Question 1

For all $x, y \in \mathcal{X}$, $0 \leq \theta \leq 1$ $(\mathcal{X} \subseteq \mathbb{R}^n)$

$$\|\theta x + (1-\theta)y\| \leq \|\theta x\| + \|(1-\theta)y\|$$

(Triangle Inequality)

and:

$$\|\theta x\| \leq \|\theta\| \times \|x\| = \theta \times \|x\| \quad (\theta \geq 0)$$

$$\|(1-\theta)y\| \leq \|1-\theta\| \times \|y\| = (1-\theta) \times \|y\| \quad (\theta \leq 1)$$

So, $\|\theta x + (1-\theta)y\| \leq \theta\|x\| + (1-\theta)\|y\|$

which means that the norm function is a convex function on $\mathbb{R}^n$

Question 2

$$\mathcal{L}(\omega, b, \xi, \mu_i, \nu_i) = \frac{1}{2}\|\omega\|^2 + C\sum_{i=1}^{m}\xi_i - \sum_{i=1}^{m}\mu_i\left[y_i(\omega^T\phi(x_i)+b) - 1 + \xi_i\right] - \sum_{i=1}^{m}\nu_i\xi_i$$

Question 3

$$\frac{\partial \mathcal{L}}{\partial \omega} = \omega - \sum_{i=1}^{m}\mu_i y_i \phi(x_i)$$

$$\frac{\partial \mathcal{L}}{\partial b} = \sum_{i=1}^{m} - y_i \mu_i$$

$$\frac{\partial \mathcal{L}}{\partial \xi_i} = C - (\nu_i + \mu_i)$$

KKT conditions:

$$-[y_i(w^T\phi(x_i)+b] \le 0$$

$$-\xi_i \le 0$$

$$\mu_i \ge 0$$

$$v_i \ge 0$$

$$\frac{\partial L}{\partial w^*} = \frac{\partial L}{\partial b^*} = \frac{\partial L}{\partial \xi_i^*} = 0$$

Question 4

The dual form is:

$$g(\mu_i, v_i) = \sum_{i=1}^{m} \mu_i - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \mu_i \mu_j y_i y_j \phi(x_i)^T \phi(x_j)$$

s.t. $\mu_i \ge 0$, $v_i \ge 0$

Question 5:

From the defination of convex set, we know:

$$x' = \frac{\theta_1}{\theta_1+\theta_2} x_1 + \frac{\theta_2}{\theta_1+\theta_2} x_2 \in C$$

Furthermore,

$$(1-\theta_3)x' + \theta_3 x_3$$

and $\theta_1+\theta_2+\theta_3=1$, so $1-\theta_3 = \theta_1+\theta_2$.

$$(1-\theta_3)x' + \theta_3 x_3 = (\theta_1+\theta_2)\frac{\theta_1}{\theta_1+\theta_2} x_1 + (\theta_1+\theta_2)\frac{\theta_2}{\theta_1+\theta_2} x_2 + \theta_3 x_3$$

$$= \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3$$

So, $\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \in C$