# Tubular-like foliations of neighbourhoods of $C^1$-submanifolds of $\mathbb{R}^n$

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## 1. Introduction of the problem

An embedded $C^2$-manifold $\mathcal{N}$ has a neighbourhood which can be foliated by perpendicular subspaces, it is so-called *tubular neighbourhood*. Situation changes when we drop differentiability assumption and consider manifolds of class $C^1$. Then the tangent space $T_x\mathcal{N}$, hence also perpendicular one, depends continuously from the point $x \in \mathcal{N}$. The following example shows that there is no chance for tubular neighbourhood of $C^1$-manifold.

**Example.** *Lines perpendicular to the graph of function $f(x) = |x|^{3/2}$ intersect each other in every neighbourhood of the graph.*

It is sufficient to find a point $p(t)$ at the intersection of the line perpendicular to graph at point $(t, f(t))$ and vertical axis O$y$, and calculate the limit $\lim_{s\to 0} p(s)$. It tuns out that

$$\lim_{t\to 0} p(t) = \lim_{t\to 0} \frac{2t + 3|t|^2 \operatorname{sgn}(t)}{3\sqrt{|t|}\operatorname{sgn}(t)} = 0$$

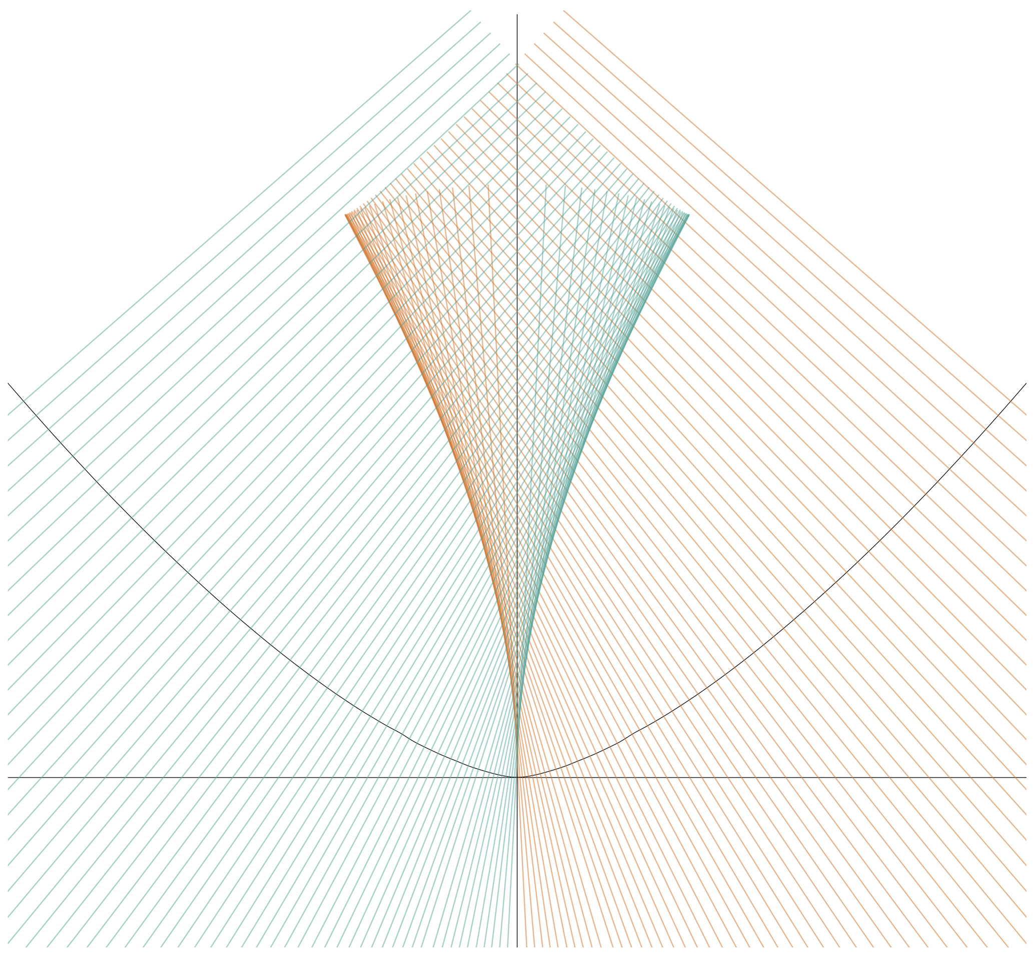


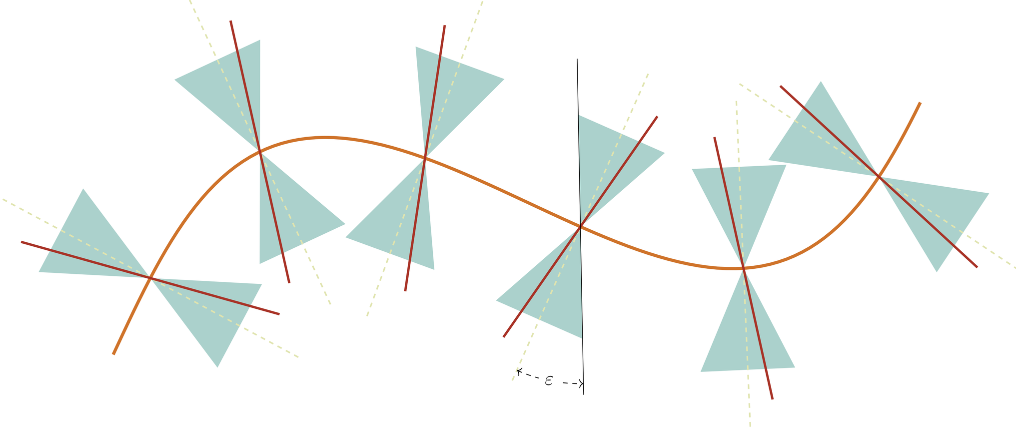
**Figure 1:** *Graph of $x \longmapsto |x|^{3/2}$ with normals lines*

We would like to produce foliated neighbourhood around $C^1$ manifold which will have as many properties of tubular as possible.

## 2. The Idea

In order to repair the foliation around $C^1$ manifold $\mathcal{N}$, in points such as above one needs to relax positioning of leaves. More precisely, we want to cover manifold $\mathcal{N}$ with a family $\mathfrak{B} = \bigcup_i B_i$ of open balls such that normal spaces in points contained in the set $B_i$ are closer than $\varepsilon$. Then there exist a partition of unity $\varphi_i$ which is subordinate to the covering $\mathfrak{B}$. Note that normal spaces at centres of adjacent balls are closer than $2\varepsilon$. Let us, for a moment, think about partition of unity as collection of weights. To every point $x \in \mathcal{N}$ we would like to assign a linear space which is a combination of normal spaces at centers of balls $B_i$ which contain point $x$, with weights $\varphi_i(x)$. And the theorem that we would like to have might be the following.

**Theorem.** *For every $\varepsilon$ there exist an open foliated neighbourhood of $C^1$ manifold such that leaves are pieces of affine subspaces of dimension equal to codimension of $\mathcal{N}$ and for every point $x \in \mathcal{N}$ the distance between the leaf at point $x$ and the normal space $T_x\mathcal{N}^\perp$ is less than $\varepsilon$.*



If we denote by $N_i$ the normal space at the center of the ball $B_i$, above interpolation can be written (formally for now)

$$\sum_{i\in I} \varphi_i(x)\, N_i \qquad (1)$$

Now we only need a good way to measure distance and take linear combinations of linear subspaces of an arbitrary dimension.

## 3. Embedding of the Grassmannian

The grassmannian $Gr(p.n)$ is the space of $p$-dimensional linear subspaces of $\mathbb{R}^n$. We embed this abstract manifold into $\mathbb{R}^{n\times n}$ by assigning to the point $q \in Gr(p,n)$ the matrix $m \in \mathbb{R}^{n\times n}$ of orthogonal projection onto subspace $q$ and let image of this embedding be denoted by $\mathcal{P}(p,n)$. Also we introduce distance between two subspaces as a usual vector length in $\mathbb{R}^{n^2}$. By replacing $N_i$ in (1) by matrix $m_i$ of orthogonal projection onto $N_i$ we obtain a matrix, but weighted sum of projection matrices usually do not belong to $\mathcal{P}(p,n)$. We will use standard tubular neighbourhood $U_{\mathcal{P}}(\varepsilon)$ of $\mathcal{P}(p,n)$ with projection along leaves $\pi : U_{\mathcal{P}}(\varepsilon) \longrightarrow \mathcal{P}(p,n)$. Finally obtaining the formula for the leaf at the point $x \in \mathcal{N}$

$$\gamma(x) = \pi\left(\sum_{i\in I} \varphi_i(x)\, m_i\right). \qquad (2)$$

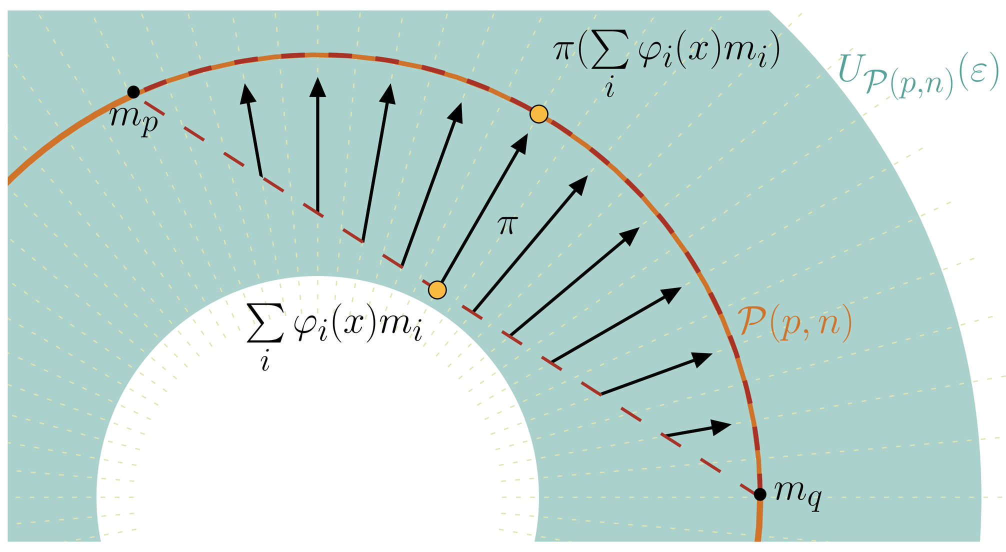



**Figure 3:** *Projection $\pi$ from the tubular neighbourhood $U_{\mathcal{P}}(\varepsilon)$ onto $\mathcal{P}(p,n)$*

One need to show the following three conditions are fulfilled in order to proceed further with Implicit function Theorem (IFT) and proving the result

1. The sum $\sum_{i\in I} \varphi_i(x)\, m_i$ is in the domain of projection $\pi$.
2. Defined above $\gamma(x)$ is $\varepsilon$-away from $T_x\mathcal{N}^\perp$.
3. $\gamma$ is $C^1$ mapping.

First and third conditions are easy and second needs triangle inequality and a simple geometric observation. Now we make use of the implicit function theorem.

## 4. Implicit function theorem part

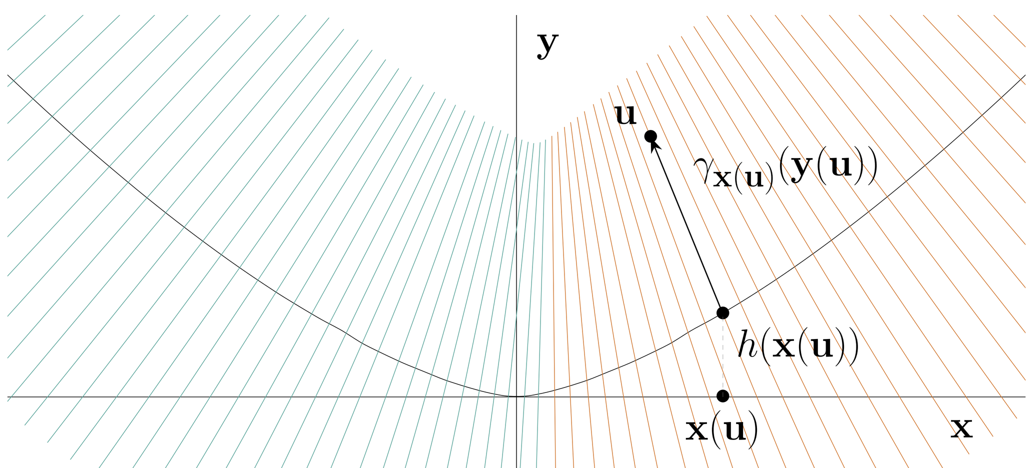



**Figure 4:** *A point $\mathbf{u}$ and the corresponding point $h(\mathbf{x}(\mathbf{u})) \in \mathcal{N}$ and leaf represented by the matrix $\gamma_{\mathbf{x}(\mathbf{u})}$.*

Equipped with above conditions, the theorem is a direct consequence of IFT. We shall employ it to the the mapping

$$F : \mathbb{R}^n_{\mathbf{x},\mathbf{y}} \times \mathbb{R}^n_{\mathbf{u}} \longrightarrow \mathbb{R}^n$$

$$\begin{aligned} F(\mathbf{x},\mathbf{y},\mathbf{u}) &= \begin{bmatrix} h_1(\mathbf{x}) \\ \vdots \\ h_n(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} \\ \quad \gamma_{\mathbf{x}} \quad \\ \\ \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \\ \vdots \\ y_p \end{bmatrix} - \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \\ &= h(\mathbf{x}) + \gamma_{\mathbf{x}}(\mathbf{y}) - \mathbf{u}, \end{aligned} \qquad (3)$$

where $h : \mathbb{R}^k_{\mathbf{x}} \to \mathbb{R}^n$ is a manifold parametrisation, $[\gamma_{\mathbf{x}}]$ is the $n \times n$ matrix of orthogonal projection onto $n-k$ dimensional subspace given by the formula (2). We should think that $\mathbf{x}$ spans a tangent space and $\mathbf{y}$ spans a normal space, then every leaf is parametrized by the variable $\mathbf{y}$. Now we would like to find functions $\mathbf{x}(\mathbf{u})$ and $\mathbf{y}(\mathbf{u})$ implicitly given by the equality

$$F(\mathbf{x},\mathbf{y},\mathbf{u}) = h(\mathbf{x}) + \gamma_{\mathbf{x}}(\mathbf{y}) - \mathbf{u} = 0.$$

Differentiating in $\mathbf{x}$ and $\mathbf{y}$ yields

$$\begin{aligned} \mathfrak{J} := \frac{\partial F}{\partial(\mathbf{x},\mathbf{y})} &= \det[\, D_{\mathbf{x}}F \mid D_{\mathbf{y}}F \,] \\ &= \det[\, D_{\mathbf{x}}(\varphi(\mathbf{x}) + \gamma_{\mathbf{x}}(\mathbf{y})) \mid D_{\mathbf{y}}\gamma_{\mathbf{x}}(\mathbf{y}) \,] \\ &= \det[\, T_{\mathbf{x}}\mathcal{N} + D_{\mathbf{x}}\gamma_{\mathbf{x}}(\mathbf{y}) \mid \gamma_{\mathbf{x}} \,], \end{aligned} \qquad (4)$$

where $T_x\mathcal{N}$ means $k$ vectors spanning the tangent space and $\gamma_{\mathbf{x}}$ is $n-k$ vectors spanning the leaf at point $\mathbf{x}$ (remains of the parametrisation of the leaf by coordinates $\mathbf{y}$). If we show that the Jacobian determinant $\mathfrak{J}$ do not vanish at point $(0_{\mathbf{x}}, 0_{\mathbf{y}}, 0_{\mathbf{u}})$, we will end the proof. Fortunately $T_x\mathcal{N}$ and $\gamma_{\mathbf{x}}$ span $\mathbb{R}^n$ and $D_{\mathbf{x}}\gamma_{\mathbf{x}}(\mathbf{y})$ does not brake the independence because it is a matrix-vector multiplication and we can take $\mathbf{y}$ small enough to prevent $\mathfrak{J}$ from being zero. From the IFT we have existence and uniqueness. Meaning that for every $\mathbf{u}$ (in neighbourhood close to $\mathcal{N}$ ) we can rewrite equality (3) as

$$h(\mathbf{x}(\mathbf{u})) + \gamma_{\mathbf{x}(\mathbf{u})}(\mathbf{y}(\mathbf{u})) = \mathbf{u}.$$

## 5. Application

In the background of our considerations there is a general problem. For two smooth closed manifolds $\mathcal{M}^{(m)}$, $\mathcal{N}^{(m+k)}$ and smooth isometric embedding $f : \mathcal{M}^{(m)} \longrightarrow \mathcal{N}^{(m+k)}$, what is the behaviour of the volume function $V(r)$ of tubular neighbourhood of radius $r$ around $f(\mathcal{M})$? In full generality, first Taylor expansion is known

$$V(r) = \omega^k \mathrm{Vol}(\mathcal{N}) r^k + O(r^{k+1}), \qquad (5)$$

where $\omega^k$ is the volume of $k-$dimensional unit disc. When the ambient manifold is flat (e.g. $\mathbb{R}^n$) Herman Weyl proved that $V(r)$ is a polynomial with coefficients equal to integrals of $k^{\text{th}}$ mean curvatures and this is a little surprising result as those are intrinsic invariants. Construction of foliated neighbourhood carried out previously can be used to prove formula (5) for $C^1$ manifolds. Lets denote our foliated neighbourhood by $\mathcal{F}(\varepsilon,\rho)$, note that it depends on two parameters $\varepsilon$ - distance of leaves to normals and $\rho$ - radius of leaves, note also that $\rho = \rho(\varepsilon)$. The strategy is to enclose set $B_{\mathcal{N}}(\rho) = \{u \mid \mathrm{dist}(\mathcal{N}, u) \leqslant \rho\}$ between two foliated neighbourhoods, a subset and a superset of $B_{\mathcal{N}}(\rho)$. Then calculate the following limit for both foliated nbds

$$\lim_{r\to 0} \frac{|\mathcal{F}(\varepsilon, r)|}{\omega^k r^k} \qquad (6)$$

and check if that limits will be the same. Capturing nbd yields two difficulties

1. The limit (6) with $\mathcal{F}(\varepsilon,\rho)$ with constant $\varepsilon$ shall not provide proper value (we hope to obtain measure volume of the manifold).
2. How much increase the radius $\rho$ of a leaf in $\mathcal{F}(\varepsilon,\rho)$ to provide inclusion $B_{\mathcal{N}}(\rho) \subset \mathcal{F}(\varepsilon, \rho + \text{sth})$ when $\varepsilon \to 0$

The first point can be easily seen by slightly perturbed parallelogram $ABCD_\rho$ with two sides parallel to span $\{(\cos(89°), \sin(89°))\}$ of the length $\rho$ and other two of length 1 parallel to $Ox$. The limit (6) with above neighbourhood of interval $[0,1]$, $ABCD_\rho$ is $\sin(89°) \neq \mathrm{Vol}([0,1])$. To solve second problem is what is the reduction of the distance between manifold $\mathcal{N}$ and $\partial\mathcal{F}(\varepsilon,\rho)$.

**Lemma.** *Define $m(\varepsilon)$ by the equality $m(\varepsilon)\rho = \mathrm{dist}(\mathcal{N}, \partial\mathcal{F}(\varepsilon,\rho))$ then*

$$\liminf_{\varepsilon\to 0} m(\varepsilon) = 1$$

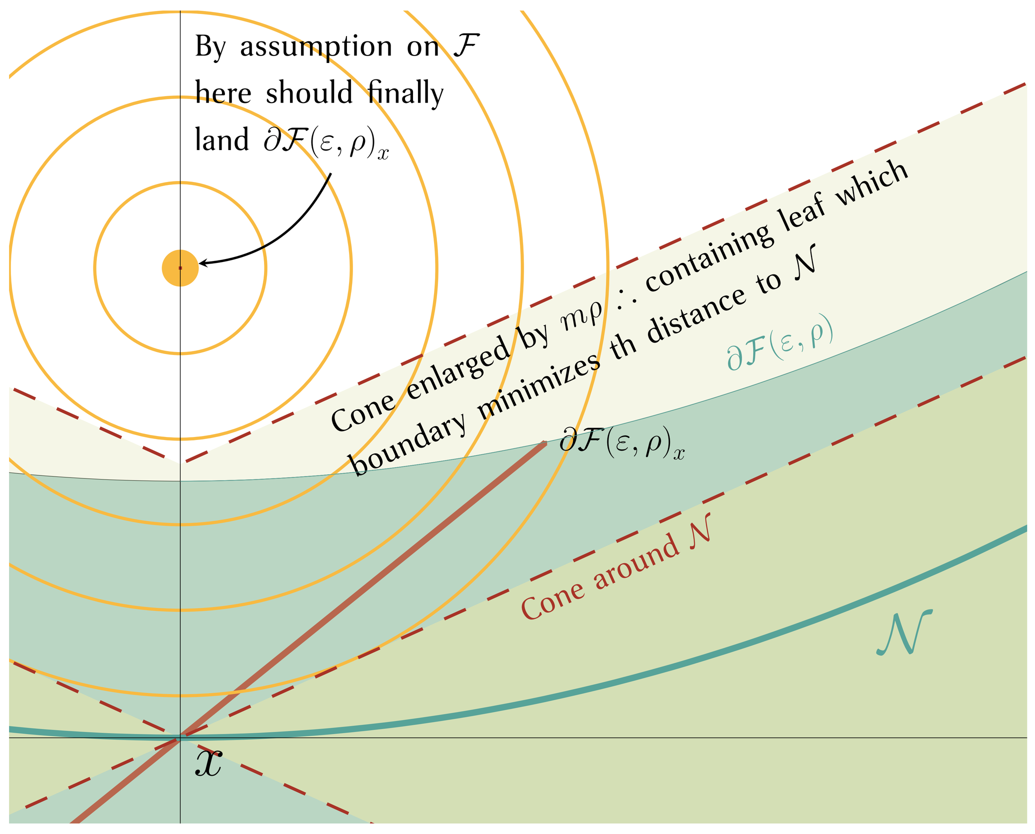



**Figure 5:** *Proof by contradiction. End of the leaf by assumption is in $m\rho$-nbd of $\mathcal{N}$ with $m < 1$ hence in in $m\rho$-nbd of the cone containing $\mathcal{N}$. While for $\varepsilon$ small enough it is close to normal and those are separate sets.*

Then we just need to calculate integrals and limit with $i \to \infty$ in the following chain of inequalities

$$|\mathcal{F}(\varepsilon,r)|/\omega^k r^k \leqslant |U_{\mathcal{N}}(r)|/\omega^k r^k \leqslant |\mathcal{F}(\varepsilon,(r + r/i))|/\omega^k r^k$$

where $\varepsilon = \varepsilon(i)$ and use squeeze theorem to show that limit (6) is the same also for $C^1$-manifolds.