Matrix Factorisation

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1 Factorisation

Let's start with some simple examples of factorisation:

$$15 = 3 * 5 \tag{1.1}$$

$$x^2 + 4x + 3 = (x+1)(x+3) \tag{1.2}$$

Factorisation consists of writing a mathematical object as a product of several factors.

2 An example

Similarly, factorisation can apply to matrices, and it is easy if we take care when doing Gaussian elimination...

Let's start from eliminating a 4×4 Pascal matrix, meanwhile taking notes of each step:

(A Pascal matrix is the matrix whose first row and first column consist of all 1, and the rest elements are the sum of its upper and left element)
The 4×4 Pascal matrix, let's call it A:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

Gaussian Elimination:

firstly for row_2

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \xrightarrow{\iota_{21} = -1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

$\iota_{21} = -1$ means that to eliminate the element a_{21}, we need to subtract $1 \times row_1$, keep going on for row_3:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \xrightarrow{\iota_{31} = -1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 1 & 4 & 10 & 20 \end{bmatrix} \xrightarrow{\iota_{32} = -2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

Now for row_4:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 1 & 4 & 10 & 20 \end{bmatrix} \xrightarrow{\iota_{41} = -1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 9 & 19 \end{bmatrix} \xrightarrow{\iota_{42} = -3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix} \xrightarrow{\iota_{43} = -3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3 A=LU

From the Gaussian elimination above, we obtain an **UPPER Triangular Matrix** , let's give it a name **U**. It is obtained from the the 4×4 Pascal matrix A.

Now take the notes down:

$\iota_{21} = -1$
$\iota_{31} = -1$, $\iota_{32} = -2$
$\iota_{41} = -1$, $\iota_{42} = -3$, $\iota_{43} = -3$

Change their sign, and fill them into an Identity matrix:
(Here we are just looking at the steps, this operation is no magic, we will get to know how it works later.)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

Here we get a beautiful **LOWER Triangular Matrix**, we will call it **L**.

As the section name suggests, $A = LU$. Let's see if it's true!

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1*1 & 1*1 & 1*1 & 1*1 \\ 1*1 & (1*1)+(1*1) & (1*1)+(1*2) & (1*1)+(1*3) \\ 1*1 & (1*1)+(2*1) & (1*1)+(2*2)+(1*1) & (1*1)+(2*3)+(1*3) \\ 1*1 & (1*1)+(3*1) & (1*1)+(3*2)+(3*1) & (1*1)+(3*3)+(3*3)+(1*1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = A$$

It's true that $A = LU$, in which:

A is the original matrix
L is the Lower triangular matrix that comes from the operations of Gaussian elimination
U is the Upper triangular matrix that comes from Gaussian elimination

4 How does the "fill in and change sign" magic work?

If I were the reader, this is the very question I want to ask: Yes Gaussian elimination is fair enough, but the simple 'fill the operation marks in and change their signs' is not trivial at all.

However, this is not magic, but comes from Elementary matrices.

Let's work through another example (which is problem 1.1 in Meriel's tutorial sheet on matrix)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$$

4.1 From Identity matrix to Elementary matrix

Before we start, we need to know how the Elementary matrix works.

First let's introduce a 3×3 matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is straightforward that $IA = A$

However, if we add one component, say, at I_{21} off the main diagonal.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.1}$$

And calculate $E_{21}A$:

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 3 & -7 & 4 \end{bmatrix} \tag{4.2}$$

What we find out is that E_{21} executed a move that we often use in Gaussian elimination:

$e_{21} = 1$ means add $1 \times row_1$ to row_2 (Try to do it yourself to find out how that exactly works!)

Now it's time to continue the calculation:

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} (E_{21}A) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 3 & -7 & 4 \end{bmatrix} \tag{4.3}$$

$$E_{31}(E_{21}A) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & -10 & -2 \end{bmatrix} \tag{4.4}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 1 \end{bmatrix} (E_{31}E_{21}A) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & -10 & -2 \end{bmatrix} \tag{4.5}$$

$$E_{32}(E_{31}E_{21}A) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & -52 \end{bmatrix} = U \tag{4.6}$$

Notice that the final matrix is in Upper triangular form

4.2 The Sign Change

From the calculation above, we obtain that:

$$(E_{32}E_{31}E_{21})A = U \tag{4.7}$$

According to the matrix multiplication rule:

$$A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U \tag{4.8}$$

We have known that $A = LU$, now it's time to verify if $(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}) = L$

For the inverse matrix to a Elementary matrix, it goes:

$$E_{21}^{-1}E_{21} = I \tag{4.9}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.10}$$

This is where the sign change comes from, as moving to the right hand side force E_{21} to become E_{21}^{-1}, e_{21}changes its sign while the main diagonal remains the same.

We can easily obtain other E^{-1} by reverting the sign.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.11}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \rightarrow E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \tag{4.12}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 1 \end{bmatrix} \rightarrow E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{bmatrix} \tag{4.13}$$

4.3 Filling in

Now let's continue to verify $(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}) = L$

Before the calculation starts, think about how Elementary works in matrix multiplication, it is a simple step you use in Gaussian elimination.

$$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{bmatrix} \tag{4.14}$$

$$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 10 & 1 \end{bmatrix} \tag{4.15}$$

Notice the final matrix is in Lower triangular form.

4.4 Verification: $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = L$

Finally, we need to calculate $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$ and sees if it equals to A.

$$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & -52 \end{bmatrix} \tag{4.16}$$

$$= \begin{bmatrix} (1*1) & (1*1) & (1*2) \\ (-1*1) & (-1*1)+(1*-1) & (-1*2)+(1*5) \\ (3*1) & (3*1)+(10*-1) & (3*2)+(10*5)+(1*-52) \end{bmatrix} \tag{4.17}$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} = A \tag{4.18}$$

$(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}) = L$ and $A = LU$

5 Ax=b, The Right Hand Side

Don't forget Meriel's problem is about $A\vec{x} = \vec{b}$!!
as it went:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -2x_2 - x_1 + 3x_3 &= 1 \\ 4x_3 - 7x_2 + 3x_1 &= 10. \end{aligned}$$

So how does $A = LU$ help us to solve $A\vec{x} = \vec{b}$?
Combine the two equations together, it is easy to say that

$$LU\vec{x} = \vec{b} \tag{5.1}$$

i.e.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & -52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix} \tag{5.2}$$

We don't know the exact vector $U\vec{x}$, but we are sure it is a 3×1vector, let's call it $\vec{c}$, suppose that

$$\vec{c} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

while

$$LU\vec{x} = L\vec{c} = \vec{b} \tag{5.3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 10 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$$

First we calculate y_1, then y_2, y_3:

$$\begin{bmatrix} y_1 = 8 \\ y_2 = 9 \\ y_3 = -104 \end{bmatrix} = \vec{c} \tag{5.4}$$

$U\vec{x} = \vec{c}$ means,

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & -52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ -104 \end{bmatrix}$$

First we calculate x_3, then x_2, x_1:

$$\begin{bmatrix} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2 \end{bmatrix} \tag{5.5}$$

In one word: Solve $L\vec{c} = \vec{b}$, then solve $U\vec{x} = \vec{c}$.