P.1

CSU11001 Homework II

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Q1:

$A=\begin{pmatrix}1&4&2\\-5&-8&-5\\6&6&5\end{pmatrix}$, it has following eigenvalues and associated eigenvectors:

$$\begin{cases}\lambda=-1, \text{ eigenvector } t\begin{pmatrix}-1\\0\\1\end{pmatrix}\\ \lambda=2, \text{ eigenvector } t\begin{pmatrix}0\\-1\\2\end{pmatrix}\\ \lambda=-3, \text{ eigenvector } t\begin{pmatrix}-1\\1\\0\end{pmatrix}\end{cases}$$

(a) Student Number: 233312$\underline{5}$0, $t=5$

$$\Rightarrow\begin{cases}\lambda=-1, \text{ eigenvector } \begin{pmatrix}-5\\0\\5\end{pmatrix}\\ \lambda=2, \text{ eigenvector } \begin{pmatrix}0\\-5\\10\end{pmatrix}\\ \lambda=-3, \text{ eigenvector } \begin{pmatrix}-5\\5\\0\end{pmatrix}\end{cases}$$

(b) For eigenvalues and eigenvectors, we have formula:

$(A-\lambda I)\vec{v}=\vec{0}$ (λ for eigenvalue, $\vec{v}$ for eigenvector)

To verify the $\lambda=-1$, $\vec{v}=\begin{pmatrix}-5\\0\\5\end{pmatrix}$ pair is correct, we can use left hand side (LHS) and right hand side (RHS) check In the given formula:

$RHS=\vec{0}=\begin{pmatrix}0\\0\\0\end{pmatrix}$;

$LHS=(A-\lambda I)\vec{v}$

$$=\left[\begin{pmatrix}1&4&2\\-5&-8&-5\\6&6&5\end{pmatrix}-(-1)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\cdot\begin{pmatrix}-5\\0\\5\end{pmatrix}$$

$$=\left[\begin{pmatrix}1&4&2\\-5&-8&-5\\6&6&5\end{pmatrix}-\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right]\begin{pmatrix}-5\\0\\5\end{pmatrix}=\left[\begin{pmatrix}1-(-1)&4&2\\-5&-8-(-1)&-5\\6&6&5-(-1)\end{pmatrix}\right]\begin{pmatrix}-5\\0\\5\end{pmatrix}$$

$$=\begin{pmatrix}2&4&2\\-5&-7&-5\\6&6&6\end{pmatrix}\begin{pmatrix}-5\\0\\5\end{pmatrix}=\begin{pmatrix}+2\times(-5)+4\times0+2\times5\\-5\times(-5)+(-7)\times0+(-5)\times5\\6\times(-5)+6\times0+6\times5\end{pmatrix}=\begin{pmatrix}-10+0+10\\25+0-25\\-30+0+30\end{pmatrix}$$

$=\begin{pmatrix}0\\0\\0\end{pmatrix}$. Therefore, LHS equals to RHS $\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$. The given conditions are correct.

Homework II

Zicheng Liang
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Q1 (C):

For matrix diagonalisation ~~(D =)~~ $D=P^{-1}AP$, given A is a 3x3 matrix and 3 linearly independent eigenvectors, we obtain that A is diagonalisable, and we can construct such a pair of P and D:

when $D=\begin{pmatrix}-1&0&0\\0&2&0\\0&0&-3\end{pmatrix}$, $P=\begin{pmatrix}-5&0&-5\\0&-5&5\\5&10&0\end{pmatrix}$ (take eigenvectors as columns corresponding to its eigenvalue's position)

~~(According to Cramer's rule)~~ P^{-1} could be obtained using cofactors: $P^{-1}=\frac{1}{det(P)}(\tilde{P})^T$

$$\tilde{P}=\begin{pmatrix}+\begin{vmatrix}-5&5\\10&0\end{vmatrix} & -\begin{vmatrix}0&5\\5&0\end{vmatrix} & +\begin{vmatrix}0&-5\\5&10\end{vmatrix}\\ -\begin{vmatrix}0&-5\\10&0\end{vmatrix} & +\begin{vmatrix}-5&-5\\5&0\end{vmatrix} & -\begin{vmatrix}-5&0\\5&10\end{vmatrix}\\ +\begin{vmatrix}0&-5\\-5&5\end{vmatrix} & -\begin{vmatrix}-5&-5\\0&5\end{vmatrix} & +\begin{vmatrix}-5&0\\0&-5\end{vmatrix}\end{pmatrix}=\begin{pmatrix}+[(-5)\times0-5\times10] & -(0\times0-5\times5) & +[0\times10-(-5)\times5]\\ -[0\times0-(-5)\times10] & +[(-5)\times0-(-5)\times5] & -[(-5)\times10-5\times0]\\ +[0\times5-(-5)\times(-5)] & -[(-5)\times5-(-5)\times0] & +[(-5)\times(-5)-0\times0]\end{pmatrix}$$

$$\tilde{P}=\begin{pmatrix}+(0-50) & -(0-25) & +(0+25)\\ -(0+50) & +(0+25) & -(-50-0)\\ +(0-25) & -(-25-0) & +(25-0)\end{pmatrix}=\begin{pmatrix}-50&25&25\\-50&25&50\\-25&25&25\end{pmatrix}$$

$$\tilde{P}^T=\begin{pmatrix}-50&-50&-25\\25&25&25\\25&50&25\end{pmatrix}=25\begin{pmatrix}-2&-2&-1\\1&1&1\\1&2&1\end{pmatrix}$$

$$det(P)=-5\times(-50)+0\times25+(-5)\times25=250-125$$
$$=125$$

$$P^{-1}=\frac{1}{det(P)}\cdot(\tilde{P})^T=\frac{1}{125}\cdot25\begin{pmatrix}-2&-2&-1\\1&1&1\\1&2&1\end{pmatrix}=\frac{1}{5}\cdot\begin{pmatrix}-2&-2&-1\\1&1&1\\1&2&1\end{pmatrix}$$

Let us do the LHS-RHS check on $D=P^{-1}AP$

$LHS=D=\begin{pmatrix}-1&0&0\\0&2&0\\0&0&-3\end{pmatrix}$; $RHS=P^{-1}AP=\frac{1}{5}\begin{pmatrix}-2&-2&-1\\1&1&1\\1&2&1\end{pmatrix}\cdot\begin{pmatrix}1&4&2\\-5&-8&-5\\6&6&5\end{pmatrix}\cdot5\begin{pmatrix}-1&0&-1\\0&-1&1\\1&2&0\end{pmatrix}$

$$RHS=\begin{pmatrix}-2&-2&-1\\1&1&1\\1&2&1\end{pmatrix}\begin{pmatrix}1&4&2\\-5&-8&-5\\6&6&5\end{pmatrix}\begin{pmatrix}-1&0&-1\\0&-1&1\\1&2&0\end{pmatrix}\xRightarrow{P^{-1}A}\begin{pmatrix}-2\times1+(-2)\times(-5)+(-1)\times6 & -2\times4+(-2)\times(-8)+(-1)\times6 & -2\times2+(-2)\times(-5)+(-1)\times5\\ 1\times1+1\times(-5)+(1)\times6 & 1\times4+1\times(-8)+1\times6 & 1\times2+1\times(-5)+1\times5\\ 1\times1+2\times(-5)+(1)\times6 & 1\times4+2\times(-8)+1\times6 & 1\times2+2\times(-5)+1\times5\end{pmatrix}$$

$$P^{-1}A=\begin{pmatrix}-2+10-6 & -8+16-6 & -4+10-5\\ 1-5+6 & 4-8+6 & 2-5+5\\ 1-10+6 & 4-16+6 & 2-10+5\end{pmatrix}=\cancel{\begin{pmatrix}14&14&11\\2&2&2\\-3&-6&-3\end{pmatrix}}=\begin{pmatrix}2&2&1\\2&2&2\\-3&-6&-3\end{pmatrix}$$

$$P^{-1}AP=\begin{pmatrix}2&2&1\\2&2&2\\-3&-6&-3\end{pmatrix}\begin{pmatrix}-1&0&-1\\0&-1&1\\1&2&0\end{pmatrix}=\cancel{(14\times(-1))}\begin{pmatrix}2\times(-1)+2\times0+1\times1 & 2\times0+2\times(-1)+1\times2 & 2\times(-1)+2\times1+1\times0\\ 2\times(-1)+2\times0+2\times1 & 2\times0+2\times(-1)+2\times2 & 2\times(-1)+2\times1+2\times0\\ -3\times(-1)+(-6)\times0+(-3)\times1 & -3\times0+(-6)\times(-1)+(-3)\times2 & -3\times(-1)+(-6)\times1+(-3)\times0\end{pmatrix}$$

Homework II

Zicheng Liang
23331250

Q1(c) continue:

$$P^{-1}AP = \begin{pmatrix} -2+1 & -2+2 & -2+2 \\ -2+2 & -2+4 & -2+2 \\ 3-3 & 6-6 & 3-6 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

That is to say: $RHS = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} = LHS$, the result we obtained is valid (D, P, P^{-1})

$$A = PDP^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

(d)

$A^4 = PD^4P^{-1}$ (This is because $A^4 = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1}) = P\cdot D\cdot(P^{-1}\cdot P)\,D \cdots\cdots P^{-1} = PD^4P^{-1}$)

$$A^4 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} (-1)^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & (-3)^4 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1\times1+0\times0+(-1)\times0 & -1\times0+0\times16+(-1)\times0 & -1\times0+0\times0+(-1)\times81 \\ 0\times1+(-1)\times0+1\times0 & 0\times0+(-1)\times16+1\times0 & 0\times0+(-1)\times0+1\times81 \\ 1\times1+2\times0+0\times0 & 1\times0+2\times16+0\times0 & 1\times0+2\times0+0\times81 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & -81 \\ 0 & -16 & 81 \\ 1 & 32 & 0 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1\times(-2)+0\times1+(-81)\times1 & -1\times(-2)+0\times1+(-81)\times2 & -1\times(-1)+0\times1+(-81)\times1 \\ 0\times(-2)+(-16)\times1+81\times1 & 0\times(-2)+(-16)\times1+81\times2 & 0\times(-1)+(-16)\times1+81\times1 \\ 1\times(-2)+32\times(+1)+0\times(+1) & 1\times(-2)+32\times1+0\times2 & 1\times(-1)+32\times1+0\times1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-81 & 2-162 & 1-81 \\ -16+81 & -16+162 & -16+81 \\ -2+32 & -2+32 & -1+32 \end{pmatrix} = \begin{pmatrix} -79 & -160 & -80 \\ 65 & 146 & 65 \\ 30 & 30 & 31 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} -79 & -160 & -80 \\ 65 & 146 & 65 \\ 30 & 30 & 31 \end{pmatrix}$$

Homework II

Zicheng Liang
23331250

Q2

(a) student ID: 2 3 3 3 1 2 5 0

The vector is $\vec{v}=\begin{pmatrix}2\\3\\1\\5\end{pmatrix}$

(b) Suppose that a, b, c, d are all real numbers, as given in the question given:

$$a\vec{v_1}+b\vec{v_2}+c\vec{v_3}+d\vec{v_4}=\vec{v}$$

i.e. $a\begin{pmatrix}1\\-1\\2\\1\end{pmatrix}+b\begin{pmatrix}3\\1\\2\\1\end{pmatrix}+c\begin{pmatrix}0\\-1\\3\\1\end{pmatrix}+d\begin{pmatrix}-1\\0\\3\\2\end{pmatrix}=\begin{pmatrix}2\\3\\1\\5\end{pmatrix}$

i.e. $\begin{cases}1\cdot a+3\cdot b+0\cdot c+(-1)d=2\\ -1\cdot a+1\cdot b+(-1)\cdot c+0\cdot d=3\\ 2\cdot a+2\cdot b+3\cdot c+3\cdot d=1\\ 1\cdot a+1\cdot b+1\cdot c+2\cdot d=5\end{cases}$ which is equivalent to $\begin{pmatrix}1&3&0&-1\\-1&1&-1&0\\2&2&3&3\\1&1&1&2\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}2\\3\\1\\5\end{pmatrix}$

In order to get the result for $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$, we can construct an augmented matrix and do Gaussian elimination

$$\left(\begin{array}{cccc|c}1&3&0&-1&2\\-1&1&-1&0&3\\2&2&3&3&1\\1&1&1&2&5\end{array}\right)\xrightarrow{R_2+R_1}\left(\begin{array}{cccc|c}1&3&0&-1&2\\0&4&-1&-1&5\\2&2&3&3&1\\1&1&1&2&5\end{array}\right)\xrightarrow{R_3-2R_1}\left(\begin{array}{cccc|c}1&3&0&-1&2\\0&4&-1&-1&5\\0&-4&3&5&-3\\1&1&1&2&5\end{array}\right)$$

$$\xrightarrow{R_3+R_2}\left(\begin{array}{cccc|c}1&3&0&-1&2\\0&4&-1&-1&5\\0&0&2&4&2\\1&1&1&2&5\end{array}\right)\xrightarrow{R_4-R_1}\left(\begin{array}{cccc|c}1&3&0&-1&2\\0&4&-1&-1&5\\0&0&2&4&2\\0&-2&1&3&3\end{array}\right)\xrightarrow{R_4+\frac{1}{2}R_2}\left(\begin{array}{cccc|c}1&3&0&-1&2\\0&4&-1&-1&5\\0&0&2&4&2\\0&0&+\frac{1}{2}&\frac{5}{2}&\frac{11}{2}\end{array}\right)$$

$$\xrightarrow{R_4-\frac{1}{4}R_3}\left(\begin{array}{cccc|c}1&3&0&-1&2\\0&4&-1&-1&5\\0&0&2&4&2\\0&0&0&\frac{3}{2}&5\end{array}\right)\xrightarrow[R_4\times\frac{2}{3}]{R_2\times\frac{1}{4},\ R_3\times\frac{1}{2}}\left(\begin{array}{cccc|c}1&3&0&-1&2\\0&1&-\frac{1}{4}&-\frac{1}{4}&\frac{5}{4}\\0&0&1&2&1\\0&0&0&1&\frac{10}{3}\end{array}\right)\xrightarrow{R_3-2R_4}\left(\begin{array}{cccc|c}1&3&0&-1&2\\0&1&-\frac{1}{4}&-\frac{1}{4}&\frac{5}{4}\\0&0&1&0&\frac{-17}{3}\\0&0&0&1&\frac{10}{3}\end{array}\right)\xrightarrow{R_2+\frac{1}{4}R_3+\frac{1}{4}R_4}\left(\begin{array}{cccc|c}1&3&0&-1&2\\0&1&0&0&\frac{2}{3}\\0&0&1&0&\frac{-17}{3}\\0&0&0&1&\frac{10}{3}\end{array}\right)$$

$$\xrightarrow{R_1-3R_2+R_4}\left(\begin{array}{cccc|c}1&0&0&0&\frac{10}{3}\\0&1&0&0&\frac{2}{3}\\0&0&1&0&\frac{-17}{3}\\0&0&0&1&\frac{10}{3}\end{array}\right)\Rightarrow\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}\frac{10}{3}\\\frac{2}{3}\\\frac{-17}{3}\\\frac{10}{3}\end{pmatrix}\Leftrightarrow\begin{cases}a=\frac{10}{3}\\b=\frac{2}{3}\\c=\frac{-17}{3}\\d=\frac{10}{3}\end{cases}$$

$\longrightarrow$ continue on P_5

Homework Ⅱ

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Q2 (b) continued:

$$\frac{10}{3}\begin{pmatrix}1\\-1\\2\\1\end{pmatrix}+\frac{2}{3}\begin{pmatrix}3\\1\\2\\1\end{pmatrix}(+)^{x}-\frac{17}{3}\begin{pmatrix}0\\-1\\3\\1\end{pmatrix}+\frac{10}{3}\begin{pmatrix}-1\\0\\3\\2\end{pmatrix}=\begin{pmatrix}2\\3\\1\\5\end{pmatrix}$$

Q2 (c):

If $\left\{\begin{pmatrix}1\\-1\\2\\1\end{pmatrix},\begin{pmatrix}3\\1\\2\\1\end{pmatrix},\begin{pmatrix}0\\-1\\3\\1\end{pmatrix},\begin{pmatrix}-1\\0\\3\\2\end{pmatrix}\right\}$ forms a basis for $\mathbb{R}^4$, it should fulfill two conditions as follow:

i) the four vectors should be linearly independent

ii) the four vectors should construct a complete set for $\mathbb{R}^4$

As we have known:

i) is equivalent to: the reduced row echelon form of matrix A has a leading one in every column.

ii) is equivalent to: the reduced row echelon form of matrix A has a leading one in every row.

As we have obtained in Q2(b), the reduced echelon form of matrix A is:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$ which satisfy the two conditions at the same time

Therefore, $\left\{\begin{pmatrix}1\\-1\\2\\1\end{pmatrix},\begin{pmatrix}3\\1\\2\\1\end{pmatrix},\begin{pmatrix}0\\-1\\3\\1\end{pmatrix},\begin{pmatrix}-1\\0\\3\\2\end{pmatrix}\right\}$ indeed forms a basis for $\mathbb{R}^4$.

Additionally, the reason why rule i) and ii)'s equivalents are valid is as follows:

By eliminating the original matrix $\begin{pmatrix}1&3&0&-1\\-1&1&-1&0\\2&2&3&3\\1&1&1&2\end{pmatrix}$ to its reduced row-echelon form $\cancel{\begin{pmatrix}1&3&0&-1\\0&1&-\frac{1}{4}&-\frac{1}{4}\\0&0&1&2\\0&0&0&1\end{pmatrix}}^{x}$, $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

we are actually checking the linear independence of the ~~matrix~~ row vectors,
because the elimination progress is actually doing linear combination, and each row vector is checked if it could be a combination of other three row vectors. If its is, then that row vector will become $\vec{0}$.

For a m×n matrix, if a row is reduced to $\vec{0}$, then it means none of the column vectors has a parameter in that dimension, hence the set of column vectors cannot span to that dimension. Finally the set is not complete for $\mathbb{R}^m$. For instance: $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$ cannot span to z axis. ⇒ ii)'s equivalent is valid.

When the matrix is eliminated to the reduced row echelon form with a leading one in every column. then only $0\cdot\vec{c_1}+0\cdot\vec{c_2}+\cdots+0\vec{c_n}=\vec{0}$ as there is a parameter for each dimension and it only exists in one column.

Therefore i)'s equivalent is valid.

Homework II

2333 1250

Q3 (a)

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$\Rightarrow$ 2×3 matrix B: $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 2 \end{pmatrix}$

(b) First reduce B to its reduced-row echelon form:

$$\begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 2 \end{pmatrix} \xrightarrow{R_2 - \frac{3}{2}R_1} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow{R_1 + 2R_2} \begin{pmatrix} 2 & 0 & 2 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_2 \times -\frac{2}{3}]{R_1 \times \frac{1}{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} \end{pmatrix}$$

there are leading ones on the first and second column.

$\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\}$ form a column space of $\mathbb{R}^2$ ($\mathcal{C}(B)$)

(c) the null space is the set of vectors $\vec{x}$ which makes all $B\vec{x} = 0$.

Assume $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, then based on the reduced row echelon form we obtained from Q3(b):

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 - \frac{1}{3}x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -x_3 \\ x_2 = \frac{1}{3}x_3 \end{cases}, \Rightarrow \vec{x} = \begin{pmatrix} -x_3 \\ \frac{1}{3}x_3 \\ x_3 \end{pmatrix}.$$ (take x_3 as $t \in \mathbb{R}$.)

$\vec{x} = \begin{pmatrix} -t \\ \frac{1}{3}t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ \frac{1}{3} \\ 1 \end{pmatrix}$ ($t \in \mathbb{R}$), therefore $\left\{ \begin{pmatrix} -1 \\ \frac{1}{3} \\ 1 \end{pmatrix} \right\}$ forms a basis for $\mathcal{N}(B)$