STU11002

Statistical Analysis I

Learning objectives

- Define the terms probability distribution and random variable.
- Calculate the expected value, variance and standard deviation for a given sample of data
- State the properties of variance.
- Given a sample of data calculate Z, the standardized version of the random variable X.
- Understand the different types of discrete and continuous probability distributions. Use these distributions to calculate their associated expected values and variance parameters.
- State the properties of the Normal distribution.
- Use the Standard Normal distribution statistical tables to calculate probabilities.

Probability distributions

What is a probability distribution?

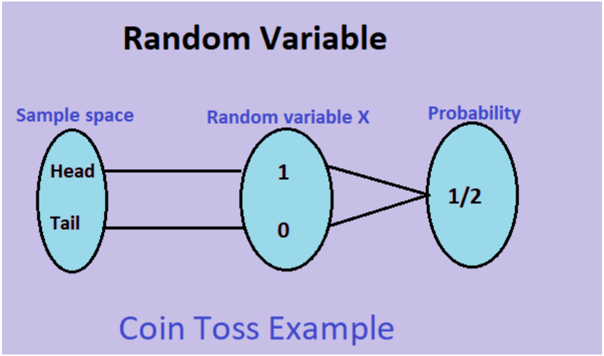
A probability distribution is a mathematical function that describes the probability of occurrence for different possible outcomes of a **random variable**.

Why is it important?

- Observed data may be similar to data arising from a particular distribution
- Simulation purposes
- Statistical theory

What is a random variable?

A random variable (r.v.) is a function defined over the sample space Ω, that associates a (real) numerical value to each one of the elementary events in the sample space.



Random variables

Different types

- **Discrete**. A discrete random variable is a r.v. that can only take a discrete set of values (in a given interval).
- **Continuous**. A continuous random variable is a r.v. that can take any value in a given interval

Example

The "roll of a dice" experiment can be described using a discrete random variable X, that takes (equally probably) integer values in the interval 1 to 6. These values are the elementary events $\{1, 2, 3, 4, 5, 6\}$, to which are associated the probabilities:
$\{P(X = 1); P(X = 2); P(X = 3); P(X = 4); P(X = 5); P(X = 6)\}$

Discrete random variables

Example - Sum of two dices

Value	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Discrete random variables

Probability distribution

The probability distribution function of a discrete r.v. X is a function that associates a probability $Pr(X = x_i)$ to each one of the possible values that X may take, x_i, for $i = 1, \dots, K$.

NOTE The r.v. is denoted using a capital letter, while its **realizations** are denoted using lower caps.

IMPORTANT:

$$\sum_{i=1}^{K} Pr(X = x_i) = \sum_{i=1}^{K} Pr(x_i) = 1; \qquad Pr(X = x_i) \geq 0$$

Discrete random variables

Cumulative distribution

The cumulative distribution function (CDF) of a discrete r.v. is a function that associates to each value x_i a **cumulative** probability $Pr(X \leq x_i)$:

$$F(x_i) = Pr(X \leq x_i) = \sum_{w \leq x_i} Pr(X \leq w)$$

Example

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$P(X \leq x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	$\frac{36}{36}$

Discrete random variables

Cumulative distribution

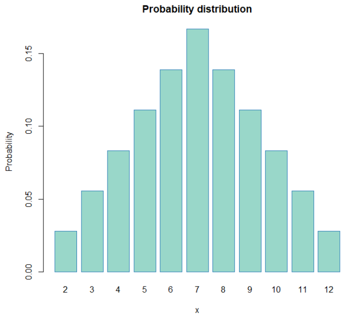
Consider a discrete r.v. in $[a, b]$. Its CDF can be defined as:

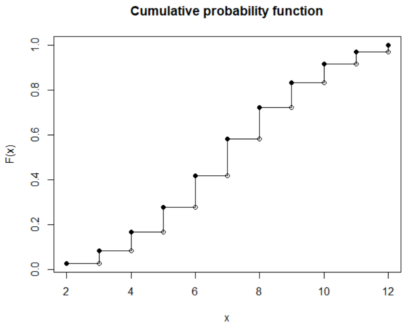
- $F(x) = 0$, for $-\infty < x < a$
- $0 \leq F(x) < 1$, for $a \leq x < b$
- $F(x) = 1$, for $b \leq x \leq \infty$

Important property: $F(x)$ is non-decreasing, meaning that:

$$x_1 < x_2 \implies F(x_1) \leq F(x_2)$$

Discrete random variables





Continuous random variables

As a continuous r.v. X can take **any value** in a given interval $[a, b]$, it does not really make sense to assign a probability to each of these single values x. However, we can associate probabilities to sub-intervals $[x_i, x_j]$:

$$Pr(x_i \leq X \leq x_j) = \int_{x_i}^{x_j} f(x)\, dx$$

The function $f(x)$ is called (probability) **density function** (pdf). The density function of a continuous r.v. X is a mathematical function whose underlying area in a given interval corresponds to the probability that X takes a value in said interval.

Continuous random variables

Properties of the density function

- $f(x)$ can not take negative values, that is: $f(x) \geq 0$. This ensures that the probability corresponding to any given interval is non-negative
- If the continuous random variable X takes values in $[a, b]$, then: $\int_a^b f(x)\,dx = 1$. That is, the total of the area under the density function is equal to 1
- The probability that X takes a specific value is 0. This is because the interval corresponding to a single, specific, value has width 0. Therefore, with continuous r.v.:

$$Pr(x_i \leq X \leq x_j) = Pr(x_i < X < x_j)$$

Continuous random variables

Cumulative distribution

Given a continuous r.v. X, its cumulative distribution function can be defined as:

$$F(x) = Pr(X \leq x) = \int_a^x f(w)\, dw$$

where a is the lower bound of the range of X. That is, the cumulative probability up to a given point x is computed as the area under the density function comprised between the lower bound of the range of X and the value x.

Note: also in the case of continuous random variables, $F(x)$ is not decreasing, meaning that:

$$x_1 < x_2 \implies F(x_1) \leq F(x_2)$$

Continuous random variables

Example

X is a continuous r.v. that takes values uniformly at random in the interval $[a, b]$. This type of r.v. is called **Uniform**, and it's distributed according to a **Uniform distribution**:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if} \quad a \leq x \leq b \\ 0 & \text{else} \end{cases}$$

Compute probabilities:

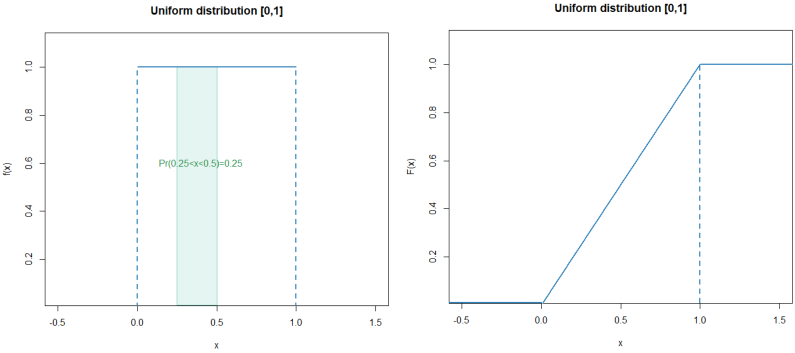
$$Pr(x_i \leq X \leq x_j) = \int_{x_i}^{x_j} \frac{1}{b-a}\, dx = \frac{x_j - x_i}{b-a}$$

Cumulative distribution function:

$$F(x) = \begin{cases} 0 & \text{if} \quad x < a \\ \frac{x-a}{b-a} & \text{if} \quad a \leq x \leq b \\ 1 & \text{if} \quad x > b \end{cases}$$

Continuous random variables

Example



Expected value

The **expected** value, $E(X)$, of a random variable X is the mean/average value of the random variable over a large number of trials.

- **Discrete r.v.**

$$E(x) = \sum_{i=1}^{K} x_i Pr(x_i)$$

- **Continuous r.v.**

$$E(x) = \int_a^b xf(x)\, dx$$

Expected value

Example - Sum of two dices

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Here the expected value can be computed as:

$$
\begin{aligned}
E(X) &= \sum_{i=1}^{11} x_i Pr(x_i) \\
&= 2\frac{1}{36} + 3\frac{2}{36} + 4\frac{3}{36} + 5\frac{4}{36} + 6\frac{5}{36} + 7\frac{6}{36} + 8\frac{5}{36} \\
&+ 9\frac{4}{36} + 10\frac{3}{36} + 11\frac{2}{36} + 12\frac{1}{36} = 7
\end{aligned}
$$

Expected value

Example - Uniform distribution

The expected value of a uniform continuous random variable with $[a, b]$ range is:

$$E(x) = \int_a^b xf(x)\,dx = \frac{1}{b-a}\left[\frac{b^2}{2} - \frac{a^2}{2}\right]$$
$$= \frac{1}{2}\frac{1}{b-a}(b-a)(b+a) = \frac{1}{2}(b+a)$$

So, for example, the expected value of a uniform random variable in $[0, 1]$ is 0.5.

Variance

The **variance**, $V(X)$, of a random variable X measures the average of the differences between the possible values of X and its expected value (where these differences are weighted by the values' probabilities).

- **Discrete r.v.**

$$V(x) = \sum_{i=1}^{K} [x_i - E(X)]^2 \, Pr(x_i)$$

- **Continuous r.v.**

$$E(x) = \int_a^b [x - E(X)]^2 \, f(x) \, dx$$

It is hence a measure of the "variability" associated to X.

Variance

- The variance can also be written as:

$$V(X) = E\left[[X - E(X)]^2\right]; \quad V(X) = E\left[X^2\right] - E\left[X\right]^2$$

- The square root of the variance is called **standard deviation**:

$$Sd(X) = \sqrt{V(X)}$$

Nice properties

- $E(Y) = E(d + sX) = d + sE(X)$
- $V(Y) = V(d + sX) = s^2 V(X)$

Variance

Example Sum of two dices

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Here the variance can be computed as:

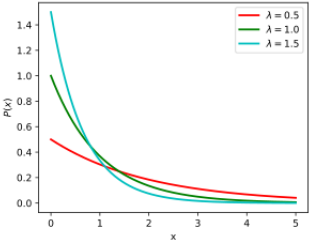
$$
\begin{aligned}
V(X) &= \sum_{i=1}^{K} [x_i - E(X)]^2 \, Pr(x_i) \\
&= (2-7)^2 \frac{1}{36} + (3-7)^2 \frac{2}{36} + (4-7)^2 \frac{3}{36} + (5-7)^2 \frac{4}{36} \\
&+ (6-7)^2 \frac{5}{36} + (7-7)^2 \frac{6}{36} + (8-7)^2 \frac{5}{36} + (9-7)^2 \frac{4}{36} \\
&+ (10-7)^2 \frac{3}{36} + (11-7)^2 \frac{2}{36} + (12-7)^2 \frac{1}{36} = 5.83
\end{aligned}
$$

Variance

Example - Exponential distribution

The density function for an **Exponential random variable** X is given by:

$$f(x) = \begin{cases} \lambda \exp(-\lambda x) & \textbf{if} \quad x \geq 0 \\ 0 \quad \text{else} \end{cases}$$



where $\lambda > 0$ is the **rate parameter**.
We can compute its expected value as:

$$E(X) = \int_0^\infty x\lambda \exp(-\lambda x)\, dx = \frac{1}{\lambda}$$

And its variance as:

$$V(X) = \int_0^\infty \left(x - \frac{1}{\lambda}\right)^2 \lambda \exp(-\lambda x)\, dx = \frac{1}{\lambda^2}$$

Standardization

Given a random variable X, its expected value $E(X)$, and its variance $V(X)$, we can compute:

$$Z = \frac{X - E(X)}{\sqrt{V(X)}} = \frac{X - E(X)}{Sd(X)}$$

where Z is another random variable, corresponding to the **standardized** X r.v.

$$E(Z) = E\left[\frac{1}{Sd(X)}(X - E(X))\right] = \frac{1}{Sd(X)}E\left[X - E(X)\right] = 0$$

$$\begin{aligned} V(Z) &= \frac{1}{Sd(X)^2}V\left(X - E(X)\right) \\ &= \frac{1}{Sd(X)^2}\left\{E\left[(X - E(X))^2\right] - E\left[X - E(X)\right]^2\right\} \\ &= \frac{1}{Sd(X)^2}\left\{Sd(X)^2 - 0\right\} = 1 \end{aligned}$$

Probability distributions

for discrete random variables

Discrete Uniform distribution

The r.v. $X \sim U(a, b)$ (where the symbol $\sim$ means "distributed as") is a r.v. that assumes integer values uniformly at random in the interval $[a, b]$:

$$Pr(X = x) = \frac{1}{b}; \quad x = a, a + 1, \ldots, b$$

where a is the minimum value that X can take, and n the total number of possible values that X can assume $(n = b - a + 1)$.

$$E(X) = \frac{a + b}{2}; \quad V(X) = \frac{n^2 - 1}{12}$$

Bernoulli distribution

The r.v. $X \sim Bern(\pi)$ is a r.v. that assumes value 1 with probability π, and value 0 with probability $(1-\pi)$:

$$Pr(X = x) = \pi^x (1-\pi)^{(1-x)}; \quad x = \{0, 1\}$$

Its expectation and variance are computed as:

$$E(X) = \sum_{i=1}^{2} x_i Pr(x_i) = 1\pi + 0(1-\pi) = \pi$$

$$V(X) = \sum_{i=1}^{2} [x_i - \pi]^2 \, Pr(x_i) = (1-\pi)^2\pi + (0-\pi)^2(1-\pi)$$
$$= \pi + \pi^3 - 2\pi^2 + \pi^2 - \pi^3 = \pi - \pi^2 = \pi(1-\pi)$$

Example: coin toss with $\pi = 0.5$ probability of a tail.

Binomial distribution

The r.v. $X \sim Binom(\pi, n)$ represents the number of successes in n independent Bernoulli trials, each one having success probability π:

$$Pr(X = x) = \binom{n}{x} \pi^x (1-\pi)^{(n-x)}; \quad x = 0, 1, \ldots, n$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ is the **binomial coefficient**.

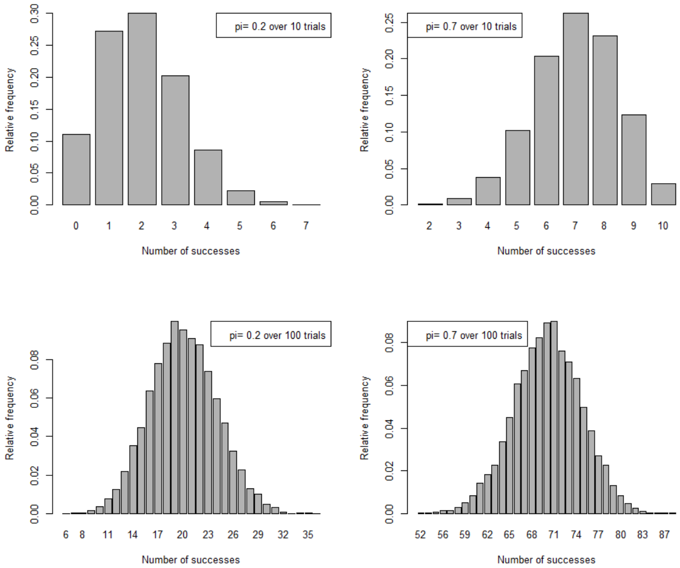
$$E(X) = E\left(X_1 + \cdots + X_n\right) = \sum_{i=1}^{n} \pi = n\pi$$

$$V(X) = V\left(X_1 + \cdots + X_n\right) = \sum_{i=1}^{n} \pi(1-\pi) = n\pi(1-\pi)$$

as the variance of the sum of independent r.v. corresponds to the sum of their variances.

Binomial distribution

Example



Binomial distribution

Note

- The distribution will tend to be symmetric around its mean value for large n values.
- The mean and the variance grow with n.

Example

The following table contains data regarding the number of exams passed in a year by a sample of 100 students (where 7 is the maximum possible number):

N.exams	0	1	2	3	4	5	6	7
Freq.	6	20	23	22	12	7	3	7
Rel. Freq.	0.06	0.20	0.23	0.22	0.12	0.07	0.03	0.07

Binomial distribution

Example

The arithmetic mean is $\bar{x}_a = 0 * 0.06 + 1 * 0.20 + 2 * 0.23 + 3 * 0.22 + 4 * 0.12 + 5 * 0.07 + 6 * 0.03 + 7 * 0.07 = 2.82$.

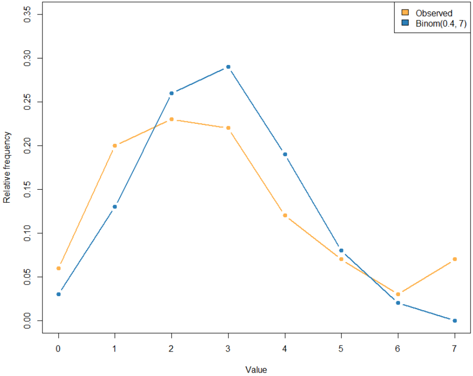
If the count of number of exams passed was the realization of a Binomial distribution with number of trials $n = 7$, then we could expect that $\bar{x}_a \approx n\pi$ (we'll see why in a few lectures). Hence: $\pi \approx 2.82/7 = 0.4$.
Your turn: Compute the expected number of exams passed under a Binomial distribution with $\pi = 0.4$ and $n = 7$?

Binomial distribution

Example

These probabilities are listed in the table below. Do you think that a *Binom*$(0.4, 7)$ is doing a good enough job in representing the observed data?

Binom	0.03	0.13	0.26	0.29	0.19	0.08	0.02	0.00



Poisson distribution

The r.v. $X \sim Pois(\lambda)$, with $\lambda > 0$, is a discrete r.v. that can take any value $x \geq 0$:

$$Pr(X = x) = \frac{\lambda^x}{x!} \exp(-\lambda); \quad x = 0, 1, 2, \ldots$$

The Poisson distribution gives the probability of an event happening a certain number of times (k) within a given interval of time or space. λ is the rate parameter. It is the expected number of events.
Note: The Poisson distribution assumes that the rate remains constant.

Its expectation and variance are given by:

$$E(X) = V(X) = \lambda$$

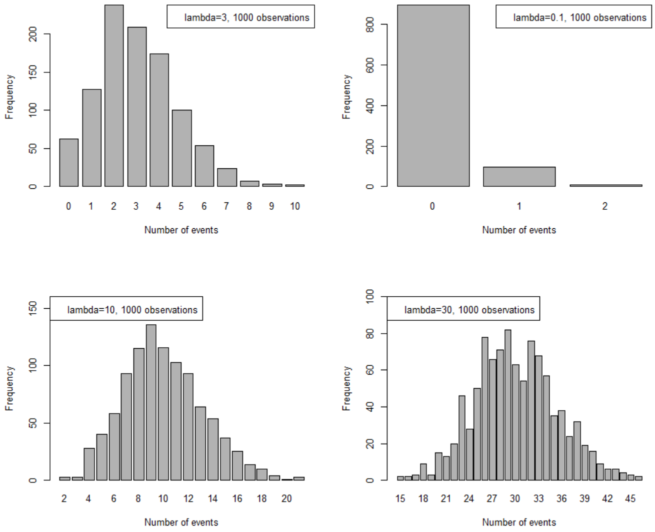
Poisson distribution

Properties

- The sum of Poisson r.vs is a Poisson r.v. That is, given $X_1 \sim Pois(\lambda_1), \dots, X_p \sim Pois(\lambda_P)$, $\sum_{p=1}^{P} X_p \sim Pois(\sum_{p=1}^{P} \lambda_p)$.
- Poisson Approximation to Binomial:
    - Given a Binomial r.v. X with success probability π and number of trials n, for large n values and small π values, X is approximately Poisson with rate $\lambda = n\pi$
    - When is this approximation appropriate? Some guidelines are: $n > 100$ and $\pi \leq 0.05$ or $\pi \leq 0.01$

Poisson distribution

Example



Probability distributions

for continuous random variables

Continuous Uniform distribution

X is a continuous r.v. that takes values uniformly at random in the interval $[a, b]$. This type of r.v. is called **Uniform**, and it's distributed according to a **Uniform distribution**, $X \sim Unif(a, b)$:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if} \quad a \leq x \leq b \\ 0 & \text{else} \end{cases}$$

Its expectation and variance are given by:

$$E(X) = \frac{1}{2}(a + b); \quad V(X) = \frac{1}{12}(b - a)^2$$

Example: You arrive into a building and are about to take an elevator to the your floor. Once you call the elevator, it will take between 0 and 40 seconds to arrive to you. We will assume that the elevator arrives uniformly between 0 and 40 seconds after you press the button. In this case a = 0 and b = 40.

Exponential distribution

If X is a continuous r.v. that takes values in the interval $[0, \infty)$ according to the following density function:

$$f(x) = \begin{cases} \lambda \exp\{-\lambda x\} & \text{if} \quad x \geq 0 \\ 0 \quad \text{else} \end{cases}$$

Then $X \sim Exp(\lambda)$ is an exponential r.v. with rate parameter $\lambda > 0$. Its expectation and variance are given by:

$$E(X) = \frac{1}{\lambda}; \quad V(X) = \frac{1}{\lambda^2}$$

Example: It can generally be used to describe the passing of time between events. So, for example, for a machine that has an average failure rate of 3 per hour, we could model the time between failures as $X \; Exp(\frac{1}{3})$.

Normal distribution

A continuous r.v. that is distributed according to a **Normal** (or **Gaussian**) distribution is denoted as $X \sim N(\mu, \sigma^2)$, and its density function is given by:

$$f(x) = \frac{1}{\sqrt{\sigma^2 2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

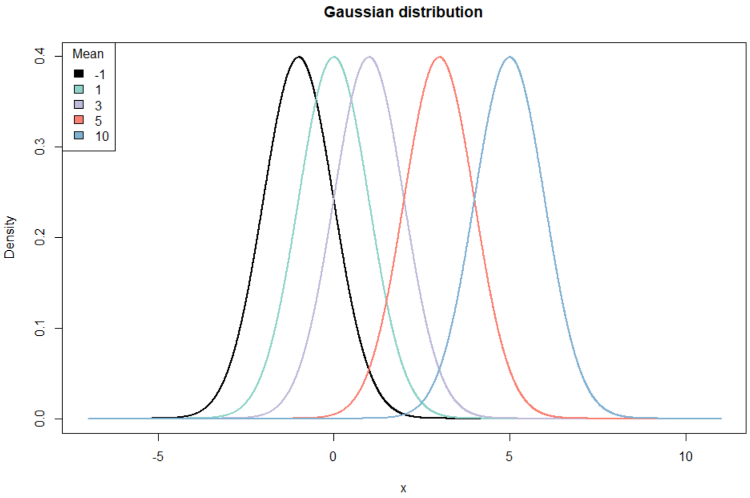
where $-\infty < \mu < \infty$ is the **mean** parameter, and $\sigma^2 > 0$ is the **variance** parameter.

Its expectation and variance are given by:

$$E(X) = \mu; \quad V(X) = \sigma^2$$

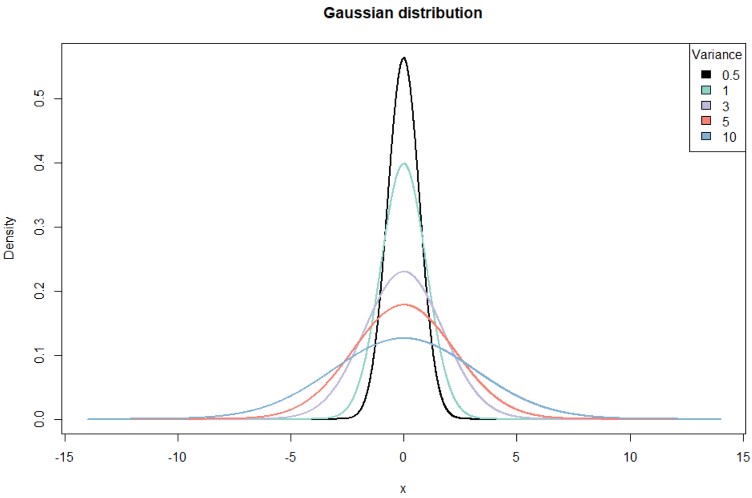
The Normal distribution is symmetric around its mean. Its also a unimodal distribution, with the mode coinciding with the mean and the median of the distribution.

Normal distribution



The mean parameter determines the **location**

Normal distribution



The variance parameter determines the **spread**

Normal distribution

Properties

- The Normal distribution is symmetric around its mean.
- Its a unimodal distribution, with the mode coinciding with the mean and the median of the distribution.
- A linear transformation of a Normal r.v. is still distributed as a Normal r.v. That is, if $X \sim N(\mu, \sigma^2)$ and $Y = a + bX$, then $Y \sim N(a + b\mu, b^2\sigma^2)$ (where $a \neq 0$)
- The sum of two Normal r.vs is a Normal r.v. with mean equal to the sum of the means and variance equal to the sum of the variances. That is, if $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, then $(X + Y) \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$
- Observations coming from a Normal distribution with mean μ and standard deviation $\sigma = \sqrt{\sigma^2}$:
  - 68% of them fall between $\mu \pm \sqrt(\sigma^2)$
  - 95% of them fall within $\mu \pm 1.96 * \sigma$
  - Approximately all of them (99.7%) fall within $\mu \pm 3\sigma$

Standard Normal

A Normal r.v. with mean $\mu = 0$ and variance $\sigma^2 = 1$, $Z \sim N(0, 1)$ is called a **Standard Normal** r.v.

Every Normal r.v. can be transformed into a Standard Normal r.v., by **standardization**. That is, if $X \sim N(\mu, \sigma^2)$, then:

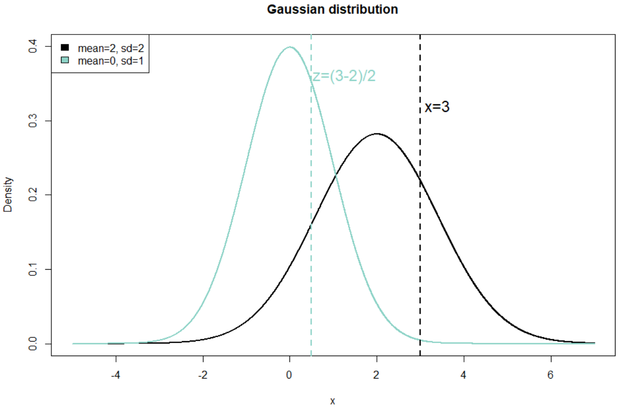
$$Z = \frac{X - \mu}{\sigma}; \quad Z \sim N(0, 1)$$

Properties

- Given the symmetry in $\mu = 0$, we have that $f(z) = f(-z)$
- Given the symmetry in $\mu = 0$, we have that $\Phi(-z) = 1 - \Phi(z)$, $z \geq 0$, where $\Phi(X) = F(X)$ denotes the cumulative distribution

Standard Normal

As we have seen, we can standardize any Normal r.v. $X \sim N(\mu, \sigma^2)$, by computing $Z = \frac{X-\mu}{\sigma}$, where $Z \sim N(0, 1)$. Therefore, a value x coming from X can be standardized as $z = (x - \mu)/\sigma$. Z values are often referred to as "Z scores".



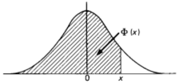
Standard Normal - Statistical tables

Statistical tables report probabilities for values under the Standard Normal distribution.

Different formats, generally column and row headings report the values for $z \geq 0$, while within the table the probability values are reported.

TABLE 4. THE NORMAL DISTRIBUTION FUNCTION

The function tabulated is $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$. $\Phi(x)$ is the probability that a random variable, normally distributed with zero mean and unit variance, will be less than or equal to x. When $x < 0$ use $\Phi(x) = 1 - \Phi(-x)$, as the normal distribution with zero mean and unit variance is symmetric about zero.



x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0·00	0·5000	**0·40**	0·6554	**0·80**	0·7881	**1·20**	0·8849	**1·60**	0·9452	**2·00**	0·97725
·01	·5040	**·41**	·6591	**·81**	·7910	**·21**	·8869	**·61**	·9463	**·01**	·97778
·02	·5080	**·42**	·6628	**·82**	·7939	**·22**	·8888	**·62**	·9474	**·02**	·97831
·03	·5120	**·43**	·6664	**·83**	·7967	**·23**	·8907	**·63**	·9484	**·03**	·97882
·04	·5160	**·44**	·6700	**·84**	·7995	**·24**	·8925	**·64**	·9495	**·04**	·97932
0·05	0·5199	**0·45**	0·6736	**0·85**	0·8023	**1·25**	0·8944	**1·65**	0·9505	**2·05**	0·97982
·06	·5239	**·46**	·6772	**·86**	·8051	**·26**	·8962	**·66**	·9515	**·06**	·98030
·07	·5279	**·47**	·6808	**·87**	·8078	**·27**	·8980	**·67**	·9525	**·07**	·98077
·08	·5319	**·48**	·6844	**·88**	·8106	**·28**	·8997	**·68**	·9535	**·08**	·98124
·09	·5359	**·49**	·6879	**·89**	·8133	**·29**	·9015	**·69**	·9545	**·09**	·98169

For example, for $z = 1.01$ (row 1.0, column 0.01), we found a probability value $Pr(0 \leq Z \leq 1.01) = 0.3438$. The cumulative probability up to z is $Pr(Z \leq z) = Pr(Z \leq 1.01) = 0.5 + Pr(0 \leq Z \leq 1.01) = 0.5 + 0.3438 = 0.8438$

Standard Normal - Statistical tables

Examples

Case A: If $z > 0$, for example $z = 0.4$, then:

$$Pr(Z \leq z) = Pr(Z \leq 0) + Pr(0 \leq Z \leq z) = 0.5 + 0.1554 = 0.6554$$

$$P(Z \geq z) = 1 - P(Z \leq z) = 1 - P(Z \leq 0.4) = 1 - 0.6554 = 0.3446$$

Case B: If $z < 0$, for example $z = -0.4$, then:

$$Pr(Z \leq z) = Pr(Z \geq -z) = 1 - P(Z \leq -z) = 0.3446$$

$$P(Z \geq z) = P(Z \leq -z) = P(Z \leq 0.4) = 0.6554$$

Standard Normal - Statistical tables

Examples - Probability of an interval

Consider $Pr(z_1 \leq Z \leq z_2)$. We can rewrite it as:

$$Pr(z_1 \leq Z \leq z_2) = Pr(Z \leq z_2) - Pr(Z \leq z_1)$$

Case A: If $z_1 > 0$ and $z_2 > 0$, e.g. $(z_1, z_2) = (0.4, 0.2)$:

$$Pr(Z \leq z_2) - Pr(Z \leq z_1) = Pr(Z \leq 0.4) - Pr(Z \leq 0.2) = 0.0761$$

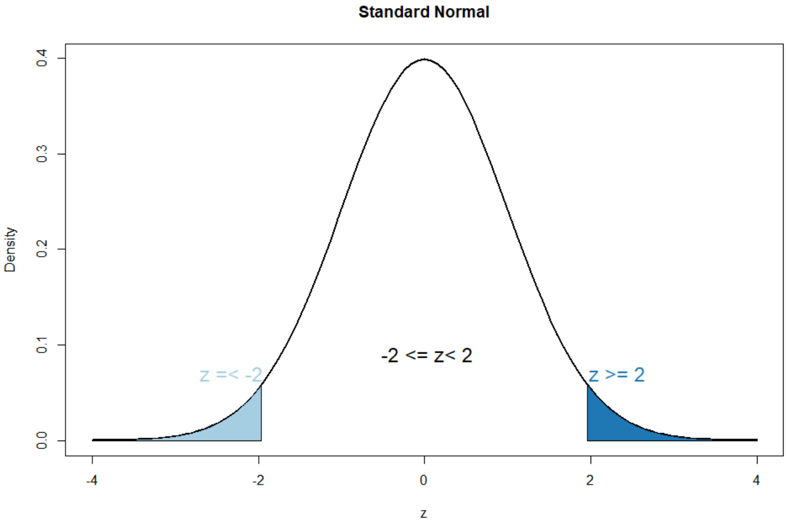
Case B: If $z_1 < 0$ and $z_2 < 0$, e.g. $(z_1, z_2) = (-0.4, -0.2)$:

$$Pr(Z \leq z_2) - Pr(Z \leq z_1) = Pr(Z \leq -z_1) - Pr(Z \leq -z_2) = 0.0761$$

Case C: If $z_1 < 0$ and $z_2 > 0$, e.g. $(z_1, z_2) = (-0.4, 0.2)$:

$$Pr(Z \leq z_2) - Pr(Z \leq z_1) = Pr(Z \leq z_2) - 1 + Pr(Z \leq -z_1) = 0.2347$$

Standard Normal - Statistical tables



Student's t-distribution

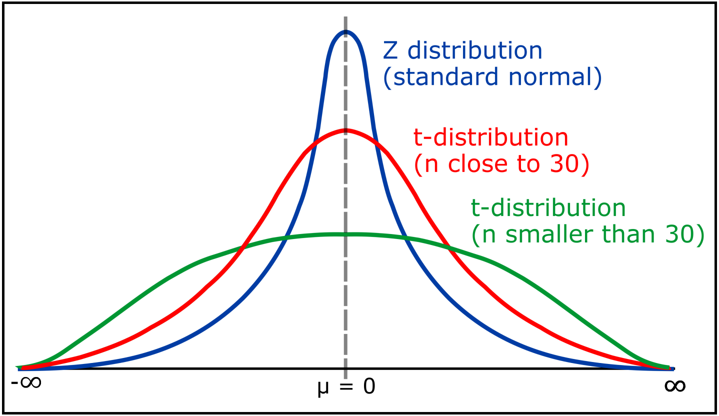
A continuous r.v. that is distributed according to a **Student's t-distribution** is denoted as $X \sim \mathcal{T}(\nu)$, where $\nu > 0$ is an integer, positive, number, denoting the **degrees of freedom** parameter.

Its expectation is defined only for $\nu \geq 2$, while its variance is defined only for $\nu \geq 3$

$$E(X) = 0; \quad V(X) = \frac{\nu}{\nu - 2}$$

The Student t-distribution origin story:
https://priceonomics.com/the-guinness-brewer-who-revolutionized-statistics/

Student's t-distribution



Like the Normal distribution, the Student's t-distribution is symmetric around its mean. The ν parameter denotes how "heavy" its tails are going to be, with lower values corresponding to heavier tails.

Exercises

Exercises

- Given a Uniform r.v. $X \sim Unif(2, 10)$, compute $Pr(3 \leq X \leq 5)$
- Consider the CDF of a discrete r.v., illustrated in the table below:

X	1	2	3	4	5	6
$Pr(X \leq x)$	0.1	0.15	0.3	0.4	0.7	1

  Compute $E(X)$ and $V(X)$
- Given $X \sim Ud(3, 5)$, compute $E(X)$ and $V(X)$
- What is the expected value of a Bernoulli r.v. with probability of success 0.2?
- What is the expected value of a Binomial r.v. with probability of success 0.2 over 17 trials?

Exercises

- What is the variance of a Poisson r.v. with rate parameter 5?
- What is the variance of the sum of the Poisson r.vs $X \sim Pois(2)$, $Y \sim Pois(4)$, and $Z \sim Pois(3)$?
- Consider a standard Gaussian distribution. Compute:
  - $Pr(Z \leq 1.3)$
  - $Pr(Z \geq -1)$
  - $Pr(-3 \leq Z \leq -1)$
  - $Pr(1.2 \leq Z \leq 1.3)$
  - $Pr(-2.1 \leq Z \leq 0.2)$