ \mathrm{d}'$ and $(\mathrm{d}, \mathrm{e}) \in r^{\mathcal{I}}$ for some role r, there exists an $\mathrm{e}' \in \Delta^{\mathcal{J}}$ such that $(\mathrm{d}', \mathrm{e}') \in r^{\mathcal{J}}$ and $\mathrm{e}\ \sigma\ \mathrm{e}'$.

Given that $(\mathcal{I}, \mathrm{d}) \precsim (\mathcal{J}, \mathrm{d}')$, we have a simulation σ from $\mathcal{I}$ to $\mathcal{J}$ such that $\mathrm{d}\ \sigma\ \mathrm{d}'$.

Now, let C be an $\mathcal{EL}$ concept description. The satisfaction of C in an interpretation $\mathcal{I}$ for a domain element d is determined by the following conditions:
- **Atomic concept:** $\mathrm{d} \in A^{\mathcal{I}}$ if A is an atomic concept and d satisfies A.
- **Inverse roles:** $(\mathrm{d}, \mathrm{e}) \in r^{-}$ if $(\mathrm{e}, \mathrm{d}) \in r$.
- **Role restrictions:** $(\mathrm{d}, \mathrm{e}) \in r$ if r is a role restriction and e satisfies the concept described by r.
- **Conjunctions:** d satisfies a conjunction $C_1 \sqcap C_2$ if it satisfies both C_1 and C_2.
- **Disjunctions:** d satisfies a disjunction $C_1 \sqcup C_2$ if it satisfies either C_1 or C_2.
- **Universal restrictions:** For every e such that $(\mathrm{d}, \mathrm{e}) \in r$, if e satisfies a concept D, then d satisfies the universal restriction $\forall r.D$.

Since σ simulates the interpretations, it ensures that if $\mathrm{d} \in C^{\mathcal{I}}$, then the corresponding elements related by σ will also satisfy the concept descriptions in $\mathcal{J}$. Therefore, if $\mathrm{d} \in C^{\mathcal{I}}$, then by the simulation relation, we have that $\mathrm{d}' \in C^{\mathcal{J}}$. This holds for all $\mathcal{EL}$ concept descriptions C. **Proof completed.**

4. The constructor **disjunction** can be added to $\mathcal{EL}$ without losing the property above. Disjunction allows for multiple possibilities to be true at the same time, which does not affect the simulation relation.

5. To show that $\mathcal{ALC}$ is more expressive than $\mathcal{EL}$, we need to demonstrate that there is a concept that can be expressed in $\mathcal{ALC}$ but not in $\mathcal{EL}$.

$\mathcal{ALC}$ extends $\mathcal{EL}$ with the constructor of negation, which allows for the explicit formation of complements of concepts. This additional constructor enables $\mathcal{ALC}$ to express more complex concepts.

Consider the concept of "not having a certain property," which can be easily expressed in $\mathcal{ALC}$ using negation but cannot be directly expressed in $\mathcal{EL}$. For instance, let's take a simple concept C defined by a single attribute p in $\mathcal{EL}$, i.e., $C \equiv p$. In $\mathcal{EL}$, there is no direct way to express the complement of C, which would be "not having property p" or "individuals that do not have property p."

In contrast, $\mathcal{ALC}$ allows us to form the complement of C using negation as $\neg C \equiv \neg p$. The ability to express such complements shows that $\mathcal{ALC}$ can describe concepts that are beyond the reach of $\mathcal{EL}$, **hence proving that $\mathcal{ALC}$ is more expressive than $\mathcal{EL}$.**