Converse Theorems for Certificates of Safety and Stability

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Abstract—**Motivated by the key role of control barrier functions (CBFs) in assessing safety and enabling the synthesis of safe controllers in nonlinear control systems, this paper presents a suite of converse results on CBFs. Given any safe set, we first identify a set of general sufficient conditions which guarantee the existence of a CBF. Our technical analysis also enables us to define an extended notion of CBF which is always guaranteed to exist if the set is safe. We next turn our attention to the problem of joint safety and stability, and give conditions under which the notions of control Lyapunov-barrier function (CLBF) and compatible control Lyapunov function (CLF) and CBF pair are guaranteed to exist. Finally, we identify conditions under which a CLBF and a compatible CLF-CBF pair can be constructed from a non-compatible CLF-CBF pair. Throughout the paper, we intersperse different examples and counterexamples to motivate our results and position them within the state of the art.**

I. Introduction

Safety-critical control is a fundamental problem in modern cyber-physical systems, with a rich set of applications ranging from autonomous driving and power systems to policy design for mitigation of epidemic spreading. Safety refers to the ability to ensure by design that the evolution of the dynamics stays within a desired set of safe states. A relatively recent but promising tool to deal with such safety specifications are control barrier functions (CBFs). An advantage of CBFs is that they do not require computing the system's reachable set, which can be computationally expensive. However, constructing CBFs is often challenging. In fact, the problem of whether a general safe set admits a CBF has received limited attention. Often, CBFs are combined with control Lyapunov functions (CLFs) to yield controllers with safety and stability guarantees. The problem of when such CBFs and CLFs coexist and can be combined has also not been thoroughly studied. Addressing these gaps is the focus of this work.

Literature Review: Control barrier functions (CBFs) [1], [2] aim to render a given set forward invariant. Despite their popularity and success in a variety of different applications [3]–[6], various fundamental questions about their properties still remain open. A key question is whether CBFs are guaranteed to exist for any safe set. Several works [2], [7]–[11] have studied such converse results, particularly for systems without inputs, for which CBFs are referred to as *barrier functions*. The main result in [8] applies to continuously differentiable vector fields without inputs and requires the existence of a continuously differentiable function that is strictly increasing along the solutions of the vector field. However, such assumptions are restrictive in many cases of interest, such as systems that admit limit cycles. The main result in [9] applies to smooth vector fields without inputs and requires the existence of a *Meyer function* [8], which is also a restrictive assumption. The work [7] shows that, when the safe set is bounded, if the system is robustly safe and has no inputs, then there exists a barrier function candidate satisfying the barrier function condition strictly at the boundary of the safe set. The paper [12] extends these converse results (also in the context of robust safety) to a more general class of systems and safe sets. Finally, the recent paper [11] studies the converse safety problem extensively in the case of systems without inputs, introducing the notion of time-varying barrier functions. Such functions are not necessarily smooth, and their existence is necessary and sufficient for safety. The paper studies also the regularity properties of such time-varying barrier functions in a variety of scenarios. Despite the importance of this work, one must note that most of the CBF literature is concerned with time-invariant (C)BFs, and the converse problem of determining what is the most general set of conditions that guarantee that a safe set admits a time-invariant (C)BF remains open. For systems with inputs, [2] (which is a survey paper discussing the main technical results regarding CBFs, as well as some of its application domains) shows that if the safe set is compact and has a regular boundary, any continuously differentiable function with the safe set as its zero superlevel set is a CBF. The proof of this result relies on Nagumo's Theorem [13] and a specific construction of the class $\mathcal{K}_\infty$ function associated with the CBF leveraging the compactness of the safe set.

A related problem, which has also received increasing attention, is that of safe stabilization. A popular approach to design safe stabilizing controllers is to combine CBFs with CLFs [4], [14]–[16]. However, in order to have provable safety and stability guarantees, these control design methods must ensure that the CBF and CLF are compatible (i.e., that there exists a control input satisfying simultaneously the associated inequalities at every point in the safe set). Alternatively, the paper [17] proposes to unify a CLF and a CBF into a unique function, called control Lyapunov-barrier function (CLBF), and applies Sontag's universal formula to derive a smooth safe stabilizing controller. However, [18] points out some limitations for the existence of such a CLBF. In a similar vein, [19], [20] generalize Brockett's necessary condition for continuous state-dependent feedback stabilization [21] in the context of feedback stabilization and safety, which in turn provides limitations on the existence of a CLBF or a compatible CLF-CBF pair. However, the converse problem, i.e., whether a CLBF or a compatible CLF-CBF pair exists

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provided that safe stabilization is possible, remains largely unexplored. A recent exception is [22], where converse results for safe stabilizability are given. However, the Lyapunov-like function derived in the main converse result [22, Theorem 8] becomes unbounded at the boundary of the safe set, as a byproduct of the proof technique utilized therein. Instead, both CLBFs and compatible CLF-CBF pairs are bounded at the boundary of the safe set. This distinction is relevant because the safe stabilizing controllers based on CLBFs or compatible CLF-CBF pairs would not be well-defined if these functions become unbounded at the boundary of the safe set.

Statement of Contributions: The contributions of this paper consist of converse theorems on the existence of CBFs for the study of safety and safe stabilization of control systems. Specifically,

(i) We provide an example that shows that for unbounded safe sets, there might be candidate CBFs (i.e., functions whose zero superlevel set is the safe set) which are not CBFs, and candidate CBFs which are. This is in contrast to the case of bounded safe sets, where all candidate CBFs are CBFs. We also provide an example that shows that the existence of a CBF does not guarantee the existence of a locally Lipschitz safe feedback controller, even if the CBF condition is satisfied strictly at every point;

(ii) Given a safe set, we provide a set of general conditions on the dynamics and the safe set under which a CBF is guaranteed to exist. These conditions include safe sets for which there exists a safe controller such that trajectories of the closed-loop system do not get arbitrarily close to the boundary of the safe set, or polynomial systems with polynomial safe set and safe feedback. We also define an extended notion of CBF, termed *extended control barrier function* (eCBF), which relies on a generalization of the notion of extended class $\mathcal{K}_\infty$ function and show that they are always guaranteed to exist for any given dynamics and safe set;

(iii) Drawing on existing results in the literature, we provide a result that shows that if the unsafe set has a bounded connected component, there does not exist a CLBF or a strictly compatible CLF-CBF pair, and if the safe set is unbounded, there does not exist a CLBF. However, for a compact safe set, we show that if there exists a controller satisfying the CBF condition strictly and another controller that is stabilizing, the safe set admits a CLBF and a strictly compatible CLF-CBF pair. We also show that if the origin is safely stabilizing, under the same conditions that we can guarantee the existence of a CBF, we can also guarantee the existence of a compatible CLF-CBF pair;

(iv) Finally, we show via a counterexample that the existence of a CLF and a CBF does not imply the existence of a strictly compatible CLF-CBF pair. On the positive side, we find sufficient conditions under which the existence of a CLF and a CBF implies the existence of a strictly compatible CLF-CBF pair and a compatible CLF-eCBF pair.

Result	Assumptions	Statement
Thm IV.3	SC1 + safety	$\exists$ CBF
Thm IV.17	safety	$\exists$ eCBF
Thm V.3 (i)	SC2 + safety + stability	$\exists$ CLBF $\exists$ strictly compatible CLF-CBF pair
Thm V.3 (ii)	SC1 + safe stabilization	$\exists$ compatible CLF-CBF pair
Thm V.3 (iii)	safe stabilization	$\exists$ compatible CLF-eCBF pair
Prop V.6	SC3 + safety + stability	$\exists$ compatible CLF-eCBF pair

TABLE I: Summary of the main results in this paper. SC1 stands for the different sufficient conditions outlined in Theorem IV.3, SC2 stands for the sufficient conditions in Theorem V.3 (i), and SC3 for the sufficient conditions in Proposition V.6.

For convenience, Table I summarizes the main results of the paper.

II. Preliminaries

In this section we introduce some notation and preliminaries on CLFs, CBFs, and CLBFs.

A. Notation

We denote by $\mathbb{Z}_{>0}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{<0}$ the set of positive integers, real, nonnegative real numbers, and negative real numbers, resp. We write $\text{Int}(\mathcal{S})$, $\partial\mathcal{S}$, $\text{clos}(\mathcal{S})$ for the interior, boundary, and closure of the set $\mathcal{S}$, resp. The symbol $\mathbf{0}_n$ stands for the n-the dimensional zero vector. Given $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm. Given two functions $b_1 : \mathbb{R} \to \mathbb{R}$, $b_2 : \mathbb{R} \to \mathbb{R}$, we say that $b_1(x) = O(b_2(x))$ near some real number a if there exist positive numbers δ and M such that $|b_1(x)| \leq M b_2(x)$ for all x with $0 < \|x - a\| < \delta$.

Given $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and a smooth function $W : \mathbb{R}^n \to \mathbb{R}$, the notation $L_f W : \mathbb{R}^n \to \mathbb{R}$ (resp. $L_g W : \mathbb{R}^n \to \mathbb{R}^m$) denotes the Lie derivative of W with respect to f (resp. g), that is $L_f W = \nabla W^T f$ (resp. $\nabla W^T g$). We denote by $\mathcal{C}^1(\mathbb{R}^n)$ and $\mathcal{C}^\infty(\mathbb{R}^n)$ the set of continuously differentiable and infinitely continuously differentiable functions in $\mathbb{R}^n$, resp. Given a multi-index $\rho = (\rho_1, \rho_2, \dots, \rho_n)$ and a smooth scalar function f, $D^\rho f(x) = \frac{\partial^{\rho_1}}{\partial x_1}\frac{\partial^{\rho_2}}{\partial x_2}\dots\frac{\partial^{\rho_n}}{\partial x_n} f(x)$. We denote by $|\rho|$ the sum of the components of ρ. Given a differentiable function $h : \mathcal{D} \to \mathbb{R}$ and $\lambda \in \mathbb{R}$, λ is a regular value of h if $\frac{\partial h}{\partial x}(x) \neq 0$ for all $x \in \{y \in \mathcal{D} : h(y) = \lambda\}$. A function $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. V is proper in a set Γ if the set $\{x \in \Gamma : V(x) \leq c\}$ is compact for any $c \geq 0$. V is proper if it is proper in its domain. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$ and of class $\mathcal{K}_\infty$ if additionally $\lim_{r\to\infty} \alpha(r) = \infty$. If $\alpha : \mathbb{R} \to \mathbb{R}$ is strictly increasing, $\alpha(0) = 0$, and $\lim_{r\to\pm\infty} \alpha(r) = \pm\infty$, we say that α is of extended class $\mathcal{K}_\infty$.

A closed set $\mathcal{C}$ is forward invariant under the dynamical system $\dot{x} = f(x)$ if any trajectory with initial condition in $\mathcal{C}$ at time $t = 0$ remains in $\mathcal{C}$ for all times $t \geq 0$. A closed set

$\mathcal{C}$ is safe for the control system $\dot{x} = f(x,u)$, with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ locally Lipschitz, if there exists a locally Lipschitz control $u_{\text{sf}} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathcal{C}$ is forward invariant for $\dot{x} = f(x, u_{\text{sf}}(x))$. Furthermore, we call u_{sf} a safe controller in $\mathcal{C}$. A point p is safely stabilizable on a set $\mathcal{C}$ if there exists a locally Lipschitz control $u_{\text{sf}} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathcal{C}$ is forward invariant for $\dot{x} = f(x, u_{\text{sf}}(x))$ and p is asymptotically stable with region of attraction containing $\mathcal{C}$. Such a controller is a *safe stabilizing* controller in $\mathcal{C}$.

B. Control Lyapunov and Barrier Functions

In this section we introduce the notions of control Lyapunov and barrier functions. Throughout the paper we consider the nonlinear control system

$$\dot{x} = f(x,u), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz, with $x \in \mathbb{R}^n$ the state and $u \in \mathbb{R}^m$ the input.

Definition II.1. (Control Lyapunov Function [23], [24]): *Given a set $\Gamma \subseteq \mathbb{R}^n$, with $\mathbf{0}_n \in \Gamma$, a continuously differentiable function $V : \Gamma \to \mathbb{R}$ is a **CLF** on Γ for the system* (1) *if it is proper in Γ, positive definite, and there exists a positive definite function $W : \Gamma \to \mathbb{R}$ such that, for each $x \in \Gamma \backslash \{0\}$, there exists a control $u \in \mathbb{R}^m$ satisfying*

$$\nabla V(x)^T f(x,u) \leq -W(x). \tag{2}$$

A Lipschitz controller $k : \mathbb{R}^n \to \mathbb{R}^m$ such that $u = k(x)$ satisfies (2) for all $x \in \Gamma \backslash \{0\}$ makes the origin of the closed-loop system asymptotically stable. Hence, CLFs provide a way to guarantee asymptotic stability.

Next we recall the notion of control barrier function (CBF) [2]. Consider the set $\mathcal{C}$ defined as the zero-superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ as follows

$$\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geq 0\}, \tag{3a}$$
$$\partial\mathcal{C} = \{x \in \mathbb{R}^n : h(x) = 0\}, \tag{3b}$$
$$\text{Int}(\mathcal{C}) = \{x \in \mathbb{R}^n : h(x) > 0\}. \tag{3c}$$

Further assume that $\text{Int}(\mathcal{C}) \neq \emptyset$. A continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ satisfying (3) is referred to as a *candidate CBF* of $\mathcal{C}$.

Definition II.2. (Control Barrier Function): *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function satisfying* (3). *The function h is a **CBF** of $\mathcal{C}$ for the system* (1) *if there exists an extended class $\mathcal{K}_\infty$ function α such that, for all $x \in \mathcal{C}$, there exists a control $u \in \mathbb{R}^m$ satisfying*

$$\nabla h(x)^T f(x,u) + \alpha(h(x)) \geq 0. \tag{4}$$

In this paper we stick with Definition II.2, which is widely used in the CBF literature, as opposed to the notion of time-varying barrier function in [11, Definition 15]. The following result establishes that the existence of a CBF of $\mathcal{C}$ certifies its safety.

Theorem II.3. (CBFs certify safety [2, Theorem 2]): *Let $\mathcal{C} \subset \mathbb{R}^n$, h be a CBF of $\mathcal{C}$ for the system* (1)*, and* 0 *be a regular value of h. Any Lipschitz controller $u_{sf} : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies*

$$u_{sf}(x) \in K_{\text{cbf}}(x) = \{u \in \mathbb{R}^m : \nabla h(x)^T f(x,u) \geq -\alpha(h(x))\} \tag{5}$$

for all $x \in \mathcal{C}$ renders the set $\mathcal{C}$ forward invariant.

The following result states that for compact safe sets the converse of Theorem II.3 also holds.

Theorem II.4. (Converse CBF result for compact safe sets [2, Theorem 3]): *Let $\mathcal{C}$ be a compact set defined as in* (3) *and assume that* 0 *is a regular value of h. If $\mathcal{C}$ is safe for system* (1)*, then $h|_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}$ is a CBF of $\mathcal{C}$.*

Note that the result not only states the existence of a CBF but also that the function defining the set $\mathcal{C}$ is itself a CBF. Next we comment on the assumptions of Theorems II.3 and II.4 and establish different connections with the literature.

Remark II.5. (Existence of Lipschitz safe controllers): As pointed out in Theorem II.3, any locally Lipschitz safe controller satisfying that CBF condition renders the set safe. However, in general, such locally Lipschitz controller might not exist even if a CBF is available. In fact, [25, Example III.5] shows that the so-called *minimum-norm* controller (obtained at every $x \in \mathbb{R}^n$ as the controller with smallest norm that satisfies (4)) can be unbounded. Since the minimum-norm controller is unbounded, this example shows that even if a CBF exists, there might not exist a locally Lipschitz controller satisfying (4). However, [25, Lemma III.2] shows that if (1) is control-affine, $\mathcal{C}$ is compact and the CBF condition (4) holds strictly at the boundary, then the minimum-norm controller is locally Lipschitz. Alternatively, if $\mathcal{C}$ is compact and there exists an open set $\mathcal{D}$ containing $\mathcal{C}$ for which the CBF condition is feasible, then a Lipschitz safe controller also exists [26, Theorem 5]. We next complement this discussion by presenting an example inspired by [27] that shows that if the system is not control-affine, even if $\mathcal{C}$ is compact and the CBF condition (4) holds strictly at the boundary, there might not exist a continuous safe controller. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be defined as $h(x,y) = -(x^2+y^2) + 10$ and take $\mathcal{C}$ as in (3). Consider the system

$$\dot{x} = x\big((u-1)^2 - (x-1)\big)\big((u+1)^2 + (x-2)\big), \tag{6a}$$
$$\dot{y} = y\big((u-1)^2 - (x-1)\big)\big((u+1)^2 + (x-2)\big). \tag{6b}$$

Let us show that h is a CBF of $\mathcal{C}$ for system (6). Take $(x,y) \in \mathcal{C}$, and note that (4) is equivalent to

$$-2(x^2+y^2)\big((u-1)^2 - (x-1)\big)\big((u+1)^2+(x-2)\big) \geq -\alpha(h(x,y)). \tag{7}$$

Note that the set of points $(x,u) \in \mathbb{R}^2$ that satisfy

$$\big((u-1)^2 - (x-1)\big)\big((u+1)^2+(x-2)\big) \leq 0, \tag{8}$$

consists of two disjoint connected sets, one formed by the points to the right of the parabola $x = 1 + (u-1)^2$, and the other formed by the set of points to the left of the parabola $x = 2 - (u+1)^2$. This set is illustrated in Figure 1. Note

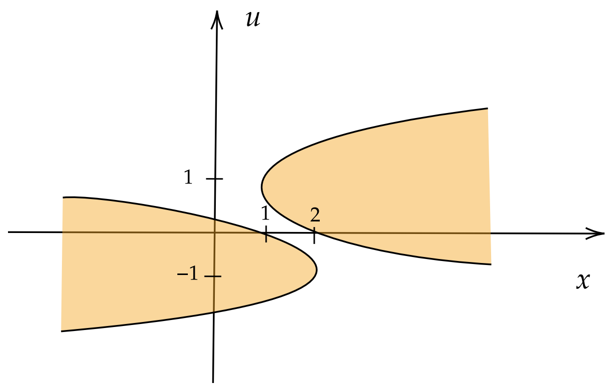

Fig. 1: Illustration of the set of points (x, u) satisfying (8). For every value of x, there are controls (colored in orange) that satisfy (8).

that the projection of these two sets onto the x axis covers the whole axis. Hence, h is a CBF. Now let $\tilde{u}_0 : \mathcal{C} \to \mathbb{R}$ be a controller such that $u = \tilde{u}_0(x, y)$ satisfies (7) at all points in $\mathcal{C}$, and let $\tilde{u} : [-\sqrt{10}, \sqrt{10}] \to \mathbb{R}$ be defined as $\tilde{u}(x) = \tilde{u}_0(x, \sqrt{10 - x^2})$. Note that since the two connected sets are disjoint, $\tilde{u}$ can not be continuous. This implies that $\tilde{u}_0$ can also not be continuous. We note also that in this example, $\mathcal{C}$ is compact and for all $x \in \mathbb{R}$, there exists $u \in \mathbb{R}$ satisfying the inequality $\big((u-1)^2 - (x-1)\big)\big((u+1)^2 + (x-2)\big) \leq 0$ strictly, which means that for all $(x, y) \in \partial\mathcal{C}$, there exists $u \in \mathbb{R}$ satisfying (7) strictly, and there also exists a neighborhood of $\mathcal{C}$ for which (7) is feasible. Hence, the only hypothesis of [25, Lemma III.2] and [26, Theorem 5] that fails is that the system is not control-affine. •

Remark II.6. (Minimal Control Barrier Functions): The work [26] shows that the regularity assumption can be dropped in both Theorems II.3 and II.4. This work identifies the minimal set of conditions that guarantee safety and defines the set of functions satisfying these conditions as *minimal (control) barrier functions* (M(C)BFs). Even though here we focus on CBFs, we also point out various connections of our results to M(C)BFs. •

When dealing with both stability and safety requirements under the dynamics (1), it is important to note that an input u might satisfy (2) but not (4), or vice versa. The following definition captures when the CLF and the CBF are compatible.

Definition II.7. (Compatibility of CLF-CBF pair [16, Definition 3]): *Let $\Gamma \subseteq \mathbb{R}^n$ be open, $\mathcal{C} \subset \Gamma$ closed, V a CLF on Γ and h a CBF of $\mathcal{C}$. Then, V and h are (strictly)* ***compatible*** *at $x \in \mathcal{C}$ if there exists $u \in \mathbb{R}^m$ satisfying* (2) *and* (4) *(strictly) simultaneously.*

We refer to V, h as a compatible pair in $\mathcal{C}$ if they are compatible at every point in $\mathcal{C}$. Similarly, V, h are a strictly compatible pair in $\mathcal{C}$ if they are strictly compatible at every point in $\mathcal{C} \backslash \{0\}$.

For control-affine systems, the work [14] shows that if V and h are strictly compatible in $\mathcal{C}$, then there exists a smooth safe stabilizing controller. If V and h are only compatible in $\mathcal{C}$, the existence of a smooth safe stabilizing controller is not guaranteed in general. However, [16] provides additional technical conditions under which the control design considered therein is locally Lipschitz and achieves safe stabilization. If instead the CBF and CLF inequalities are included as hard constraints of an optimization problem, one can use the theory of parametric optimization to obtain conditions under which the resulting optimization-based controller satisfies desirable regularity properties without requiring strict compatibility, cf. [28].

C. Control Lyapunov-Barrier Functions

Here we recall the notion of Control Lyapunov-Barrier Function (CLBF) introduced in [17] to design safe stabilizing controllers. Although the original definition is for control-affine systems, here we present it for general control systems (1).

Definition II.8. (Control Lyapunov-Barrier Function [17, Definition 2]): *A proper and lower-bounded function $\bar{V} \in \mathcal{C}^1(\mathbb{R}^n)$ is a Control Lyapunov-Barrier Function (**CLBF**) of $\mathbb{R}^n \backslash \mathcal{C}$ if it satisfies*

$$\bar{V}(x) > 0 \quad \forall x \in \mathbb{R}^n \backslash \mathcal{C}, \tag{9a}$$

$$\mathcal{U} := \{x \in \mathbb{R}^n : \bar{V}(x) \leq 0\} \neq \emptyset, \tag{9b}$$

$$(\overline{\mathcal{C} \backslash \mathcal{U}}) \cap (\overline{\mathbb{R}^n \backslash \mathcal{C}}) = \emptyset, \tag{9c}$$

$$\inf_{u \in \mathbb{R}^m} \nabla \bar{V}(x)^T f(x, u) < 0 \quad \forall x \in \mathcal{C} \backslash \{0\}. \tag{9d}$$

In the case where $\mathcal{U} = \mathcal{C}$, the conditions (9a), (9b) are reminiscent of (3) (with the sign changed), and (9d) is a more restrictive version of the inequality in Definition II.2. On the other hand, the requirement that $\bar{V}$ is proper and (9d) resemble the definition of CLF (cf. Definition II.1), although in this case we do not require $\bar{V}$ to be positive definite. Finally, condition (9c) is technical and guarantees that trajectories never enter the unsafe set even if their initial value of $\bar{V}$ is positive (cf. [17, Theorem 2]). This condition is trivially satisfied in the case $\mathcal{U} = \mathcal{C}$. Given a CLBF, [17, Proposition 3] provides an explicit construction of a safe stabilizing controller in $\mathcal{C}$. In particular, this means that if there exists a CLBF of $\mathbb{R}^n \backslash \mathcal{C}$, then $\mathcal{C}$ is safe.

III. Problem Statement

We consider a control system of the form (1) and a safe set $\mathcal{C}$ described by a differentiable function h as in (3). We are broadly motivated by questions about the existence of functions certifying stability and safety. Specifically, our goal is to answer the following questions:

(P1) If $\mathcal{C}$ is safe for the system, does it always admit a CBF? This problem corresponds to the converse of Theorem II.3, to which Theorem II.4 provides an answer in case $\mathcal{C}$ is compact. Here we intend to establish a more general result;

(P2) Under what conditions can the existence of a CLBF or a (strictly) compatible CLF-CBF pair be guaranteed? These problems are motivated by the fact that in either case feedback controllers that achieve safe stabilization can be designed under appropriate technical conditions, as described in Section II.

(P3) Does the existence of a (not necessarily compatible) CLF-CBF pair imply the existence of a (strictly) compatible CLF-CBF pair? This question shares its motivation with (P2) and the ease afforded by identifying a CLF and a CBF independently of each other.

We address these problems in the remainder of the paper, providing sufficient conditions under which each of them can be solved.

IV. Converse Results for Safety

In this section, we address problem (P1). Note that Theorem II.4 already provides a partial answer for the case when $\mathcal{C}$ is compact. The treatment of this section establishes more general results. We start with an example showing that, given an arbitrary safe set $\mathcal{C}$, not every candidate CBF of $\mathcal{C}$ is a CBF of $\mathcal{C}$.

Example IV.1. (Choice of candidate CBF matters for unbounded safe sets): Consider the function $h:\mathbb{R}^2\to\mathbb{R}$ given by $h(x,y)=x$ and the set $\mathcal{C}$ defined as in (3). Consider the system

$$\begin{aligned}\dot{x}&=xy+1,\\ \dot{y}&=-y+u.\end{aligned}$$

Since $\frac{d}{dt}h(x,y)=1$ when $x=0$ for any choice of u, by Nagumo's Theorem [13], $\mathcal{C}$ is safe. However, h is not a CBF. To show this, assume that it is. Therefore, there exists an extended class $\mathcal{K}_\infty$ function α satisfying

$$\nabla h(x,y)^T\begin{pmatrix}xy\\ -y+u\end{pmatrix}=xy+1\geq-\alpha(x).$$

Note that, for any fixed $x>0$, $\alpha(x)$ is constant, but by taking y sufficiently negative, xy can be arbitrarily negative, and the inequality $xy+1\geq-\alpha(x)$ will not be satisfied. Since this argument holds for any extended class $\mathcal{K}_\infty$ function α, h is not a CBF. Similarly, by assuming that α is a minimal function [26, Definition 1] also shows that h is not a *minimal control barrier function* [26, Definition 3].

However, one can show that the function $\tilde{h}(x,y)=e^y x$, which also satisfies (3), is in fact a CBF. Indeed, note that

$$\frac{d}{dt}\tilde{h}(x,y)=\nabla\tilde{h}(x,y)^T\begin{pmatrix}xy+1\\ -y+u\end{pmatrix}=e^y+e^y xu.$$

Taking $u=0$ for all $(x,y)\in\mathbb{R}^2$ makes $\tilde{h}$ satisfy (4) for any class $\mathcal{K}_\infty$ function α. △

The relevance of this example is in showing that the assumption that $\mathcal{C}$ be compact is critical for Theorem II.4 (as well as [26, Corollary 2]) to hold, since this result establishes that any candidate CBF of $\mathcal{C}$ is a CBF of $\mathcal{C}$. The extension of this result to unbounded safe sets therefore requires adjustments in the technical approach. Interestingly, we should point out that some aspects of Lyapunov theory for stability are also not fully understood in the case of unbounded attractors [29].

Remark IV.2. (Other counterexamples in the literature): We explain here the relative value and qualitative differences of Example IV.1 with respect to other counterexamples in the literature. [4, Remark 8] gives an example of a safe set for which a differentiable function satisfying (3) is not a CBF, but does not specify whether there exists another function with the same properties that is. [11, Example 1] provides an example of a safe set with empty interior which does not admit a continuous barrier certificate [1] that is only a function of the state. However, the system considered does not have control inputs and the notion of barrier certificate is different from the standard notion of CBF considered here (for which, for instance, safe sets have non empty interior). Finally, the counterexample in [11, Example 5] defines a safe set that is not expressible as the superlevel set of a differentiable function. •

A. Converse Theorem for CBFs

Example IV.1 shows that, for an arbitrary safe set $\mathcal{C}$, not every function satisfying (3) is a CBF, and in turn also raises the question of whether a CBF might even exist. The following result states conditions under which this is the case.

Theorem IV.3. (Converse CBF result for arbitrary sets): *Given a control system* (1)*, let $\mathcal{C}$ be a set for which there exists a continuously differentiable function $h:\mathbb{R}^n\to\mathbb{R}$ satisfying* (3)*. Suppose that $\mathcal{C}$ is safe and any of the following assumptions hold:*

(i) *there exists an extended class $\mathcal{K}_\infty$ function α and a function $u_*:\mathbb{R}^n\to\mathbb{R}^m$ such that, for all $r\geq0$,*

$$\inf_{\{x\in\mathbb{R}^n:h(x)\in[0,r]\}}\nabla h(x)^T f(x,u_*(x))\geq-\alpha(r);\quad(10)$$

(ii) *there exists a locally Lipschitz safe controller $u_0:\mathbb{R}^n\to\mathbb{R}^m$ and a positive function $\nu:Int(\mathcal{C})\to\mathbb{R}_{>0}$ such that, for any $x_0\in Int(\mathcal{C})$, the trajectory $x(\cdot)$ of $\dot{x}=f(x,u_0(x))$ with initial condition at x_0, satisfies $h(x(t))\geq\nu(x_0)>0$ for all $t\geq0$;*

(iii) *the function f is continuously differentiable, there exists a continuously differentiable safe controller $\hat{u}:\mathbb{R}^n\to\mathbb{R}^m$, positive integers $M_2\in\mathbb{Z}_{>0}$, $N_2\in\mathbb{Z}_{>0}$, and positive constants $\{b_j\}_{j\in\{1,\dots,M_2\}}$, and $\{c_k\}_{k\in\{1,\dots,N_2\}}$ such that*

$$\left\|\nabla(\|f(x,\hat{u}(x))\|^2)\right\|^2\leq\sum_{j=0}^{M_2}c_j\left\|f(x,\hat{u}(x))\right\|^j,\quad(11a)$$

$$\|\nabla h(x)\|^2\leq\sum_{k=0}^{N_2}b_k\left\|f(x,\hat{u}(x))\right\|^k,\quad(11b)$$

for all $x \in \mathcal{C}$;

(iv) *the set $\mathcal{C}$ is compact.*

Then, there exists a CBF of $\mathcal{C}$.

Proof. Note that (iv) is simply Theorem II.4.

To show (i), note that if (10) holds, then for any $x \in \mathbb{R}^n$, we have $\nabla h(x)^T f(x, u_*(x)) \geq -\alpha(h(x))$, and hence h is a CBF of $\mathcal{C}$.

We now prove (ii) and divide the proof in two steps. First, we construct a function that is differentiable almost everywhere, satisfies (3) and for which the CBF condition (4) holds at all points where it is differentiable. Second, we smoothen this function and obtain an actual CBF.

First step: construction of a *CBF almost everywhere*. For this step, we rely on the techniques in [30], which studies the connection between Hamilton-Jacobi reachability and CBFs. Let us consider the cost function $V : \mathbb{R}^n \times \mathbb{R}_{<0} \to \mathbb{R}$

$$V(x,t) = \min_{s \in [t,0]} h(x(s)),$$

which captures the minimum value of h along the trajectory $x(\cdot)$ that solves (1), with initial condition x, initial time $t < 0$ and control u_0 (note that as opposed to [30], here we omit the maximization over all possible controllers and simply use u_0). As explained in [30, Section II.D], by extending the definition of V for infinite time as $V_{-\infty}(x) := \lim_{t \to -\infty} V(x,t)$, we obtain a time-invariant function whose zero-superlevel set is the largest forward invariant set of $\dot{x} = f(x, u_0(x))$ contained in $\mathcal{C}$. In our case, since u_0 is a safe controller, $\mathcal{C} = \{x \in \mathbb{R}^n : V_{-\infty}(x) \geq 0\}$. Moreover, since u_0 is such that all trajectories of $\dot{x} = f(x, u_0(x))$ with initial condition $x_0 \in \text{Int}(\mathcal{C})$ satisfy $h(x(t)) \geq \nu(x_0)$ for all $t \geq 0$, it follows that $V(x_0, t) \geq \nu(x_0)$ for all $t < 0$ and therefore $V_{-\infty}(x_0) \geq \nu(x_0) > 0$ for all $x_0 \in \text{Int}(\mathcal{C})$. As also noted in [30, Section II.D], for all points in $\mathcal{C}$ where the gradient of $V_{-\infty}$ exists,

$$\nabla V_{-\infty}(x)^T f(x, u_0(x)) \geq -\alpha(V_{-\infty}(x)), \tag{12}$$

for any smooth extended class $\mathcal{K}_\infty$ function α. Since $V_{-\infty}$ might not be differentiable at some points, it might not be a valid CBF. However, since f, u_0 and h are locally Lipschitz, $V_{-\infty}$ is locally Lipschitz (cf. [30, Remark 1]), and therefore by Rademacher's Theorem $V_{-\infty}$ is differentiable almost everywhere [31, Theorem 7.11].

Second step: smoothing. The rest of the proof smoothens $V_{-\infty}$ to obtain a valid CBF. To do so, we follow a procedure closely related to that of [32] for smoothing Lyapunov functions. Let us start by showing that we can smoothen $V_{-\infty}$ at the interior of $\mathcal{C}$ and guarantee (12) for all $x \in \text{Int}(\mathcal{C})$ for the smoothened version of $V_{-\infty}$. Indeed, by [32, Theorem B.I] there exists a smooth $\Psi : \text{Int}(\mathcal{C}) \to \mathbb{R}$ such that for all $x \in \text{Int}(\mathcal{C})$,

$$|V_{-\infty}(x) - \Psi(x)| < \min\{\frac{1}{2} V_{-\infty}(x), 1\},$$
$$\nabla \Psi(x)^T f(x, u_0(x)) \geq -2\alpha(V_{-\infty}(x)).$$

Since $V_{-\infty}(x) > 0$ for all $x \in \text{Int}(\mathcal{C})$, then $\Psi(x) > V_{-\infty}(x) - \frac{1}{2} V_{-\infty}(x) = \frac{1}{2} V_{-\infty}(x) > 0$ for all $x \in \text{Int}(\mathcal{C})$. Now extend Ψ at $\partial\mathcal{C}$ so that $\Psi(x) = 0$ for all $x \in \partial\mathcal{C}$. It follows that Ψ defined in this way is smooth in $\text{Int}(\mathcal{C})$ and continuous in $\mathcal{C}$. Moreover, since α is increasing, $2\alpha(V_{-\infty}(x)) \leq 2\alpha(2\Psi(x))$. Hence, by defining $\bar{\alpha}(r) = 2\alpha(2r)$, $\bar{\alpha}$ is smooth, extended class $\mathcal{K}_\infty$ and for all $x \in \text{Int}(\mathcal{C})$ it holds that $\nabla \Psi(x)^T f(x, u_0(x)) \geq -\bar{\alpha}(\Psi(x))$. Now, extend Ψ in $\mathbb{R}^n \backslash \mathcal{C}$ in such a way that Ψ is smooth in $\mathbb{R}^n \backslash \mathcal{C}$ and $\Psi(x) < 0$ for all $x \in \mathbb{R}^n \backslash \mathcal{C}$ so that Ψ is continuous in $\mathbb{R}^n$ and smooth in $\mathbb{R}^n \backslash \partial\mathcal{C}$. Let us now use Ψ to construct a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ that is smooth in all of $\mathbb{R}^n$. In order to do so, let us show that there exists an extended class $\mathcal{K}_\infty$ function β such that $\Phi := \beta \circ \Psi$ is smooth in all of $\mathbb{R}^n$. The proof follows that of [32, Lemma 4.3], where the main difference is that in our case Ψ takes positive and negative values. For $i \in \mathbb{Z}_{>0}$, let K_i be compact subsets of $\mathbb{R}^n$ such that $\partial\mathcal{C} \subset \bigcup_{i=1}^{\infty} K_i$. For any $k \in \mathbb{Z}_{>0}$, let:

$$I_k^+ := \left(\frac{1}{k+2}, \frac{1}{k} \right), \quad I_k^- := \left(-\frac{1}{k}, -\frac{1}{k+2} \right).$$

Pick also smooth $\mathcal{C}^\infty(\mathbb{R})$ functions $\gamma_k^+ : \mathbb{R} \to [0,1]$, $\gamma_k^- : \mathbb{R} \to [-1,0]$ satisfying

- $\gamma_k^+(t) = 0$ if $t \notin I_k^+$,
- $\gamma_k^+(t) > 0$ if $t \in I_k^+$,
- $\gamma_k^-(t) = 0$ if $t \notin I_k^-$,
- $\gamma_k^-(t) < 0$ if $t \in I_k^-$.

Define also for any $k \in \mathbb{Z}_{>0}$,

$$\mathcal{G}_k := \{x \in \mathbb{R}^n : x \in \bigcup_{i=1}^{k} K_i, \quad \Psi(x) \in \text{clos}(I_k^+ \cup I_k^-)\}.$$

Observe that $\mathcal{G}_k$ is compact for all $k \in \mathbb{Z}_{>0}$ (because the sets K_i are compact and Ψ is continuous) and hence for each $k \in \mathbb{Z}_{>0}$ there exists $c_k \in \mathbb{R}$ satisfying:

(i) $c_k \geq 1$,

(ii) $c_k \geq |(D^\rho \Psi)(x)|$ for any multi-index $|\rho| \leq k$ and $x \in \mathcal{G}_k$,

(iii) $c_k \geq |\gamma_k^{(i)}(t)|$ for any $i \leq k$ and any $t \in \mathbb{R}_{>0}$.

Choose the sequence $\{d_k\}_{k \in \mathbb{Z}_{>0}}$ to satisfy

$$0 < d_k < \frac{1}{2^k (k+1)! c_k^k}, \quad k \in \mathbb{Z}_{>0}.$$

Now, define

$$\gamma(t) = \sum_{k=1}^{\infty} d_k (\gamma_k^+(t) + \gamma_k^-(t)) + \delta(t)$$

where $\delta : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty(\mathbb{R})$ function such that $\delta \equiv 0$ on $[-\frac{1}{3}, \frac{1}{3}]$, $\delta \geq 1$ on $[\frac{1}{2}, \infty)$ and $\delta \leq -1$ on $(-\infty, -\frac{1}{2}]$. By following an argument analogous to the one in the proof of [32, Lemma 4.3], we can show that γ is smooth in $\mathbb{R} \backslash \{0\}$ and $\lim_{t \to 0} \gamma^{(i)}(t) = 0$ for all $i \geq 1$. Moreover, $\beta(t) := \int_0^t \gamma(s) ds$ is an extended class $\mathcal{K}_\infty$, smooth in $\mathbb{R} \backslash \{0\}$, and satisfies $\lim_{t \to 0} \beta^{(i)}(t) = 0$ for all $i \geq 1$. Again by following the proof of [32, Lemma 4.3], one can show that $\Phi = \beta \circ \Psi$ is smooth in $\mathbb{R}^n$. This follows by considering sequences converging to points in $\partial\mathcal{C}$ and showing that the derivatives of any order of Φ along these sequences converge to zero.

Finally, now that we know that Φ is smooth and satisfies $\text{Int}(\mathcal{C}) = \{x \in \mathbb{R}^n : \Phi(x) > 0\}$, $\partial\mathcal{C} = \{x \in \mathbb{R}^n : \Phi(x) = 0\}$, let us show that Φ is a CBF. Indeed, for all $x \in \text{Int}(\mathcal{C})$, since $\Phi = \beta \circ \Psi$, it holds that

$$\nabla\Phi(x)^T f(x, u_0(x)) \geq -\beta'(\Psi(x))\bar{\alpha}(\Psi(x))$$

Note that $\beta'(r)\bar{\alpha}(r) > 0$ for $r > 0$, $\beta'(0)\bar{\alpha}(0) = 0$ and $\beta'(r)\bar{\alpha}(r)$ is smooth for all $r \in \mathbb{R}$. Therefore, $\beta'(r)\bar{\alpha}(r)$ can be upper bounded for $r \geq 0$ by a smooth extended class $\mathcal{K}_\infty$ function $\hat{\alpha}$. Finally, since β is an extended class $\mathcal{K}_\infty$ function, it is invertible and we can define $\breve{\alpha}(r) := \hat{\alpha} \circ \beta^{-1}(r)$, which is also smooth and extended class $\mathcal{K}_\infty$, so that for all $x \in \mathcal{C}$ it holds that

$$\nabla\Phi(x)^T f(x, u_0(x)) \geq -\breve{\alpha}(\Phi(x)).$$

Hence Φ is a CBF of $\mathcal{C}$. In fact $u_0(x)$ satisfies the CBF inequality (4) for all $x \in \mathcal{C}$.

Let us now show (iii). First, let us show that without loss of generality, we can assume that h is bounded in $\mathcal{C}$. Indeed, if h is not bounded, consider the function $\tilde{h} : \mathbb{R}^n \to \mathbb{R}$ as $\tilde{h}(x) = h(x)e^{-h(x)}$. Note that $\tilde{h}$ is continuously differentiable (because h is also continuously differentiable), bounded in $\mathcal{C}$ and, since $e^{h(x)} > 0$ for all $x \in \mathcal{C}$, $\mathcal{C} = \{x \in \mathbb{R}^n : \tilde{h}(x) \geq 0\}$, $\partial\mathcal{C} = \{x \in \mathbb{R}^n : h(x) = 0\}$, $\text{Int}(\mathcal{C}) = \{x \in \mathbb{R}^n : h(x) > 0\}$. Furthermore,

$$\nabla\tilde{h}(x) = \nabla h(x)e^{-h(x)}(1 - h(x)), \tag{13}$$

for all $x \in \mathcal{C}$, and it satisfies an inequality analogous to (11b). Indeed, let $\tilde{M} > 0$ be such that $e^{-2h(x)}(1+h(x))^2 < \tilde{M}$ for all $x \in \mathcal{C}$. Then, by defining $\tilde{b}_k := \tilde{M}b_k$ for $k \in \{0, 1, \dots, N_2\}$, and using (13) we have

$$\|\nabla\tilde{h}(x)\|^2 \leq \sum_{k=0}^{N_2} \tilde{b}_k \|f(x, \hat{u}(x))\|^2,$$

for all $x \in \mathcal{C}$. Therefore, without loss of generality, we assume that h is bounded in $\mathcal{C}$. Let $M > 0$ be such that $h(x) \leq M$ for all $x \in \mathcal{C}$. Now, consider the function $\bar{h}(x) = e^{-\|f(x,\hat{u}(x))\|^2}h(x)$. Note that $\bar{h}$ is bounded, continuously differentiable and $\mathcal{C} = \{x \in \mathbb{R}^n : \bar{h}(x) \geq 0\}$, $\text{Int}(\mathcal{C}) = \{x \in \mathbb{R}^n : \bar{h}(x) > 0\}$ and $\partial\mathcal{C} = \{x \in \mathbb{R}^n : \bar{h}(x) = 0\}$. Moreover,

$$\begin{aligned}\nabla\bar{h}(x)^T f(x, \hat{u}(x)) &= e^{-\|f(x,\hat{u}(x))\|^2}\nabla h(x)^T f(x, \hat{u}(x)) \\ &\quad - h(x)e^{-\|f(x,\hat{u}(x))\|^2}\nabla(\|f(x, \hat{u}(x))\|^2)^T f(x, \hat{u}(x)),\end{aligned}$$

and, by using the bounds in (11), the fact that h is bounded in $\mathcal{C}$, and the Cauchy-Schwartz inequality, we get that for all $x \in \mathcal{C}$, the following holds:

$$\begin{aligned}&\nabla\bar{h}(x)^T f(x, \hat{u}(x)) \geq \\ &\quad - e^{-\|f(x,\hat{u}(x))\|^2}\sqrt{\sum_{k=0}^{N_2} b_k \|f(x, \hat{u}(x))\|^{k+1}} \\ &\quad - e^{-\|f(x,\hat{u}(x))\|^2} M \sqrt{\sum_{j=0}^{M_2} c_j \|f(x, \hat{u}(x))\|^{j+1}}.\end{aligned}$$

Thus,

$$\alpha(r) = -\inf_{\substack{x \text{ s.t.} \\ \bar{h}(x)\in[0,r]}} \nabla\bar{h}(x)^T f(x, \hat{u}(x))$$

is finite for all $r \geq 0$. This follows from the fact that even if $\|f(x, \hat{u}(x))\|$ is unbounded as $\|x\| \to \infty$, the exponential term in $\|f(x, \hat{u}(x))\|$ will dominate the polynomial terms in $\|f(x, \hat{u}(x))\|$. Moreover, since $\hat{u}$ is safe, and $\bar{h}$ satisfies conditions (3), any trajectory $x(\cdot)$ of $\dot{x} = f(x, \hat{u}(x))$ for which $x(t^*) \in \partial\mathcal{C}$ for some $t^* \in \mathbb{R}$ necessarily satisfies $\nabla\bar{h}(x)^T f(x, \hat{u}(x)) \geq 0$ by Nagumo's Theorem [13]. This implies that $\alpha(0) \leq 0$, and α can be upper bounded by a class $\mathcal{K}_\infty$ function α_0, from where we have $\nabla\bar{h}(x)^T f(x, \hat{u}(x)) \geq -\alpha_0(\bar{h}(x))$ for all $x \in \mathcal{C}$, and hence $\bar{h}$ is a CBF of $\mathcal{C}$. In fact $\hat{u}(x)$ satisfies the CBF inequality (4) for all $x \in \mathcal{C}$. $\square$

Remark IV.4. (Minimal CBFs and smoothness properties): Even though 0 is not a regular value of the CBF Φ constructed in Theorem IV.3 (ii), Φ is a minimal CBF because the extended class $\mathcal{K}_\infty$ function $\breve{\alpha}$ is smooth [26, Corollary 1]. Moreover, if in Theorem IV.3 (iii), we add the assumption that 0 is a regular value of h, then 0 is also a regular value of $\bar{h}$ (the CBF constructed in the proof) and $\bar{h}$ is a minimal CBF [26, Section III]. This implies that the CBFs constructed in Theorem IV.3 can be used for control design according to [26, Theorem 4] and [2, Theorem 2]. Moreover, even if the minimal CBFs constructed in Theorem IV.3 (iii) and (iv) are not $\mathcal{C}^\infty(\mathbb{R}^n)$, by applying the smoothing procedure outlined in the proof of Theorem IV.3 (ii) to any differentiable minimal CBF, we can construct another minimal CBF that is $\mathcal{C}^\infty(\mathbb{R}^n)$. •

Remark IV.5. (Class of systems and safe sets satisfying (11)): As shown in the proof of (iii), condition (11) guarantees that $\bar{h}$ (as defined therein) satisfies that the function $x \to \nabla\bar{h}(x)^T f(x, \hat{u}(x))$ is lower bounded in the set $\{x \in \mathbb{R}^n : \bar{h}(x) \in [0, r]\}$, avoiding the issues faced in Example IV.1. Condition (11) is satisfied by a large class of systems, including polynomial systems for which there exists a polynomial safe feedback and safe sets $\mathcal{C}$ for which there exists a polynomial function h satisfying (3). •

Remark IV.6. (Comparison with time-varying barrier functions): The result in [11, Theorem 2] guarantees the existence of a time-varying barrier function, cf. [11, Definition 15], under more general assumptions than Theorem IV.3. Despite the importance of this result, the construction of such time-varying barrier functions is in general complicated and requires computing an appropriately defined reachable set. Moreover, such functions are in general not differentiable even if the dynamics are smooth [11, Theorem 4]. The added time dependence and the lack of control input and extended class $\mathcal{K}$ function make the notion of time-varying barrier functions substantially different from the notion of control barrier function considered here, which is the one widely employed in the safety-critical control literature [2]–[4]. It is also not apparent from the proof of [11, Theorem 2] when the obtained barrier function is time-independent. We also point out that the proof technique in Theorem IV.3 is different from that of [11, Theorem 2], and as pointed out in Remark IV.4, can be used to obtain barrier functions which are $\mathcal{C}^\infty(\mathbb{R}^n)$. •

Remark IV.7. (Robust safety): Here we comment on the relationship between robust safety, cf. [12, Definition 2], and the assumptions in Theorem IV.3 (ii), motivated by the fact that robust safety is also a sufficient condition for the existence of a certain notion of barrier function (different from the one adopted here because it does not utilize extended class $\mathcal{K}_\infty$ functions) [12, Theorems 1 and 2]. In particular, next we provide an example where the conditions in Theorem IV.3 (iii) hold but robust safety does not hold. Consider the scalar system $\dot{x} = xu$, safe set $\mathcal{C} = \{x \in \mathbb{R} : x \geq 0\}$, and safe controller $k(x) = 1$ for all $x \in \mathbb{R}$. Any trajectory of the closed-loop system with initial condition in $\mathrm{Int}(\mathcal{C})$ diverges to infinity and therefore the assumptions of Theorem IV.3 (iii) hold. However, the system is not robustly safe, since for any scalar function $\epsilon : \mathbb{R} \to \mathbb{R}_{>0}$, the trajectory of $\dot{x} = x - \epsilon(x)$ with initial condition at the origin enters the unsafe set (because $\epsilon(0) > 0$). •

Remark IV.8. (Strict positivity in the interior of the safe set): Definition II.2 requires h to be strictly positive in $\mathrm{Int}(\mathcal{C})$. However, other definitions available in the literature (e.g., [26, Theorem 1]) only require $\mathcal{C} := \{x \in \mathbb{R}^n : h(x) \geq 0\}$. With this definition of CBF it can be shown that any safe set admits a CBF. This follows by adapting the proof of Theorem IV.3 (ii) to the case where there might be points $x \in \mathrm{Int}(\mathcal{C})$ for which $V_{-\infty}(x) = 0$, i.e., finding a smooth approximation Ψ of $V_{-\infty}(x)$ at $\{x \in \mathbb{R}^n : V_{-\infty}(x) \neq 0\}$ and then extending Ψ smoothly at $\{x \in \mathbb{R}^n : V_{-\infty}(x) = 0\}$ as done in the proof of Theorem IV.3 (ii). •

Remark IV.9. (Asymptotic stability of safe set): As shown in [4, Proposition 2], if $u_{\text{sf}} : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz controller satisfying the CBF condition (4) for all $x \in \mathcal{D}$, where $\mathcal{D}$ is an open set containing $\mathcal{C}$, then the set $\mathcal{C}$ is asymptotically stable. By appropriately strengthening the conditions in Theorem IV.3, we can also guarantee the existence of a CBF valid in an open set containing $\mathcal{C}$, and hence certifying asymptotic stability of $\mathcal{C}$. Indeed,

(i) in Theorem IV.3 (i), if there exists $r_0 < 0$ such that (10) is satisfied for all $r \geq r_0$, then h is a CBF and (4) is feasible (by using $u = u_*(x)$) for all $x \in \mathbb{R}^n$ with $h(x) \geq r_0$;
(ii) in Theorem IV.3 (ii), under the additional assumption that there exists an open set $\mathcal{D}$ containing $\mathcal{C}$ for which any trajectory of $\dot{x} = f(x_0, u_0(x))$ with initial condition in $\mathcal{D}\backslash\mathrm{Int}(\mathcal{C})$ either converges to $\mathcal{C}$ in finite time or asymptotically, by following the same proof technique we can show that Φ (as obtained in the proof) is a CBF and (4) is feasible (by using $u = u_0(x)$) for all $x \in \mathcal{D}$;
(iii) in Theorem IV.3 (iii), if there exists an open set $\mathcal{D}$ containing $\mathcal{C}$ for which (11) holds for all $x \in \mathcal{D}$, then the same proof technique shows that there exists a CBF of $\mathcal{C}$ and the corresponding inequality (4) is also feasible in $\mathcal{D}$;
(iv) in Theorem IV.3 (iv), since $\mathcal{C}$ is compact, if it is asymptotically stable, by [32, Theorem 2.9] there exists a Lyapunov function with respect to $\mathcal{C}$. Therefore, h (which is guaranteed to be a CBF by Theorem II.4) can be extended outside of $\mathcal{C}$ using this Lyapunov function (smoothly, using the arguments in [32, Proposition 4.2] and Theorem IV.3 (ii)). •

Even though Theorem IV.3 extends significantly Theorem II.4 regarding the class of safe sets and systems for which a CBF exists, it is an open problem to determine whether an even more general result holds true. The following example is not covered by any of the cases in Theorem IV.3.

Example IV.10. (Example not covered by converse CBF theorem): Here we provide an example of a control system and safe set not covered by any of the cases in Theorem IV.3. Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = e^x \sin(x)$, and let $\mathcal{C}$ be the corresponding safe set as defined in (3). Note that h is continuously differentiable. Define the dynamics by letting $f(x, u) = h(x)$ (since systems without control are a special case of systems with control, Theorem IV.3 still applies). Note that $\mathcal{C}$ is safe because all points in $\partial\mathcal{C}$ are equilibrium points, and since h is continuously differentiable, trajectories of the dynamical system are unique, which means that no trajectory can leave the safe set. Furthermore, since $\dot{x} > 0$ whenever $x \in \mathrm{Int}(\mathcal{C})$, trajectories of the dynamical system with initial condition in $\mathrm{Int}(\mathcal{C})$ converge to $\partial\mathcal{C}$ and therefore item (ii) does not hold. Similarly, since $\mathcal{C}$ is not compact, item (iv) does not hold either. Now let us show that item (i) does not hold, which in turn also implies that item (iii) can not hold (because item (iii) implies item (i), as shown in the proof of Theorem IV.3). For any $k \in \mathbb{Z}_{>0}$, let $\underline{a}_k = (2k+1)\pi - \frac{\pi}{4}$, $\bar{a}_k = (2k+1)\pi$, and note that

$$h(\underline{a}_k) \geq \frac{e^{-\frac{\pi}{4}}\sqrt{2}}{2}, \ \cos(\underline{a}_k) = -\frac{\sqrt{2}}{2}, \ h(\bar{a}_k) = 0, \ \cos(\bar{a}_k) = -1,$$

and $\cos(a) < 0$ for all $a \in [\underline{a}_k, \bar{a}_k]$. Therefore, by letting $r = e^{-\frac{\pi}{4}}\frac{\sqrt{2}}{2}$, there exists a sequence $(a_k)_{k \in \mathbb{Z}_{>0}}$ (with $a_k \in ((2k+1)\pi - \frac{\pi}{4}, (2k+1)\pi)$ for all $k \in \mathbb{Z}_{>0}$) such that $\frac{r}{2} < h(a_k) < r$, and $\cos(a_k) < 0$ for all $k \in \mathbb{Z}_{>0}$. Now, note that

$$h'(a_k)f(a_k) = h(a_k)^2 + e^{a_k}\cos(a_k)h(a_k).$$

Since $\sin(a_k) < \frac{r}{e^{a_k}} < 1$, it follows that $|\cos(a_k)| = \sqrt{1 - \sin^2(a_k)} > \sqrt{1 - \left(\frac{r}{e^{a_k}}\right)^2}$, and

$$h'(a_k)f(a_k) < r^2 - e^{a_k}\frac{r}{2}\sqrt{1 - \left(\frac{r}{e^{a_k}}\right)^2}.$$

Therefore, since $\lim_{k\to\infty} a_k = \infty$,

$$\lim_{k\to\infty} h'(a_k)f(a_k) = -\infty,$$

which means that (i) does not hold. •

B. Extended Control Barrier Functions

In this section, we show that Theorem II.4 remains valid when we drop the compactness assumption provided that one employs a slight generalization of the notion of CBF. The latter requires a generalization of the notion of extended class $\mathcal{K}_\infty$.

Definition IV.11. (Extended class $\mathcal{KK}$ function): *A continuous function $\alpha : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KK}$ if $\alpha(\cdot, s)$ and $\alpha(r, \cdot)$ are strictly increasing for all $s \geq 0$, $r \in \mathbb{R}$, respectively and $\alpha(0, s) = 0$ for all $s \geq 0$. It is of extended class $\mathcal{K}_\infty\mathcal{K}$ if, additionally, $\lim_{r\to\pm\infty} \alpha(r, s) = \pm\infty$ for all $s \geq 0$.*

We are ready to introduce the notion of *Extended Control Barrier Functions*.

Definition IV.12. (Extended Control Barrier Function): *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and let $\mathcal{C}$ be defined as in* (3)*. The function h is an extended control barrier function (**eCBF**) of $\mathcal{C}$ if there exists an extended class $\mathcal{K}_\infty\mathcal{K}$ function α such that, for all $x \in \mathcal{C}$, there exists a control $u \in \mathbb{R}^m$ satisfying:*

$$\nabla h(x)^T f(x, u) \geq -\alpha(h(x), \|x\|). \tag{14}$$

Note that eCBFs allow for the time derivative of h to become arbitrarily negative as $\|x\|$ approaches infinity, as long as such derivative stays nonnegative at the boundary of $\mathcal{C}$. The following result relates the notions of CBF and eCBF.

Proposition IV.13. (Relationship between CBFs and eCBFs): *A CBF of $\mathcal{C}$ is also an eCBF of $\mathcal{C}$. Moreover, if $\mathcal{C}$ is compact, an eCBF of $\mathcal{C}$ is a CBF of $\mathcal{C}$.*

Proof. Let h be a CBF of $\mathcal{C}$, i.e., there exists an extended class $\mathcal{K}_\infty$ function α such that, for all $x \in \mathcal{C}$, there exists $u \in \mathbb{R}^m$ satisfying (4). Define

$$\hat{\alpha}(r, s) := \alpha(r)(s + 1)$$

and note that it is an extended class $\mathcal{K}_\infty\mathcal{K}$ function. Since $\hat{\alpha}(h(x), \|x\|) \geq \alpha(h(x))$ for all $x \in \mathcal{C}$, it follows that for all $x \in \mathcal{C}$ there exists $u \in \mathbb{R}^m$ satisfying (14).

Now, suppose that $\mathcal{C}$ is compact and let h_e be an eCBF of $\mathcal{C}$, i.e., there exists a class $\mathcal{K}_\infty\mathcal{K}$ function α_e such that, for all $x \in \mathcal{C}$, there exists $u \in \mathbb{R}^m$ satisfying (14). Define, for $r \geq 0$,

$$\tilde{\alpha}_e(r) := \sup_{\{x\in\mathbb{R}^n:\ 0\leq h_e(x)\leq r\}} \alpha_e(r, \|x\|).$$

Since $\mathcal{C}$ is compact, $\tilde{\alpha}_e(r)$ is finite for all $r \geq 0$. Furthermore, it is strictly increasing, satisfies $\tilde{\alpha}_e(0) = 0$, and satisfies $\lim_{r\to\infty} \alpha(r) = \infty$, so it can be extended to $\mathbb{R}_{<0}$ so that is of extended class $\mathcal{K}_\infty$. Since for all $x \in \mathcal{C}$, $\tilde{\alpha}_e(h(x)) \geq \alpha(h(x), \|x\|)$, we have that, for all $x \in \mathcal{C}$, there exists $u \in \mathbb{R}^m$ satisfying (4), i.e., h_e is a CBF. □

According to Proposition IV.13, eCBFs coincide with CBFs when the safe set is compact. As we show later, the notion of eCBF is a more suitable notion to deal with safe sets that are unbounded. Similarly to Definition II.7, we can also define a notion of compatibility for eCBFs (instead of CBFs) and CLFs.

Definition IV.14. *A CLF V and an eCBF h are **compatible** at $x \in \mathcal{C}$ if there exists $u \in \mathbb{R}^m$ satisfying* (2) *and* (14) *simultaneously. We refer to both functions as compatible in $\mathcal{C}$ if they are compatible at every point in $\mathcal{C}$.*

Since eCBFs also enforce the satisfaction of Nagumo's Theorem [13], they can be used to certify safety, as stated in the following result. We omit its proof, which follows an argument analogous to that of [2, Theorem 2].

Proposition IV.15. (eCBFs certify safety): *Let $\mathcal{C} \subset \mathbb{R}^n$, h an eCBF of $\mathcal{C}$, and 0 a regular value of h. Any Lipschitz controller $k : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies $k(x) \in K_{ecbf}(x) := \{u \in \mathbb{R}^m : \nabla h(x)^T f(x, u) + \alpha(h(x), \|x\|) \geq 0\}$ for all $x \in \mathcal{C}$ renders the set $\mathcal{C}$ forward invariant.*

We also point out that the control designs proposed in [4], [14], [16] can easily be adapted using eCBFs instead of CBFs.

Next we show that the flexibility added by the class $\mathcal{K}_\infty\mathcal{K}$ function allows eCBFs to resolve some of the issues faced by CBFs.

Example IV.16 (Examples IV.1 and IV.10 revisited)**.** We show here that $h(x, y) = x$ is an eCBF for Example IV.1. Take $\alpha(r, s) = rs$ as the extended class $\mathcal{K}_\infty\mathcal{K}$ function in (14). It is straightforward to check that $\nabla h(x, y)^T \left(\begin{smallmatrix} xy+1 \\ -y+u \end{smallmatrix}\right) = xy + 1 \geq -x\sqrt{x^2 + y^2} = -\alpha(h(x, y), \|(x, y)\|)$ for $x \geq 0$ and hence (14) is satisfied for all points in $\mathcal{C}$. By a similar argument, the function h defined in IV.10 is also an eCBF for the dynamics defined therein. △

The following result states that the existence of an eCBF is also necessary for a set to be safe, generalizing Theorem II.4 to safe sets that might be unbounded.

Theorem IV.17. (Converse eCBF result for safe sets): *Given a control system* (1)*, let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with $\mathcal{C}$ defined as in* (3) *and with 0 a regular value of h. If $\mathcal{C}$ is safe, then h is an eCBF.*

Proof. For $r \geq 0$, $c \geq 0$, define the set $S_{r,c} := \{x \in \mathbb{R}^n : 0 \leq h(x) \leq r, \|x\| \leq c + c_{\min}\}$, where $c_{\min} \geq 0$ is taken so that $S_{0,0} \neq \emptyset$ (for instance, one can set $c_{\min}$ as the distance from the origin to $\partial\mathcal{C}$). Since $S_{0,0} \subseteq S_{r,c}$ for any $r \geq 0$, $c \geq 0$, this guarantees that $S_{r,c} \neq \emptyset$ for all $r \geq 0, c \geq 0$. Next, define

$$\hat{\alpha}(r, c) := -\min_{x\in S_{r,c}} \nabla h(x)^T f(x, \hat{u}(x)), \tag{15}$$

where $\hat{u} : \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz controller that renders $\mathcal{C}$ safe (which exists, by assumption). Since $S_{r,c}$ is compact for all $r \geq 0$, $c \geq 0$, $\hat{\alpha}(r, c)$ is finite for all $r \geq 0$, $c \geq 0$. Note that $\hat{\alpha}$ is non-decreasing in both its first and second arguments. Moreover, since $\mathcal{C}$ is forward invariant under $\dot{x} = f(x, \hat{u}(x))$ and 0 is a regular value of h, by Nagumo's theorem [13], $\hat{\alpha}(0, c) \leq 0$ for all $c \geq 0$. Hence, we can find a class $\mathcal{K}_\infty\mathcal{K}$ function α such that $\alpha(r,c) \geq \hat{\alpha}(r,c)$ for all $r \geq 0, c \geq 0$. This ensures that

$$\nabla h(x)^T f(x, \hat{u}(x)) \geq -\alpha(h(x), \|x\|).$$

for all $x \in \mathcal{C}$, hence completing the proof. □

Remark IV.18. (Comparison with time-varying barrier functions-cont'd): As mentioned in Remark IV.6, [11, Theorem 2] provides a necessary and sufficient condition for safety in terms of so-called time-varying barrier functions, which might however be difficult to construct and utilize in practice to design safe controllers. Instead, in the less general setting considered here, Theorem IV.17 ensures that if $\mathcal{C}$ is safe, any scalar continuously differentiable function satisfying (3) and having 0 as a regular value is an eCBF. This ensures that h is time-invariant and continuously differentiable, and instead of computing a complicated reachable set, only requires finding a scalar continuously differentiable function satisfying (3), with 0 being a regular value of it. •

V. Converse Theorems for Joint Safety and Stability

In this section we address problems (P2) and (P3) in Section III. We start by studying under what conditions the existence of either (i) a compatible CLF-CBF pair or (ii) a CLBF is guaranteed. Our motivation comes from the fact that in either case locally Lipschitz feedback controllers that achieve safe stabilization can be designed under appropriate technical conditions, cf. Section II. Throughout this section, we assume $\mathbf{0}_n \in \text{Int}(\mathcal{C})$.

Our starting point is the result in [33, Theorem 11], which shows that if the set $\mathbb{R}^n \backslash \mathcal{C}$ is bounded, then a CLBF of $\mathbb{R}^n \backslash \mathcal{C}$ can not exist. In fact the proof of [33, Theorem 11] shows that if $\mathbb{R}^n \backslash \mathcal{C}$ is bounded, a locally Lipschitz safe stabilizing controller can not exist. More generally, the same argument shows that if $\mathbb{R}^n \backslash \mathcal{C}$ has a bounded connected component, then a safe stabilizing controller can not exist.

Proposition V.1. (Topological obstruction for existence of compatible CLF-CBF pair): *If the set $\mathbb{R}^n \backslash \mathcal{C}$ has a bounded connected component, then there does not exist a strictly compatible CLF-CBF pair on $\mathcal{C}$.*

Proof. Suppose there exits a strictly compatible CLF-CBF pair on $\mathcal{C}$. Then, by using the universal formula in [14] (which is constructed by computing the centroid of the set of controls satisfying the CLF and CBF conditions), one can construct a smooth safe stabilizing controller on $\mathcal{C}$. But, since $\mathbb{R}^n \backslash \mathcal{C}$ contains a bounded connected component, by the argument used in [33, Theorem 11], this can not be possible, reaching a contradiction. □

Even though Proposition V.1 only ensures the non-existence of a *strictly* compatible CLF-CBF pair, it also shows that even if a compatible CLF-CBF pair exists, one would not be able to leverage it to design a locally Lipschitz controller that safely stabilizes the system. The above result explains why the recent body of literature [16], [34]–[37] on locally Lipschitz controllers that achieve safe stabilization obtain closed-loop systems with undesirable equilibrium points in the boundary of the safe set, when the set of unsafe states has a bounded connected component. We note also that the proof of Proposition V.1 relies on [33, Theorem 11], which shows that if $\mathbb{R}^n \backslash \mathcal{C}$ is bounded, there can not exist a smooth safe stabilizing controller. A similar result (for analytic vector fields) is also available in [38, Proposition 3].

The next result identifies another scenario where a CLBF does not exist.

Proposition V.2. (No CLBF exists for unbounded safe sets): *Suppose $\mathcal{C} \neq \mathbb{R}^n$ is unbounded. Then, there does not exist a CLBF of $\mathbb{R}^n \backslash \mathcal{C}$.*

Proof. If $\mathbb{R}^n \backslash \mathcal{C}$ is bounded, there does not exist a CLBF of $\mathbb{R}^n \backslash \mathcal{C}$ by [18]. If $\mathbb{R}^n \backslash \mathcal{C}$ is unbounded, assume by contradiction that there exists a CLBF $\bar{V}$. As shown in [22, Remark 13], condition (9c) requires that $\partial\mathcal{C}$ is the 0-level set of $\bar{V}$. Indeed, if $x \in \partial\mathcal{C}$ is such that $\bar{V}(x) > 0$, then there exists a sequence $\{x_n\}_{n\in\mathbb{Z}_{>0}}$ converging to x such that $\bar{V}(x_n) > 0$ (and hence $x_n \notin \mathcal{U}$) and $x_n \in \mathcal{C}$ for all $n \in \mathbb{Z}_{>0}$. This means that $x \in \overline{(\mathcal{C}\backslash\mathcal{U})} \cap \overline{(\mathbb{R}^n\backslash\mathcal{C})}$, which is impossible by condition (9c). Finally, note that it is not possible for $\partial\mathcal{C}$ to be a 0-level set of $\bar{V}$ because $\bar{V}$ is proper, implying that all of its level sets are compact. □

Next, we turn our attention to identifying conditions under which either a CLBF or a compatible CLF-CBF pair exists provided that the origin is safely stabilizable.

Theorem V.3. (Converse result on safe stabilization): *Given a control system* (1)*, let $\mathcal{C}$ be a set for which there exists a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ satisfying* (3)*. Then,*

(i) *if $\mathcal{C}$ is compact, h is proper, and there exists a locally Lipschitz controller $u_{str} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\nabla h(x)^T f(x, u_{str}(x)) > 0$ for all $x \in \partial\mathcal{C}$, as well as a stabilizing controller $u_{st} : \mathbb{R}^n \to \mathbb{R}^m$ such that the origin is asymptotically stable for the closed-loop system $\dot{x} = f(x, u_{st}(x))$ with region of attraction containing $\mathcal{C}$, then there exists a CLBF of $\mathbb{R}^n \backslash \mathcal{C}$ and a strictly compatible CLF-CBF pair in $\mathcal{C}$;*

(ii) *if the origin is safely stabilizable on $\mathcal{C}$ with u_{ss} a safe stabilizing controller, and either the condition in Theorem IV.3 (i) holds with $u_* = u_{ss}$, the condition in Theorem IV.3 (ii) holds with $u_0 = u_{ss}$, the condition in Theorem IV.3 (iii) holds with $\hat{u} = u_{ss}$, or the condition in Theorem IV.3 (iv) holds, then there exists a compatible CLF-CBF pair on $\mathcal{C}$;*

(iii) *if the origin is safely stabilizable on $\mathcal{C}$, there exists a compatible CLF-eCBF pair on $\mathcal{C}$.*

Proof. **We first show (i).** Note that u_{str} is a safe controller. Since for all $x \in \partial\mathcal{C}$, $\nabla h(x)^T f(x, u_{\text{str}}(x)) > 0$, $\mathcal{C}$ is compact, and $\nabla h(x)^T f(x, u_{\text{str}}(x))$ is continuous as a function of x, there exists $\epsilon > 0$ such that $\nabla h(x)^T f(x, u_{\text{str}}(x)) > 0$ over $\mathcal{T} := \{x \in \mathbb{R}^n : 0 \leq h(x) \leq \epsilon\}$ and such that $\mathbf{0}_n \notin \mathcal{T}$ (this is possible because by assumption, $\mathbf{0}_n \in \text{Int}(\mathcal{C})$). Since the origin is asymptotically stable for the closed-loop system $\dot{x} = f(x, u_{\text{st}}(x))$ with region of attraction containing an open set containing $\mathcal{C}$, and since the region of attraction is an open set [39, Lemma 8.1], by [39, Theorem 4.17], this implies that there exists a CLF V on an open set containing $\mathcal{C}$ and, furthermore, $\nabla V(x)^T f(x, u_{\text{st}}(x)) < 0$ for all $x \in \mathcal{C} \backslash \{\mathbf{0}_n\}$. Since V and h are continuous, $\mathcal{C}$ is compact and $\{x \in \mathbb{R}^n : h(x) = \frac{\epsilon}{2}\} \subset \mathcal{T}$, there exists $\lambda > 0$ sufficiently large such that $\{x \in \mathbb{R}^n : \frac{1}{\lambda} V(x) + h(x) = \frac{\epsilon}{2}\} \cap \mathcal{T} \neq \emptyset$. Let $\Pi = \{x \in \mathbb{R}^n : \frac{1}{\lambda} V(x) + h(x) \geq \frac{\epsilon}{2}\} \cap \mathcal{C}$. Figure 2 illustrates the different sets defined up to this point. Since $\{x \in \mathbb{R}^n : \frac{1}{\lambda} V(x) + h(x) = \frac{\epsilon}{2}\} \cap \mathcal{T} \neq \emptyset$, it follows that Π is nonempty, is contained in $\mathcal{C}$, and is compact. Note that $\mathbf{0}_n \in \Pi$ (because $\mathbf{0}_n \in \mathcal{C} \backslash \mathcal{T}$). Now, define

$$\tilde{V}(x) = \begin{cases} \frac{1}{\lambda} V(x) + h(x) & \text{if} \quad x \in \Pi, \\ \frac{\epsilon}{2} & \text{else.} \end{cases}$$

Recall that $\nabla V(x)^T f(x, u_{\text{st}}(x)) < 0$ for all $x \in \text{Int}(\Pi) \backslash \{\mathbf{0}_n\}$. By smoothing $\tilde{V}$ using the smoothing argument in the proof of Theorem IV.3 (ii), there exists a smooth function $\Psi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} \nabla \Psi(x)^T f(x, u_{\text{st}}(x)) \leq \big(\frac{1}{\lambda} \nabla V(x) + \nabla h(x)\big)^T f(x, u_{\text{st}}(x)\big) \\ - \frac{1}{2\lambda} \nabla V(x)^T f(x, u_{\text{st}}(x)), \end{aligned}$$

for all $x \in \text{Int}(\Pi)$, and $\Psi(x) = \frac{\epsilon}{2}$, $\nabla \Psi(x) = 0$ for all $x \in \mathbb{R}^n \backslash \text{Int}(\Pi)$. Next, we show that $\bar{V}(x) = -h(x) + \Psi(x) - \frac{\epsilon}{2}$ is a CLBF of $\mathbb{R}^n \backslash \mathcal{C}$. First, note that $\bar{V}$ is proper because h is proper and $\bar{V}(x) = -h(x)$ for all $x \in \mathbb{R}^n \backslash \text{Int}(\Pi)$, $\bar{V}(x) > 0$ for all $x \in \mathbb{R}^n \backslash \mathcal{C}$, and hence (9a) holds. Moreover, for $x \in \mathcal{C} \backslash \text{Int}(\Pi)$, since $\nabla \bar{V} \equiv 0$ and $\mathcal{C} \backslash \text{Int}(\Pi) \subset \mathcal{T}$, it follows that $\nabla \bar{V}(x)^T f(x, u_{\text{str}}(x)) = -\nabla h(x)^T f(x, u_{\text{str}}(x)) < 0$. For $x \in \text{Int}(\Pi) \backslash \{\mathbf{0}_n\}$,

$$\begin{aligned} \nabla \bar{V}(x)^T f(x, u_{\text{st}}(x)) &= (-\nabla h(x) + \nabla \Psi(x))^T f(x, u_{\text{st}}(x)) \\ &\leq \frac{1}{2\lambda} \nabla V(x)^T f(x, u_{\text{st}}(x)) < 0. \end{aligned}$$

Hence, (9d) holds. Moreover, note that $\mathcal{C} \backslash \Pi \neq \emptyset$ and $\bar{V}(x) = -h(x) < 0$ in $\mathcal{C} \backslash \Pi$. Hence, $\mathcal{U} := \{x \in \mathbb{R}^n : \bar{V}(x) \leq 0\} \neq \emptyset$ and (9b) holds. Moreover, since again $\bar{V}(x) = -h(x)$ in $\mathcal{C} \backslash \Pi$, $\mathcal{C} \backslash \mathcal{U} \subset \Pi$. This means that $\overline{(\mathcal{C} \backslash \mathcal{U})} \cap \overline{(\mathbb{R}^n \backslash \mathcal{C})} = \emptyset$ and (9c) holds. This shows that $\bar{V}$ is a CLBF of $\mathbb{R}^n \backslash \mathcal{C}$.

Next, let us show that there exists a strictly compatible CLF-CBF pair in $\mathcal{C}$. We do so by using the CLBF $\bar{V}$. Since $\bar{V}$ is lower-bounded, it achieves its minimum value at a point p. Note that p must be the origin, because otherwise, by (9d) $\nabla \bar{V}(p) \neq 0$, which would mean that p is not a local minimum. Let $\hat{V}(x) = \bar{V}(x) - \bar{V}(0)$. Note that $\hat{V}$ is proper, positive definite and, for each $x \in \mathcal{C}$, there exists a control $u \in \mathbb{R}^m$ satisfying (2) strictly. Indeed, this follows by considering a controller $\breve{u} : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $\nabla \bar{V}(x)^T f(x, \breve{u}(x)) < 0$ for all $x \in \mathcal{C} \backslash \{\mathbf{0}_n\}$ and taking $W(x) := -\frac{1}{2} \bar{V}(x)^T f(x, \breve{u}(x))$

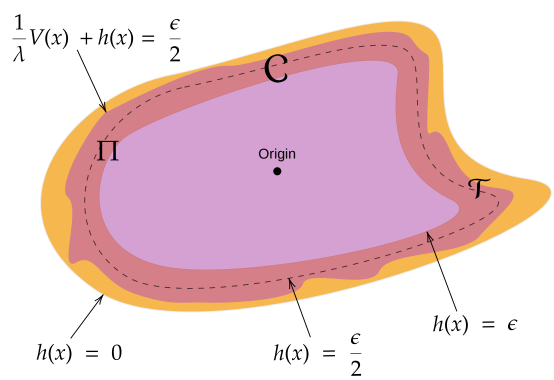

Fig. 2: Illustration of the different sets defined in the proof of Theorem V.3(i). The set $\mathcal{C}$ is the union of the orange, dark purple and light purple regions. The set $\mathcal{T}$ is the union of the orange and dark purple regions, and the set Π is the union of the dark and light purple regions.

for $x \neq \mathbf{0}_n$ and $W(0) = 0$ in Definition II.1. Therefore, $\hat{V}$ is a CLF. Now, let $\hat{h}(x) = -\bar{V}(x)$. It is easy to check that $\hat{h}$ is a CBF of $\mathcal{C}$ because it is a candidate CBF of $\mathcal{C}$ and $\mathcal{C}$ is compact (cf. Theorem II.4). Now, (2) and (4) read as

$$\nabla \bar{V}(x)^T f(x, u) + W(x) \leq 0, \tag{16a}$$
$$-\nabla \bar{V}(x)^T f(x, u) + \alpha(-\bar{V}(x)) \geq 0. \tag{16b}$$

Now note that $\breve{u}(x)$ satisfies (16) strictly for all $x \in \mathcal{C}$. Hence, $\hat{V}$ and $\hat{h}$ are a strictly compatible CLF-CBF pair.

To show (ii) and (iii), we reason as follows. Since u_{ss} is a safe stabilizing controller, $\mathcal{C}$ is forward invariant and the origin is asymptotically stable for the closed-loop system $\dot{x} = f(x, u_{\text{ss}}(x))$ with $\mathcal{C}$ contained in an open set contained in its region of attraction. By [39, Theorem 4.17], there exists a Lyapunov function V for the closed-loop system $\dot{x} = f(x, u_{\text{ss}}(x))$, and u_{ss} satisfies (4). Now, if condition (ii) in Theorem IV.3 holds with $u_0 = u_{\text{ss}}$, there exists a CBF h^* of $\mathcal{C}$. As shown in the proof of Theorem IV.3(ii), u_{ss} satisfies the associated CBF condition (4) for all $x \in \mathcal{C}$. Similarly, if condition (iii) in Theorem IV.3 holds with $\hat{u} = u_{\text{ss}}$, there exists a CBF h^* of $\mathcal{C}$. As shown in the proof of Theorem IV.3(iii), u_{ss} satisfies the associated CBF condition (4) for all $x \in \mathcal{C}$. Finally, if condition (iv) in Theorem IV.3 holds, there exists a CBF h^* of $\mathcal{C}$, and as shown in the proof of [4, Proposition 3], any safe controller (in particular, u_{ss}) satisfies (4) for an appropriately defined extended class $\mathcal{K}_\infty$ function α. Hence, for every $x \in \mathcal{C}$, $u_{\text{ss}}(x)$ satisfies inequalities (2) and (4), which means that V and h^* are compatible, showing (ii).

Moreover, since $\mathcal{C}$ is safe under u_{ss}, Theorem IV.17 implies that there exists an eCBF $\hat{h}$ of $\mathcal{C}$. Moreover, as shown in the proof of Theorem IV.17, any locally Lipschitz safe controller (in particular, u_{ss}) satisfies (14) for all $x \in \mathcal{C}$. Since $u_{\text{ss}}(x)$ satisfies (2) and (14) simultaneously, V and $\hat{h}$ are a compatible CLF-eCBF pair, showing (iii). $\square$

Theorem V.3 (i) is consistent with Proposition V.1, because it only ensures the existence of a CLBF if $\mathcal{C}$ is compact.

Remark V.4. (On CLBFs and compatible pairs): It is worth noting how Theorem V.3(i)-(iii) provide existence results under decreasingly restrictive hypotheses. In fact, the conditions in Theorem V.3 under which a CLBF is guaranteed to exist always guarantee the existence of a compatible CLF-CBF pair, but the converse does not hold. To see this, note that from Proposition V.2 and Theorem V.3, that unbounded sets containing a safely stabilizable point do not admit a CLBF, but they can admit a compatible CLF-CBF pair if either of the conditions in Theorem IV.3(ii), (iii), or (iv) hold. Instead, compact safe sets that contain a safely stabilizable point and satisfy the strict inequality condition in Theorem V.3 (i) for some controller admit both a CLBF and a compatible CLF-CBF pair (because if the safe set is compact the assumptions of Theorem V.3 (ii) hold). •

Next we address problem (P3) in Section III. The following example shows that in general, even if there exists a CBF of $\mathcal{C}$ and a CLF on an open set containing $\mathcal{C}$, there might not exist a strictly compatible CLF-CBF pair in $\mathcal{C}$.

Example V.5. (Safety and stability separately do not imply safe stabilization): Consider the control-affine system:

$$\dot{x} = -xu, \tag{17a}$$

$$\dot{y} = -yu. \tag{17b}$$

Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and $\mathcal{V}$ a neighborhood of $p = (0, 5)$ with the following properties:

- $h(x, y) = 1 - (x + 1)^2 - (y - 5)^2$ in $\mathcal{V}$,
- $h(0, 0) > 0$,
- $\{x \in \mathbb{R}^n : h(x) \geq 0\}$ is compact.

Let $\mathcal{C}$ be defined as in (3). The controller $u_{\text{sf}} : \mathbb{R}^2 \to \mathbb{R}$ given by $u_{\text{sf}}(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$ renders the set $\mathcal{C}$ safe. Since $\mathcal{C}$ is compact, by Theorem IV.3 (iv) it follows that h is a CBF of $\mathcal{C}$. Moreover, the origin is globally asymptotically stabilizable, since the controller $u_{\text{st}} : \mathbb{R}^2 \to \mathbb{R}$ given by $u_{\text{st}}(x, y) = 1$ for all $(x, y) \in \mathbb{R}^2$ makes the origin globally asymptotically stable. Moreover, $V(x, y) = \frac{1}{2}(x^2 + y^2)$ is a CLF in $\mathbb{R}^2$. However, any locally Lipschitz controller $\hat{u} : \mathbb{R}^2 \to \mathbb{R}$ such that $\hat{u}(0, 5) \neq 0$ steers the trajectory starting at the point $(0, 5)$ away from $\mathcal{C}$. Indeed, note that the y axis is forward invariant and hence $x(t) = 0$ for all $t \geq 0$. Moreover, the solution of (17) is differentiable and by performing a Taylor expansion of first order, for time $0 < \epsilon \ll 1$, the solution of (17) satisfies

$$y(\epsilon) = 5 - 5u(p)\epsilon + O(\epsilon^2).$$

This implies that

$$h(x(\epsilon), y(\epsilon)) = -25u(p)^2\epsilon^2 + O(\epsilon^3),$$

and therefore $h(x(\epsilon), y(\epsilon)) < 0$ for small enough ϵ. Hence, there does not exist a safe stabilizing controller in $\mathcal{C}$. Therefore, even though h is a CBF of $\mathcal{C}$ and V is a CLF in $\mathbb{R}^2$, there does not exist a strictly compatible CLF-CBF pair. Indeed, if that were the case the control design provided in [14] would yield a safe stabilizing controller, which does not exist. Note that this example does not preclude the existence of a compatible CLF-CBF pair. However, even if such a compatible pair exists, one would not be able to use it to obtain a safe stabilizing controller. △

Note that the cause of difficulty in Example V.5 is the point $p = (0, 5)$, which is such that $\nabla h(x)^T f(x, u) = 0$ for any $u \in \mathbb{R}^m$. Instead, using Theorem V.3 (i), we know that if there exists a locally Lipschitz controller $u_{\text{str}} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\nabla h(x)^T f(x, u_{\text{str}}(x)) > 0$ for all $x \in \partial\mathcal{C}$, $\mathcal{C}$ is compact and there exists a stabilizing controller with region of attraction containing the safe set, then there exists a strictly compatible CLF-CBF pair. Note that the proof of Theorem V.3 (i) heavily relies on the compactness of $\mathcal{C}$. Next, we provide a similar result for non-compact $\mathcal{C}$ but restricted to control-affine systems.

Proposition V.6. (Existence of compatible CLF-eCBF pair): *Given an open set Γ such that $\mathcal{C} \subseteq \Gamma$, let h be an eCBF of $\mathcal{C}$ with 0 as a regular value and V be a CLF on Γ. Further assume that the dynamics are control-affine, so that $\dot{x} = a(x) + g(x)u$, with $a : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ locally Lipschitz. Let $\mathcal{P} := \{x \in \mathbb{R}^n : L_gV(x) = \kappa L_gh(x),\ \kappa > 0\}$. Assume that $\mathcal{P} \cap \partial\mathcal{C}$ is contained in*

$$\{x \in \partial\mathcal{C} : L_gh(x) \neq \mathbf{0}_m,\ L_aV(x) < \frac{L_gV(x)^T L_gh(x)}{\|L_gh(x)\|^2} L_ah(x)\}.$$

Then, there exists a compatible CLF-eCBF pair in $\mathcal{C}$ with 0 a regular value of the eCBF.

Proof. The proof relies on the characterization of compatibility for a CLF and a CBF, as provided in [16, Lemma 5.2], and which naturally extends to CLFs and eCBFs. This result says that V and h are compatible at $x \in \mathcal{C}$ if and only if $L_gV(x)$ and $L_gh(x)$ are linearly dependent, $L_gV(x)^T L_gh(x) > 0$ and $L_fV(x) + W(x) > \frac{L_gV(x)^T L_gh(x)}{\|L_gh(x)\|^2}(L_fh(x) + \alpha(h(x)))$. By continuity of L_gV and L_gh, there exists a neighborhood $\mathcal{T}$ of $\partial\mathcal{C}$ such that $\mathcal{P} \cap \mathcal{T} \subseteq \{x \in \mathbb{R}^n : L_gh(x) \neq \mathbf{0}_m,\ L_aV(x) < \frac{L_gV(x)^T L_gh(x)}{\|L_gh(x)\|^2} L_ah(x)\}$. By [16, Lemma 5.2], V and h are compatible in $\mathcal{C} \cap \mathcal{T}$. Next, define $\mathcal{S} = \{x \in \mathcal{C}\backslash(\mathcal{T} \cup \{\mathbf{0}_n\}) : L_gV(x)^T L_gh(x) = \mathbf{0}_m\}$. Given $y \in \mathcal{S}$, if $L_gV(y)$ and $L_gh(y)$ are linearly independent, V and h are compatible at y (cf. [16, Lemma 5.2]). If instead $L_gV(y)$ and $L_gh(y)$ are linearly dependent, $L_gV(y) = L_gh(y) = \mathbf{0}_m$ and V and h are compatible at y because V is a CLF and h an eCBF. Moreover, there exists a neighborhood $\bar{\mathcal{S}}$ of $\mathcal{S}$ such that V and h are compatible in $\bar{\mathcal{S}}$. This is because for any $y \in \mathcal{S}$, if $L_gV(y)$ and $L_gh(y)$ are linearly independent, there exists a neighborhood $\mathcal{V}(y)$ of y where $L_gV(z)$ and $L_gh(z)$ are linearly independent for all $z \in \mathcal{V}(y)$, and hence by [16, Lemma 5.2], V and h are compatible at z. If instead $L_gV(y)$ and $L_gh(y)$ are linearly dependent, we can assume without loss of generality that $L_ah(y) + \alpha(h(y)) > 0$ and $L_aV(y) + W(y) < 0$ (otherwise, define $\tilde{\alpha}(s) := \frac{1}{2}\alpha(s)$ and $\tilde{W}(x) = \frac{1}{2}W(x)$), hence making V and h compatible in a neighborhood of y. Now, we only need to show that V and h are compatible at $\mathcal{C}\backslash(\mathcal{T} \cup \bar{\mathcal{S}})$. Define

$S_{r,c} := \{x \in \mathbb{R}^n : 0 \leq h(x) \leq r,\ \|x\| \leq c + c_{\min}\}$, where $c_{\min}$ is taken so that $S_{0,0} \neq \emptyset$, and define α as follows:

$$\alpha(r,c) := \sup_{x \in S_{r,c} \backslash (\mathcal{T} \cup \bar{\mathcal{S}})} \Big\{ (L_a V(x) + W(x)) \frac{\|L_g h(x)\|^2}{L_g V(x)^T L_g h(x)} - L_a h(x) \Big\}.$$

Since $S_{r,c} \backslash (\mathcal{T} \cup \bar{\mathcal{S}})$ is bounded for all $r \geq 0$, $c \geq 0$, and there exists a positive constant $\iota_{r,c} > 0$ such that $L_g V(x)^T L_g h(x) > \iota_{r,c}$ for all $x \in S_{r,c}$, $\alpha(r,c)$ is finite for all $r \geq 0$, $c \geq 0$. Hence, there exists a class $\mathcal{K}_\infty \mathcal{K}$ function $\hat{\alpha}$ such that $\hat{\alpha}(h(x), \|x\|) \geq \alpha(h(x), \|x\|)$ for all $x \in \mathcal{C} \backslash (\mathcal{T} \cup \bar{\mathcal{S}})$. Hence, by [16, Lemma 5.2], for all $x \in \mathcal{C} \backslash (\mathcal{T} \cup \bar{\mathcal{S}})$ there exists $u \in \mathbb{R}^m$ satisfying

$$L_a V(x) + L_g V(x) u + W(x) \leq 0,$$
$$L_a h(x) + L_g h(x) u + \hat{\alpha}(h(x), \|x\|) \geq 0,$$

and hence V and h are compatible in all of $\mathcal{C}$. $\square$

The conditions in Proposition V.6 are only sufficient. In other words, there could exist weaker conditions ensuring the existence of a compatible CLF-eCBF pair.

Remark V.7. (Origin at the boundary of safe set): The treatment above relies on the assumption that the origin belongs to $\mathrm{Int}(\mathcal{C})$. The extension of our results to the case when the origin is instead at $\partial\mathcal{C}$ remains an open problem. In fact, establishing whether in such case there exists a CLBF of $\mathbb{R}^n \backslash \mathcal{C}$, or a (strictly) compatible CLF-CBF pair in $\mathcal{C}$ requires facing additional technical challenges. For example, the construction of the CLBF in the proof of Theorem V.3 relies on the fact that $\mathbf{0}_n$ does not belong to the set $\mathcal{T}$ and belongs to the set Π. Otherwise, the CLBF $\bar{V}$ (as defined therein) does not satisfy condition (9d) at the origin. •

VI. Conclusions

We have provided converse theorems on the existence of CBFs for the study of safety and safe stabilization of control systems. Regarding safety, we have shown that for unbounded safe sets not all candidate CBFs are CBFs, in contrast to what happens for bounded safe sets. Next, we have derived a general set of conditions under which a CBF is guaranteed to exist for any given safe set. We have also extended the definition of CBF conveniently to introduce eCBFs, and we have shown that any safe set admits an eCBF. Regarding safe stabilization, we have established an alternate set of conditions under which a CLBF, a (strictly) compatible CLF-CBF pair, and compatible CLF-eCBF pairs can or can not exist. Finally, we have shown via a counterexample that the existence of a CLF and a CBF does not imply in general the existence of a strictly compatible CLF-CBF pair, but we have found sufficient conditions under which this holds. Future work will focus on tightening the conditions identified in our results and in extending the results to nonsmooth barrier functions and discontinuous controlled dynamics.

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