Besicovitch-Federer theorem for $\mathscr{C}^1$ mappings

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1. Introduction &

Throughout the whole poster $\Sigma \subset \Omega$ will be purely m-unrectifiable set with finite Hausdorff measure $\mathscr{H}^m(\Sigma) < +\infty$, where Ω is domain of $\mathbb{R}^n$. Let $Gr(n,m)$ be the Grassmannian manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$, with rotationally invariant measure $\gamma_{n,m}$. Recall the classical *Besicovitch-Federer projection theorem.*

Theorem. *Σ is purely m-unrectifiable iff $\mathscr{H}^m(P_V\Sigma) = 0$ for $\gamma_{n,m}$ almost all $V \in Gr(n,m)$, where $P_V : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal projection onto V.*

Now we can ask:

♠ What can we say about mappings that "locally look like" orthogonal projection?

♣ What property of space of mappings will mimic the $\gamma_{n,m}$-almost all property on $Gr(n,m)$?

♠

Let $\mathscr{C}^1_{=m}(\Omega, \mathbb{R}^n)$ be the class of continuously differentiable maps, such that the rank of Jacobian matrix is equal to m. For functions in $\mathscr{C}^1_{=m}$ we know what "locally look like" means.

Lemma (Constant rank theorem)**.** *Suppose $f \in \mathscr{C}^k_{=m}(\Omega, \mathbb{R}^n)$. For $x \in \Omega$ there exist*

open sets
$$x \in U_x \subset \Omega \text{ and } f(x) \in W_{f(x)} \subset \mathbb{R}^n,$$
and $\mathscr{C}^k$-diffeomorphisms
$$\phi_x : U_x \to \mathbb{R}^n \text{ and } \psi_{f(x)} : W_{f(x)} \to \mathbb{R}^n,$$
such that on the set $\phi_x(U_x)$ we have
$$(\psi_{f(x)} \circ f \circ \phi_x{}^{-1}) = P_V.$$

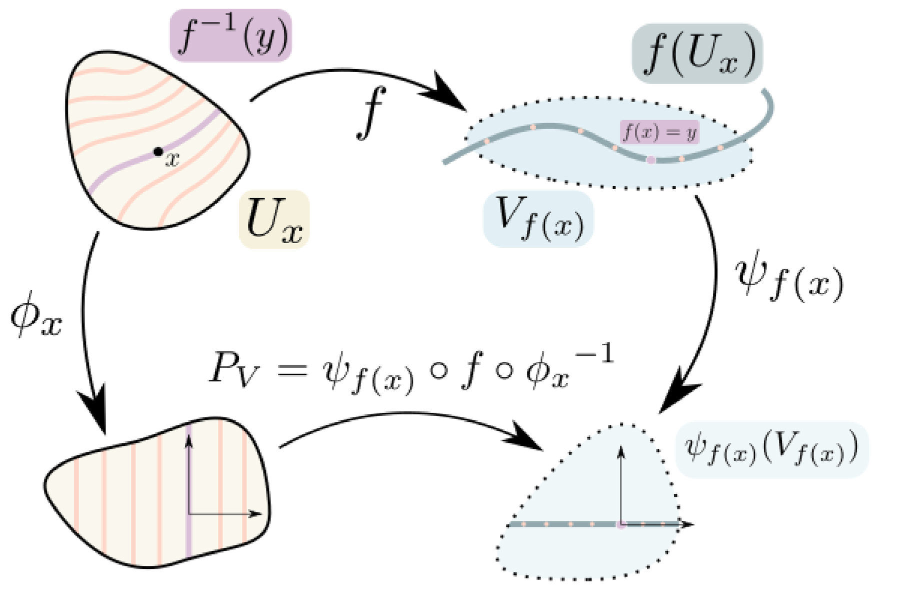

Figure 1: *Picture commentary explaining constant rank theorem.*

Fix $x \in \Sigma$, for now denote $\psi = \psi_{f(x)}$, $\phi = \phi_x$, and sets U and W analogously. We shall try to change mapping f on the set U in such a way that the image of Σ under this modified function is of zero $\mathscr{H}^m$-measure. Observe that we can write the mapping f in the form:

$$\begin{aligned} f &= \psi^{-1} \circ \psi \circ f \circ \phi^{-1} \circ \phi \\ &= \psi^{-1} \circ P_{V(x)} \circ \phi. \end{aligned} \tag{1}$$

Now since the pure m-unrectifiability is kept by bi-Lipschitz diffeomorphisms, the set $\phi(\Sigma)$ is also unrectifiable, hence we want to change $P_{V(x)}$ to $P_{\widetilde{V}(x)}$ such that

$$\mathscr{H}^m\left(P_{\widetilde{V}(x)}\phi(\Sigma)\right) = 0$$

We can do this change by introducing a small rotation θ because composition of rotation and projection is equal to projection on the rotated subspace i.e.

$$P_V \circ \theta = \theta \circ P_{\theta^{-1}(V)}.$$

We have to choose a rotation such that

$$\mathscr{H}^m\left(P_{\theta^{-1}(V)}\phi(\Sigma)\right) = 0, \tag{2}$$

but since almost all m-planes are like that we can find rotation arbitrarily close to identity and fulfilling condition (2).

Note that ϕ^{-1} (hence also $P_{V(x)}$ in (1)) is only defined on $\phi(U)$ and we do not have control on $\phi(U) \cap \theta\phi(U)$. Therefore, in order to remove this inconvenience, we assign to each point $x \in \Omega$ radius $r > 0$ such that $B(x,r) \subset \phi_x(U_x)$ and the neighbourhood

$$U(x,r) \coloneqq \phi_x^{-1}(B(0,r)) \subset U_x.$$

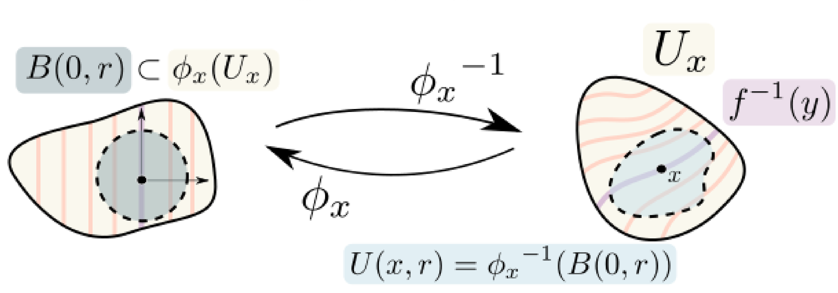

Figure 2: *The definition of $U(x,r)$.*

The modified mapping $f_\theta \coloneqq f \circ \Xi_\theta$ is the composition:

$$f \circ \Xi_\theta \coloneqq f \circ \phi^{-1} \circ \theta \circ \phi.$$

Note also that both ψ and θ are bi-Lipschitz and hence they carry $\mathscr{H}^m$-zero sets to such. Therefore we have

$$\begin{aligned} 0 &= \mathscr{H}^m(P_{\theta^{-1}(V)}(\phi(\Sigma))) \\ &= \mathscr{H}^m(\theta \circ P_{\theta^{-1}(V)}(\phi(\Sigma))) \\ &= \mathscr{H}^m(\psi^{-1} \circ P_V \circ \theta(\phi(\Sigma))) \\ &= \mathscr{H}^m(\psi^{-1} \circ \psi \circ f \circ \phi^{-1} \circ \theta \circ \phi(\Sigma)) \\ &= \mathscr{H}^m(f \circ \Xi_\theta(\Sigma)). \end{aligned}$$

♣

Choosing θ sufficiently close to identity we can provide that f_θ is as close to f in $\mathscr{C}^1$ topology as we wish. Thus we changed "almost all" property on m-planes to local density in $\mathscr{C}^1$.

Now the challenge is to glue together neighbourhoods $U(x,r)$ in order to obtain modification on the whole set Ω. On intersecting domains $\{U(x_i,r_i)\}_{i=1,2}$ rotations guide points to different places and we are unable to define one Ξ on the sum.

2. The Cover & Poking holes in Swiss cheese

Luckily for us $U(x,r)$ has $\mathscr{C}^1$ boundary, and it is enough to perform a procedure which will avoid inability-to-define problem mentioned above. What we would like to have is a countable cover of Σ by disjoint open sets in $\mathbb{R}^n$ such that every boundary intersects Σ on the set of $\mathscr{H}^m$-measure zero. The second part is easy because there are unaccountably many (param. r) disjoint $(n-1)$-dimensional boundaries and $\mathscr{H}^m(\Sigma) < \infty$, thus only countably many are "bad". Covering up to the measure zero is also easy: we can find countable cover of the set Σ by $V_i = U(x_i, r_i)$ in such a way that

$$\mathscr{H}^m \llcorner \Sigma(\Omega \setminus \bigcup_i V_i) = 0 \quad \text{where every } V_i \text{ have the property}$$
$$\mathscr{H}^m \llcorner \Sigma(\partial V_i) = 0.$$

To produce disjoint family first define auxiliary family

$$\widetilde{\mathfrak{U}} = \left\{ \begin{aligned} &\mathrm{int}(V_1),\ \mathrm{int}(V_2) \setminus \overline{V_1},\ \mathrm{int}(V_3) \setminus \overline{(V_1 \cup V_2)},\ \ldots \\ &\qquad \ldots, \mathrm{int}(V_n) \setminus \overline{\bigcup_{i<n} V_i},\ \ldots \end{aligned} \right\}.$$

Define $\mathfrak{U}$ as the family of connected components of the elements of $\widetilde{\mathfrak{U}}$ and denote elements of $\mathfrak{U}$ by $(U_i)_{i\in\mathbb{N}}$.

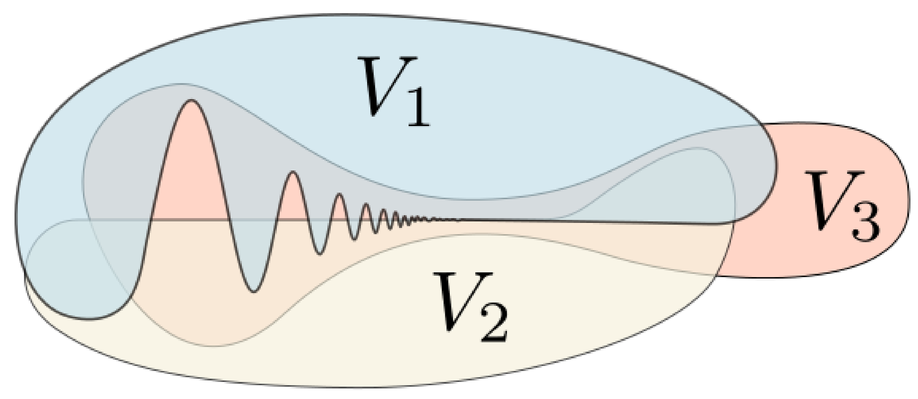

Figure 3: *Example of countably many connected components in the 3rd element of $\widetilde{\mathfrak{U}}$.*

Take $U_1 \in \mathfrak{U}$ (treat it like generic element of $\mathfrak{U}$), we have to provide a $\mathscr{C}^1$ diffeomorphism $\zeta : U_1 \to U_1$ with identity on the boundary such that $\mathscr{H}^n(f \circ \zeta(\Sigma \cap U_1)) = 0$. We will describe iterative construction of diffeomorphisms such that the limit is the desired one.

1) Divide U_1 into four regions, a "hole" in the middle and three "collars":
$$\begin{aligned} H_{3\delta} &\coloneqq U_1 \setminus B_{3\delta}(\partial U_1) \\ \mathrm{col}_\delta &\coloneqq U_1 \cap B_\delta(\partial U_1) \end{aligned}$$
$\mathrm{col}_{(2\delta,\delta)}$ and $\mathrm{col}_{(3\delta,2\delta)}$
where
$$\mathrm{col}_{(s,t)} \coloneqq col_s \setminus col_t$$

2) Set U_1 is part of $V_i = U(x_i,r_i)$. From previous section we have a set $Z_i \subset SO(n)$ of full measure such that $\mathscr{H}^m(f \circ \Xi_\theta(\Sigma \cap V_i)) = 0$ for all $\theta \in Z_i$. Choose such $\theta_1 \in Z_i$ that
$$\Xi_{\theta_1}(H_{3\delta}) \subset U_1 \setminus \mathrm{col}_{2\delta}.$$
Construct a mapping ξ_1 such that
$$\begin{cases} \xi_1 = \Xi_{\theta_1} & \text{on } H_{3\delta} \\ \xi_1 = \mathrm{id} & \text{on } \mathrm{col}_\delta \end{cases}$$

3) Let $U_{1,2} = U_1 \setminus \xi_1(H_{3\delta})$, and go to the first point with $U_{1,2}$ instead of U_1 and construct ξ_2.

Note. *The set $H_{3\delta}$ and the mapping ξ_1 are picked in such a way that*

$$\mathscr{H}^m\left(f \circ \xi_1(H_{3\delta} \cap \Sigma)\right) = 0.$$

Thus we can forget about the set $\xi_1(H_{3\delta})$. All further mappings will be equal to identity on this set.

One has to be careful executing above points. We have to set appropriate parameters that control the measure of the set Σ in collars that we are working with and the distance to the identity of consecutive ξ_i's in order to make the composition converge in $\mathscr{C}^1$. We introduce now a way to construct diffeomorphism ξ_1.

3. One picture lemma

To describe ξ_1 we need to know that for every $\theta \in SO(n)$ there exist vector field $V(\theta)$ such that $\theta(x) = \Phi_{V(\theta)}(x,1)$, where $\Phi_V(x,t)$ is the trajectory of the vector field V at the time t starting at the point x. Note that the composition

$$\Xi_\theta(x,t) \coloneqq \phi^{-1} \circ \Phi_{V(\theta)}(x,t) \circ \phi(x).$$

is a flow, hence it generates a vector field on domain $U(x_i,r_i)$ that corresponds to ϕ, call this vector field V_θ. Multiply the vector field V_θ by smooth cutoff function

$$\begin{cases} \vartheta(x) = 1 & \text{for } x \in U_1 \setminus \mathrm{col}_{2\delta} \\ \vartheta(x) \in [0,1] & \text{for } x \in \mathrm{col}_{(2\delta,\delta)} \\ \vartheta(x) = 0 & \text{for } x \in \mathrm{col}_\delta \end{cases}$$

Let $W_\theta = \vartheta(x) \cdot V_\theta$. The flow $\Phi_{W_\theta}(x,t)$ is continuous group of diffeomorphisms such that $\Phi_{W_\theta}(x,0) = \mathrm{id}$ hence there exist a positive ε such that $\Phi_{W_\theta}(H_{3\delta},t) \subset H_{2\delta}$ for all $t \in (-\varepsilon,\varepsilon)$.

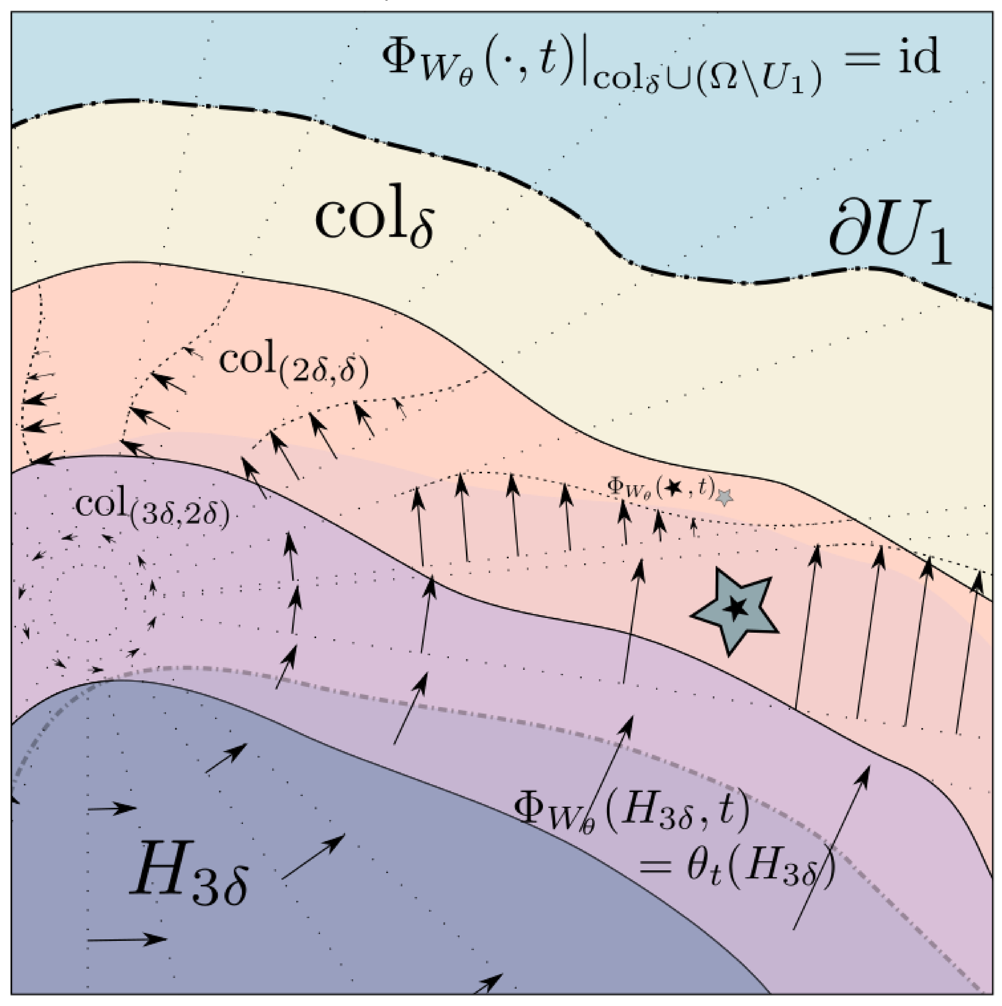

Figure 4: *One picture lemma*

From continuity we will also control the amount of the measure in the collars at each consecutive step. Let $\sigma_1 = \mathscr{H}^m(U_1 \cap \Sigma)$, since the $\mathscr{H}^m$-measure of $\Sigma \cap \partial U_1$ is zero,

$$\mathscr{H}^m(\Sigma \cap \mathrm{col}_{3\delta}) \longrightarrow 0 \text{ as } \delta \to 0.$$

We can choose δ so small that $\mathscr{H}^m(\Sigma \cap \mathrm{col}_{3\delta}) \le \sigma_1/3$ and from continuity of Φ_{W_θ} there exist t so small that

$$\mathscr{H}^m(\Phi_{W_\theta}(\Sigma \cap \mathrm{col}_{3\delta}, s) \le \sigma_i/2 \text{ for all } 0 < s < t$$

At the second step, in order to produce ξ_2, we take $\sigma_2 = \mathscr{H}^m(\xi_1(\Sigma) \cap U_{1,2})$ and do the same thing. Consequently at each step we "remove" at least half of the measure of the remains of Σ and it turns out in the limit the measure of $f \circ \zeta(\Sigma)$ vanishes.

4. The Limit

We are almost done with the proof of the theorem which answers questions ♠ and ♣.

Theorem. *For a purely m-unrectifiable set Σ contained in an open set $\Omega \subset \mathbb{R}^n$, such that $\mathscr{H}^m(\Sigma) < \infty$, let*

$$\mathscr{A}(\Sigma) \coloneqq \left\{ f \in \mathscr{C}^1_{=m}(\Omega,\mathbb{R}^n) \mid \mathscr{H}^m(f(\Sigma)) > 0 \right\}.$$

Then the interior of the set $\mathscr{A}(\Sigma)$ in the $\mathscr{C}^1_{=m}(U,\mathbb{R}^n)$ topology is empty.

Strategy of the proof is to find in every $\mathscr{C}^1$-neighbourhood of $f \in \mathscr{A}(\Sigma)$ an element that is not in $\mathscr{A}$. The only thing left to prove is the convergence of

$$\xi_n \circ \xi_{n-1} \circ \ldots \circ \xi_1$$

Differentiable dependence on initial conditions guarantees that for any ϵ we will find time t so small that $\|\xi_1 - \mathrm{id}\|_{\mathscr{C}^1} \le \epsilon$. Now only thing left to do is to pick $\varepsilon > 0$ and your favourite convergent series $\sum \epsilon_i \le \varepsilon$ and note the inequality for two mappings h and k that are ϵ_h and ϵ_k close to identity, then

$$\begin{aligned} \|f \circ g - g\|_{\mathscr{C}^1} &= \|f \circ g - g\|_{\mathscr{C}^0} + \|D(f \circ g) - D(g)\|_{\mathscr{C}^0} \\ &= \|f - \mathrm{id}\|_{\mathscr{C}^0} + \|Df(g).Dg - D(g)\|_{\mathscr{C}^0} \\ &\le \epsilon_f + \|Df(g) - \mathrm{id}\|_{\mathscr{C}^0}\|Dg\|_{\mathscr{C}^0} \\ &\le \epsilon_f + \epsilon_f(1 + \epsilon_g) \end{aligned}$$

In the three step poking holes procedure take ξ_i that are $\varepsilon_i/3$-close to identity and use the following telescopic sum

$$\begin{aligned} &\|\zeta_n \circ \ldots \circ \zeta_1 - \mathrm{id}\|_{\mathscr{C}^1} \\ &\quad = \|\zeta_n \circ \ldots \circ \zeta_1 + (-\zeta_{n-1} \circ \ldots \circ \zeta_1 + \zeta_{n-1} \circ \ldots \circ \zeta_1) + \\ &\quad + (-\zeta_{n-2} \circ \ldots \circ \zeta_1 + \zeta_{n-2} \circ \ldots \circ \zeta_1) + \ldots \\ &\quad \ldots + (-\zeta_1 + \zeta_1) - \mathrm{id}\| \end{aligned}$$

and the triangle inequality like above to prove that in fact this sequence is a Cauchy sequence in $\mathscr{C}^1$. ■

The opposite implication to one in theorem above is also true.