Proof of 2-D Inverse Matrix Formula

Zicheng Liang

13th October 2023

1 Matrices AB

Given

$$MatrixA = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{1.0.1}$$

$$MatrixB = \begin{bmatrix} b_{11} & b_{22} \\ b_{21} & b_{22} \end{bmatrix} \tag{1.0.2}$$

Let's assume that B = A^{-1}, therefore AB = AA^{-1} = I

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.0.3}$$

2 Equations

Therefore we get four equations:

$$a_{11}b_{11} + a_{12}b_{21} = 1 \tag{2.0.1}$$

$$a_{11}b_{12} + a_{12}b_{22} = 0 \tag{2.0.2}$$

$$a_{21}b_{11} + a_{22}b_{21} = 0 \tag{2.0.3}$$

$$a_{21}b_{12} + a_{22}b_{22} = 1 \tag{2.0.4}$$

2.1 b_{11}

from equation (2.0.3) we know that:

$$a_{21}b_{11} = -a_{22}b_{21} \tag{2.1.1}$$

i.e.

$$b_{21} = -b_{11}\frac{a_{21}}{a_{22}} \tag{2.1.2}$$

Apply equation (2.1.2) to equation (2.0.1), we get:

$$a_{11}b_{11} - b_{11}\frac{a_{21}a_{12}}{a_{22}} = 1 \tag{2.1.3}$$

i.e.

$$b_{11}(\frac{a_{11}a_{22} - a_{21}a_{12}}{a_{22}}) = 1 \tag{2.1.4}$$

i.e.

$$b_{11} = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}} \tag{2.1.5}$$

2.2 b_{12}

from equation(2.0.2) we know that:

$$b_{22} = -b_{12}\frac{a_{11}}{a_{12}} \tag{2.2.1}$$

Apply equation (2.2.1) to equation (2.0.4), we get:

$$a_{21}b_{12} - b_{12}\frac{a_{11}a_{22}}{a_{12}} = 1 \tag{2.2.2}$$

i.e.

$$b_{12} = \frac{-a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \tag{2.2.3}$$

2.3 b_{21}

from equation(2.0.3) we know that:

$$b_{11} = -b_{21}\frac{a_{22}}{a_{21}} \tag{2.3.1}$$

Apply equation (2.3.1) to equation (2.0.1), we get:

$$-b_{21}\frac{a_{11}a_{22}}{a_{21}} + a_{12}b_{21} = 1 \tag{2.3.2}$$

i.e.

$$b_{21} = \frac{-a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \tag{2.3.3}$$

2.4 b_{22}

from equation(2.0.2) we know that:

$$b_{12} = -b_{22}\frac{a_{12}}{a_{11}} \tag{2.4.1}$$

Apply equation (2.4.1) to equation (2.0.4), we get:

$$-b_{22}\frac{a_{12}a_{21}}{a_{11}} + a_{22}b_{22} = 1 \tag{2.4.2}$$

i.e.

$$b_{22} = \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} \tag{2.4.3}$$

3 A^{-1}

$$MatrixB = \begin{bmatrix} b_{11} & b_{22} \\ b_{21} & b_{22} \end{bmatrix} \tag{3.0.1}$$

As we assumed Matrix B = Matrix A^{-1}

Apply $b_{11}, b_{12}, b_{21}, b_{22}$ above to MatrixB, i.e. Matrix A^{-1}:

$$MatrixA^{-1} = \begin{bmatrix} \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{-a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ \frac{-a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{bmatrix} \tag{3.0.2}$$

Extract scalar

$$\frac{1}{a_{11}a_{22} - a_{12}a_{21}} \tag{3.0.3}$$

then

$$MatrixA^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{3.0.4}$$