Notes on the Change of Variables in KdV-Burgers

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September 29, 2017

1 Changes of Variables

The KdV-Burgers equation is:

$$u_t + uu_x = \epsilon^2 u_{xxx} + \alpha u_{xx}. \tag{1}$$

We will transform this to a nondimensional form. Choose a unit velocity U. Then define nondimensional x' and t' as:

$$x' = \frac{\sqrt{U}}{\epsilon} x \tag{2}$$

$$t' = \frac{U^{3/2}}{\epsilon} t. \tag{3}$$

Then,

$$f_x = \frac{\partial f}{\partial x'} \frac{dx'}{dx} = f_{x'} \frac{\sqrt{U}}{\epsilon} \tag{4}$$

$$f_t = \frac{\partial f}{\partial t'} \frac{dt'}{dt} = f_{t'} \frac{U^{3/2}}{\epsilon}. \tag{5}$$

Plugging these into the above, we find:

$$\begin{aligned}
u_t &= -\frac{1}{2}(u^2)_x + \epsilon^2 u_{xxx} + \alpha u_{xx} \\
\frac{U^{3/2}}{\epsilon} u_{t'} &= -\frac{1}{2}(u^2)_{x'} \frac{\sqrt{U}}{\epsilon} + \epsilon^2 u_{x'x'x'} \frac{U^{3/2}}{\epsilon^3} + \alpha u_{x'x'} \frac{U}{\epsilon^2} \\
\frac{\epsilon}{U^{5/2}} \frac{U^{3/2}}{\epsilon} u_{t'} &= -\frac{1}{2} \frac{\epsilon}{U^{5/2}} (u^2)_{x'} \frac{\sqrt{U}}{\epsilon} + \frac{\epsilon}{U^{5/2}} \epsilon^2 u_{x'x'x'} \frac{U^{3/2}}{\epsilon^3} + \alpha \frac{\epsilon}{U^{5/2}} u_{x'x'} \frac{U}{\epsilon^2} \\
u'_{t'} &= -\frac{1}{2}(u'^2)_{x'} + u'_{x'x'x'} + \frac{\alpha}{\epsilon\sqrt{U}} u'_{x'x'}.
\end{aligned}$$

If we define $R = \frac{\epsilon\sqrt{U}}{\alpha}$ and drop the primes, we are left with:

$$u_t = -\frac{1}{2}(u^2)_x + u_{xxx} + \frac{1}{R} u_{xx}. \tag{6}$$

This is the nondimensionalized version of KdV-Burgers.

2 Effect on Fourier Transforms

In these changed variables, the domain is no longer $[0, 2\pi]$. Instead, it is $[0, L]$ where $L = 2\pi\frac{\sqrt{U}}{\epsilon}$. Let's see how we can factor this in to our FFT routine. The current FFT routine defines:

$$f(x) = \sum_k f_k e^{ikx}, \tag{7}$$

which means

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\,\mathrm{d}x. \tag{8}$$

This quantity (8) is what is calculated by the FFT algorithm in Matlab. We want to use new basis functions that are orthogonal in our new units. They are:

$$\phi_k(x') = e^{ik\frac{2\pi}{L}x'}. \tag{9}$$

Let's demonstrate what happens when we compute the inner product of one of these basis functions with another on our new domain:

$$\int_0^L \phi_k(x')\phi_j^*(x')\,\mathrm{d}x' = \int_0^L e^{i(k-j)\frac{2\pi}{L}x'}\,\mathrm{d}x'.$$

We'll now do a subsitution $x = \frac{2\pi}{L}x'$, $\mathrm{d}x' = \frac{L}{2\pi}\mathrm{d}x$, $x(0) = 0$, $x(L) = 2\pi$:

$$\int_0^L e^{i(k-j)\frac{2\pi}{L}x'}\,\mathrm{d}x' = \int_0^{2\pi} e^{i(k-j)x}\frac{L}{2\pi}\,\mathrm{d}x = \begin{cases} L & k = j \\ 0 & k \neq j. \end{cases} \tag{10}$$

Our expansion of $f(x)$ in the new basis would be

$$f(x') = \sum_k f_k e^{ik\frac{2\pi}{L}x'}. \tag{11}$$

Now, the FFT in the new units would be defined as:

$$\int_0^L f(x')e^{-ij\frac{2\pi}{L}x'}\mathrm{d}x' = \sum_k f_k \int_0^L e^{i(k-j)\frac{2\pi}{L}x'}\,\mathrm{d}x'. \tag{12}$$

Using the results from (10), we find:

$$\sum_k f_k \int_0^L e^{i(k-j)\frac{2\pi}{L}x'}\,\mathrm{d}x' = \sum_k L f_k \delta_{kj} = L f_j. \tag{13}$$

Therefore, we have shown:

$$f_k = \frac{1}{L}\int_0^L f(x')e^{-ik\frac{2\pi}{L}x'}\,\mathrm{d}x. \tag{14}$$

Using the same substitution as above, we find:

$$\begin{aligned}
f_k &= \frac{1}{L}\int_0^L f(x')e^{-ik\frac{2\pi}{L}x'}\,\mathrm{d}x' \\
&= \frac{1}{L}\int_0^{2\pi} f\left(\frac{L}{2\pi}x\right)e^{-ikx}\frac{L}{2\pi}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} f\left(\frac{L}{2\pi}x\right)e^{-ikx}\,\mathrm{d}x.
\end{aligned}$$

This is simply the standard FFT of a function evaluated at the points $\frac{L}{2\pi}x$, which comprises the points x'. Thus, no modifications of the FFT need to be made. In order to check if any modifications to the IFFT need to be made, we plug this result into the original sum:

$$\sum_k f_k e^{ik\frac{2\pi}{L}x'} = \sum_k \left(\frac{1}{2\pi}\int_0^{2\pi} f\left(\frac{L}{2\pi}x\right)e^{-ikx}\,\mathrm{d}x\right)e^{ik\frac{L}{2\pi}x'}.$$

This is the IFFT of a function defined at the points $\frac{L}{2\pi}x$, which are the points x'. Thus, the IFFT also does not need any modification due to the change of variables.

3 Fourier Transform of KdV-Burgers in New Variables

We now take the nondimensionalized version of KdV-Burgers, expand each u as $u(x,t) = \sum_k u_k(t)e^{ik\frac{2\pi}{L}x}$:

$$\begin{aligned}
u_t &= -\frac{1}{2}(u^2)_x + u_{xxx} + \frac{1}{R}u_{xx} \\
\frac{\partial}{\partial t}\left[\sum_k u_k e^{ik\frac{2\pi}{L}x}\right] &= -\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\sum_p u_p e^{ip\frac{2\pi}{L}x}\right)\left(\sum_q u_q e^{iq\frac{2\pi}{L}x}\right)\right] + \frac{\partial^3}{\partial x^3}\left[\sum_k u_k e^{ik\frac{2\pi}{L}x}\right] \\
&\quad + \frac{1}{R}\frac{\partial^2}{\partial x^2}\left[\sum_k u_k e^{ik\frac{2\pi}{L}x}\right] \\
\sum_k \frac{du_k}{dt}e^{ik\frac{2\pi}{L}x} &= -\frac{1}{2}\frac{\partial}{\partial x}\left[\sum_k\sum_{p+q=k} u_p u_q e^{ik\frac{2\pi}{L}x}\right] - \sum_k i\left(\frac{2\pi k}{L}\right)^3 u_k e^{ik\frac{2\pi}{L}x} \\
&\quad - \frac{1}{R}\sum_k \left(\frac{2\pi k}{L}\right)^2 u_k e^{ik\frac{2\pi}{L}x} \\
\sum_k \frac{du_k}{dt}e^{ik\frac{2\pi}{L}x} &= -\frac{1}{2}\sum_k\sum_{p+q=k} i\left(\frac{2\pi k}{L}\right) u_p u_q e^{ik\frac{2\pi}{L}x} - \sum_k i\left(\frac{2\pi k}{L}\right)^3 u_k e^{ik\frac{2\pi}{L}x} \\
&\quad - \frac{1}{R}\sum_k \left(\frac{2\pi k}{L}\right)^2 u_k e^{ik\frac{2\pi}{L}x}.
\end{aligned}$$

Now we compute the inner product of each side with $\phi_j = e^{ij\frac{2\pi}{L}x}$

$$\begin{aligned}
\sum_k \frac{du_k}{dt} \int_0^L e^{ik\frac{2\pi}{L}x} e^{-ij\frac{2\pi}{j}x}\,\mathrm{d}x =& -\frac{1}{2}\sum_k \sum_{p+q=k} i\left(\frac{2\pi k}{L}\right) u_p u_q \int e^{ik\frac{2\pi}{L}x} e^{-ij\frac{2\pi}{L}x}\,\mathrm{d}x \\
& - \sum_k i\left(\frac{2\pi k}{L}\right)^3 u_k \int_0^L e^{ik\frac{2\pi}{L}x} e^{-ij\frac{2\pi}{L}x}\,\mathrm{d}x \\
& - \frac{1}{R}\sum_k \left(\frac{2\pi k}{L}\right)^2 u_k \int_0^L e^{ik\frac{2\pi}{L}x} e^{-ij\frac{2\pi}{L}x}\,\mathrm{d}x \\
L\frac{du_j}{dt} =& -L\frac{i\hat{j}}{2}\sum_{p+q=j} u_p u_q - Li\hat{j}^3 u_j - L\frac{\hat{j}^2}{R}u_j \\
\frac{du_k}{dt} =& \frac{i\hat{k}}{2}\sum_{p+q=k} u_p u_q - i\hat{k}^3 u_k - \frac{\hat{k}^2}{R}u_k,
\end{aligned}$$

where $\hat{k} = \frac{2\pi k}{L}$. From this, it appears I need to redefine $k \to \hat{k}$, and make no other changes. Importantly, the FFT and IFFT algorithms will work correctly without any changes.