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Prawie tubularne otoczenia rozmaitości klasy C^k w $\mathbb{R}^n$ dla $k \geq 1$

Praca magisterska
na kierunku MATEMATYKA

Praca wykonana pod kierunkiem
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Oświadczenie kierującego pracą

Potwierdzam, że niniejsza praca została przygotowana pod moim kierunkiem i kwalifikuje się do przedstawienia jej w postępowaniu o nadanie tytułu zawodowego.

Data Podpis kierującego pracą

Oświadczenie autora (autorów) pracy

Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowiązującymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur związanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Oświadczam ponadto, że niniejsza wersja pracy jest identyczna z załączoną wersją elektroniczną.

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Abstract

We begin with the investigation of an embedding of the Grassmannian into the space of matrices. Then we construct a foliated neighbourhood - which is similar to classic tubular neighbourhood around the manifold $\mathcal{M}$ of class at least C^1. In our neighbourhood every leaf is an open disc of an affine subspace and it is close to the normal space at the given point. Then using our foliated neighbourhoods we prove the equality

$$\lim_{\epsilon \to 0} \frac{|U_{\mathcal{M}}(\epsilon)|}{w^p \, \epsilon^p} = Vol(\mathcal{M}),$$

where in the numerator $|U_{\mathcal{M}}(\epsilon)|$ denotes the volume of a tube of radius ϵ around $\mathcal{M}$, the constant w_p is equal to the volume of a unit ball in $\mathbb{R}^p$ and the number p is the codimension of $\mathcal{M}^k \subset \mathbb{R}^{k+p}$.

Słowa kluczowe

tubular neighbourhood, foliation, volume of tubes, embedded differentiable manifold

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

28A75 - Length, area, volume, other geometric measure theory
53C12 - Foliations
53A07 - Higher-dimensional and -codimensional surfaces in Euclidean n-space

Tytuł pracy w języku angielskim

Tubular-like neighbourhoods for C^k submanifols of $\mathbb{R}^n$ with $k \geq 1$

Contents

Introduction

An embedded manifold $\mathcal{M}$ which belongs to a differentiability class 2 or higher has a neighbourhood which can be foliated by subspaces perpendicular to it. This is the so-called tubular neighbourhood. When the manifold is of class C^1, all tangent thus also perpendicular spaces depend continuously from the point $x \in \mathcal{M}$. Then it may happen that perpendicular spaces cross each other in every neighbourhood of $\mathcal{M}$. This is the case if we treat the graph of the function $f(x) = x^{3/2}$ as a manifold in $\mathbb{R}^2$. In this work we will build a foliated neighbourhood around C^1 manifolds with a relaxed perpendicularity condition, that is, all leaves shall not be normal but lay closer than a fixed number η to normal spaces. A sketch of construction of such neighbourhood can be found in the book by F. Hélein who included it in [He] as a list of exercises. In the first chapter of this work we solve them, providing necessary details. With foliated neighbourhood in hand we shall prove that one can calculate the volume of C^1 manifold of codimension k by finding the limit

$$\lim_{\epsilon \to 0} \frac{|U_{\mathcal{M}}(\epsilon)|}{w_k\, \epsilon^k}.$$

In other words, if we denote by $V_{\mathcal{M}}(\epsilon) = |U_{\mathcal{M}}(\epsilon)|$ the function which to argument ϵ assigns the volume of a tube of radius ϵ around $\mathcal{M}$. Then the k^{th} coefficient of the Taylor series of $V_{\mathcal{M}}$ is the same for manifolds that are of class C^1 and for those which are of higher classes. A volume function described above (in different settings: ambient space is flat or has constant curvature or submanifold is nicely curved) is discussed by H.Weyl in [W] and by A. Gorin in [G]. Beside mentioning those works here in introduction, we will not refer to them as they focus on smooth submanifolds and we are generally in the C^1 case.

Notation & abbreviations

- $C^k(U)$ is the class of functions which are k times continuously differentiable on a given set U.
- $\mathrm{Gr}(p, n)$ stands for the set of all p-dimensional linear subspaces (also called p-planes) in $\mathbb{R}^n$, it is called the Grassmannian.
- $\mathcal{P}(p, n)$ is the set $\{A \in M_{n\times n}(\mathbb{R}) \,|\, A = A^\top = A^2,\ \mathrm{tr}(A) = p\}$.
- $M_{n\times m}(k)$ is a space of matrices of n rows and m columns with coefficients from a given field k.
- $SO(n) = \{A \in GL(n) \,|\, AA^\top = A^\top A = \mathrm{Id},\ \det(A) = 1\}$ is the special orthogonal group.

- $M(\mathrm{id})^{\mathcal{A}}_{\mathcal{B}}$ denotes the matrix of coordinate change from $\mathcal{B}$ to $\mathcal{A}$. We will use it as follows $M(\phi)^{\mathcal{B}}_{\mathcal{B}} = M(\mathrm{id} \circ \phi \circ \mathrm{id})^{\mathcal{B}}_{\mathcal{B}} = M(\mathrm{id})^{\mathcal{B}}_{\mathcal{A}} M(\phi)^{\mathcal{A}}_{\mathcal{A}} M(\mathrm{id})^{\mathcal{A}}_{\mathcal{B}}$
- $\|A\|_{HS} = \sqrt{\sum_{ij} a_{ij}^2}$ is called the Hilbert–Schmidt norm of a matrix A.
- $d(A, B) = \|A - B\|_{HS}$ where A and B are matrices of the same size.
- (v, w) is a vector $(v_1, \ldots, v_p, w_1, \ldots, v_{n-p})$ in $\mathbb{R}^n$, where $v \in \mathbb{R}^p$ and $w \in \mathbb{R}^{n-p}$
- ONB - Orthonormal Basis
- nbd. - neighbourhood

Chapter 1

Grassmannian and foliated neighbourhood of C^1 manifolds

First we will study the set whose elements are p-dimensional linear subspaces of an n-dimensional vector space. This is the so-called Grassmannian and we will denote it by $Gr(p,n)$. This set is where leaves of our foliation will come from. In order to find convex combinations of different p-planes we shall embed $Gr(p,n)$ into space of matrices $M_{n\times n}(\mathbb{R})$. Denote by $\mathcal{P}(p,n)$ a subset of matrices $A \in M_{n\times n}$ which fulfil $A = A^{\perp} = A^2$ and have trace equal to p.

1.1. Grassmannian as matrices of projections

Claim 1. *$Gr(p,n) = \mathcal{P}(p,n)$ as sets.*

Proof. To every element $Y \in \mathrm{Gr}(p,n)$ we associate a linear homomorphism of the orthogonal projection onto Y and call it Π_Y. Let the collection of vectors $\mathcal{A}^Y = \{a_1^Y, \ldots, a_p^Y\}$ and $\mathcal{A}^{Y^\perp} = \{a_1^{Y^\perp}, \ldots, a_{n-p}^{Y^\perp}\}$ form ONBs of $Y \subset \mathbb{R}^n$ and $Y^\perp \subset \mathbb{R}^n$ respectively. In the basis $\mathcal{A} = \{\mathcal{A}^Y, \mathcal{A}^{Y^\perp}\}$ matrix of the projection Π_Y is

$$P := M(\Pi_Y)^{\mathcal{A}}_{\mathcal{A}} = \begin{array}{cc} & \begin{array}{cc} \quad p \quad & \quad n-p \quad \end{array} \\ \begin{array}{c} p \\ \\ n-p \end{array} & \begin{pmatrix} 1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ 0 & & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array}. \tag{1.1}$$

In the standard basis $\mathrm{st} = \{e_1, \ldots, e_n\}$, this projection is given by

$$M(\Pi_Y)^{st}_{st} = M(\mathrm{id})^{st}_{\mathcal{A}} M(\Pi_Y)^{\mathcal{A}}_{\mathcal{A}} M(\mathrm{id})^{\mathcal{A}}_{st} = C^{-1}PC$$

where C is an element of $SO(n)$. To prove the claim we will represent every element $A \in \mathcal{P}(p,n)$ to be in the form $C^{-1}PC$. We will use properties: $C^{-1} = C^{\perp}$ fulfilled by every matrix in $SO(n)$ and an idempotence of the projection $\Pi_Y \circ \Pi_Y = \Pi_Y$. We now check if $C^{-1}PC$

meets conditions required to be in the set $\mathcal{P}(p,n)$. First

$$(C^{-1}PC)(C^{-1}PC) = C^{-1}P(CC^{-1})PC = C^{-1}P\,\mathrm{Id}\,PC = C^{-1}PC$$

as $P^2 = P$, and secondly

$$C^{-1}PC = C^{\perp}PC^{-1^{\perp}} = (C^{-1}PC)^{\perp}.$$

A trace of every matrix $C^{-1}PC$ is p because the trace is independent of the basis. Thus, we have shown that every element in the set $\mathcal{P}(p,n)$ comes from the projection onto subspace, therefore we have shown an epimorphism $Gr(p,n) \longrightarrow \mathcal{P}(p,n)$. It's clearly a monomorphism since for every two p-subspaces Y, S in $\mathbb{R}^n$ there exists an index i such that images $\Pi_Y(e_i)$ and $\Pi_S(e_i)$ are different, which implies that their matrices are different in the i^{th} column. Q.E.D

1.2. Smoothness

For the future use of the tubular neighbourhood theorem we will check that the set $\mathcal{P}(p,n)$ is a smooth manifold.

Claim 2. $\mathcal{P}(p,n) \subset \mathbb{R}^{n^2}$ *is a smooth manifold.*

Proof. Let us denote by E_x the eigenspace of a projection matrix $x \in \mathcal{P}(p,n)$. From now on A and B denote two k-dimensional linear spaces ($\cong \mathbb{R}^k$), where $k = n - p$. We make the following assumptions:

$i_\square : \square \longrightarrow \mathbb{R}^n$ are linear embeddings for $\square = A,\, B$;
$\{b_1, \dots, b_k, b_{k+1}, \dots, b_n\}$ is the ONB of $\mathbb{R}^n$ such that first k vectors forms the base of $i_B(B)$;
$U_A := U(i_A) = \{x \in \mathcal{P}(p,n) \,|\, E_x \cap i_A(A) = 0\}$.

Choose i_A, i_B and let $i_{p_0} : p_0 \longrightarrow \mathbb{R}^n$ be a linear embedding of the p-dimensional space p_0 such that

$$i_{p_0}(p_0) \in U_A \cap U_B.$$

Let us introduce a **coordinate chart** for $\mathcal{P}(p,n)$. For every $x \in U_A$ we define the map

$$h_A(x) : p_0 \longrightarrow A$$

to be the restriction of the projection $\pi_x : A \oplus E_x \longrightarrow A$ along E_x to the subspace p_0.

It turns out that the mapping $h_A : U_A \longrightarrow \mathrm{Hom}(p_0, A)$ is a bijection between the open set U_A and the set of linear transformations $\mathrm{Hom}(p_0, A)$. Using the bases of $p_0 = \mathrm{span}\{e_1, \dots, e_p\}$ and A one can identify $\mathrm{Hom}(p_0, A)$ with the space $M_{p\times k}(\mathbb{R})$. To prove the claim, we have to show that **transition map** i.e. the composition $h_B \circ h_A^{-1}$ in the diagram

$$\begin{tikzcd} & U_A \cap U_B \arrow[dl, "h_A"'] \arrow[dr, "h_B"] & \\ M_{p\times k}(\mathbb{R}) \arrow[rr, "h_B \circ h_A^{-1}"] & & M_{p\times k}(\mathbb{R}) \end{tikzcd}$$

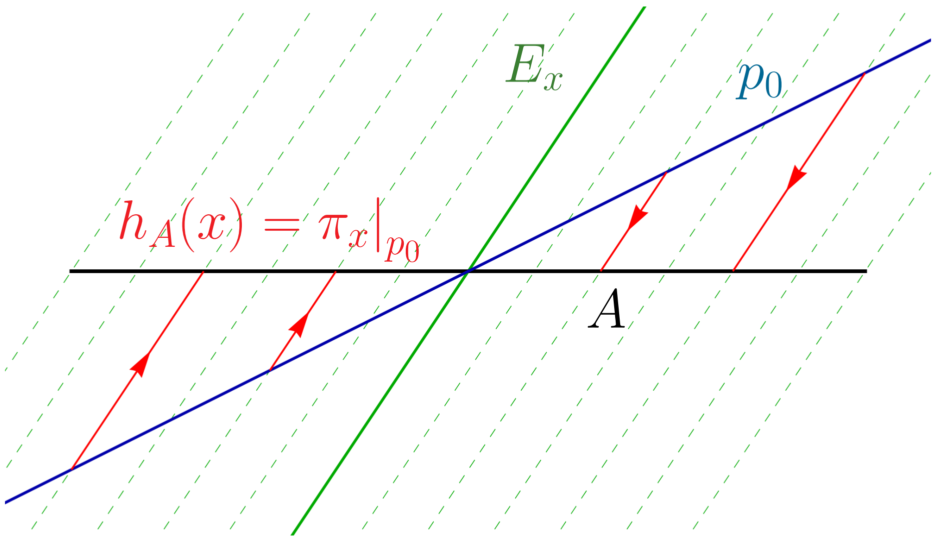

Figure 1.1: Projection $h_A(x)$. Notice that p_0 and A are fixed and E_x represents point from manifold $\mathcal{P}(p,n)$. Therefore reader should imagine that E_x is different every time (always intersecting 0) as we pick different points from $U_A \cap U_B$

is smooth.
In this arrangement the transition map is the following chain of actions. Start with a matrix $x \in M_{p\times k}$ which, following figure (1.1), describes the plane

$$h_A^{-1}(x) = \operatorname{span}\{i_{p_0}(e_1) - i_A(x(e_1)), \ldots, i_{p_0}(e_p) - i_A(x(e_p))\}.$$

Then the mapping h_B restricts the projection $i_B(B) \oplus h_A^{-1}(x) \longrightarrow i_B(B)$ along $h_A^{-1}(x)$ to the subspace p_0. In order to obtain above projection we first describe some matrices. Here

$$\beta_B = \begin{bmatrix} \top & & \top \\ b_{k+1} & \cdots & b_n \\ \bot & & \bot \end{bmatrix}$$

is the $p \times n$ matrix consisting of p vectors written vertically and spanning an **orthogonal compliment** of the subspace B. Denote by

$$\alpha_A = \begin{bmatrix} \top & & \top \\ i_{p_0}(e_1) - i_A(x(e_1)) & \cdots & i_{p_0}(e_p) - i_A(x(e_p)) \\ \bot & & \bot \end{bmatrix}$$

the $p \times n$ matrix consisting of p vectors which span the subspace $h_A^{-1}(x)$. Then the matrix $P : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ of the projection $i_B(B) \oplus h_A^{-1}(x) \longrightarrow i_B(B)$ along $h_A^{-1}(x)$, following [M], is given by the formula

$$P = \beta_B({\alpha_A}^T \beta_B)^{-1} {\alpha_A}^T.$$

Finally restrict this mapping to the plane p_0 and write the result in the base of B to obtain a $p \times k$ matrix i.e. composition $h_B \circ h_A^{-1}$. Using the matrix P the result will be

$$(h_B \circ h_A^{-1})(x) = {i_B}^{-1} \circ P(x) \circ i_{p_0}$$

Since the operation $\cdot \longrightarrow \cdot^{-1}$, used inside the definition of P, is smooth in the Lie group $GL(p)$, right hand side in the above equality depends smoothly form the x, thus we proved that $\mathcal{P}(p,n)$ is a smooth manifold. Q.E.D

1.3. Classical Tubular neighbourhood

In this section we will define the classical tubular neighbourhood $\mathcal{V} \supset \mathcal{P}(p,n)$ and the projection

$$\pi : \mathcal{V} \longrightarrow \mathcal{P}(p,n)$$

such that

$$\mathrm{d}(A, \pi(A)) = \mathrm{d}(A, \mathcal{P}(p,n)).$$

Lets start with an abstract definition of the tubular nbd of $M \subset V$, where M is a submanifold of V.

Definition 1. *By* ***tubular neighbourhood*** *of a compact manifold M contained in V we will call the triple (f, ξ, U), where $\xi = (p, E, M)$ is a vector bundle $p : E \to M$, $f : E \longrightarrow V$ and $U \subset E$ is a nbd of the zero-section such that $f : U \longrightarrow V$ is an embedding, $f|_M = \mathbf{1}_M$ and $f(U)$ is an open nbd of M in V.*

Definition 2. *Let $p : \gamma_{k,n} \longrightarrow Gr(k,n)$ be the vector bundle with the fibre $p^{-1}(P)$ consisting of pairs (P, x) where $x \in P$.*

We will show that a smooth compact manifold $M \subset \mathbb{R}^m$ has the tubular neighbourhood. Let $k = m - \dim M$ and the mapping $v : M \longrightarrow \mathrm{Gr}(k,m)$ stands for the C^∞ field of k-planes $v(x) = T_xM^\perp$. The pull-back diagram

$$\begin{tikzcd}
v^*\gamma_{k,n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \gamma_{k,n} \arrow[d, "p"] \\
M \arrow[r, "v"] & \mathrm{Gr}(k,n)
\end{tikzcd}$$

defines a new vector bundle, whose total space is equal to

$$E = v^*\gamma_{k,n} = \{(x,y) \in M \times \mathbb{R}^m \mid y \in v(x)\}.$$

Define the mapping $f : E \longrightarrow \mathbb{R}^m$ by

$$f(x,y) = x + y \quad \text{in the coordinates above.} \tag{1.2}$$

A tangent space T_xE along the zero-section $(x,0)$ can be decomposed as following

$$T_{(x,0)}E = T_xM \oplus v(x).$$

It is clear from (1.2) that Df is non singular on $T_{(x,0)}E$. Since the plane $v(x)$ and the tangent space T_xM are transversal, Jacobian matrix Df has rank m. From the **Inverse function theorem** [S] every point of the zero-section $(x,0)$ has a neighbourhood $V_x \subset E$ such that $f : V_x \longrightarrow \mathbb{R}^m$ is a homomorphism. The manifold $M \subset E$ lies within the open set

$\bigcup_{x \in M} V_x = V$. On the open set $U = f(V)$ we have coordinates (x, y) taken from $V \subset E$. Define a C^∞ projection

$$\pi : U \longrightarrow M$$
$$\pi(x, y) = x.$$

which satisfies required condition because the set $\{x = const\}$ defines a line, normal to the manifold M i.e. geodesic in $\mathbb{R}^m$ and geodesics are locally minimising paths. In particular, since we assumed that $\mathcal{M}$ is compact, there exist $\epsilon > 0$ and the set

$$U_\epsilon = U_\epsilon(\mathcal{P}(p, n)) = \{y \in M_{n \times n}(\mathbb{R}^n) \mid d(y, \mathcal{P}(p, n)) < \epsilon\} \tag{1.3}$$

such that (f, ξ, U) is the tubular neighbourhood of $\mathcal{P}(p, n) \in \mathbb{R}^{n^2}$, where $f(U) = U_\epsilon$ and $\xi = (p^*, v^*\gamma_{k,n}, \mathcal{M})$.

1.4. Covering manifold by balls

From now on, $\mathcal{N} \subset \mathbb{R}^n$ denotes the k-dimensional compact manifold of class at least C^1. Let A_x be the $n \times n$ matrix of the orthogonal projection onto $N_x\mathcal{N} = T_x\mathcal{N}^\perp$ and $B_x(s)$ denotes the n-dimensional ball centred at the point $x \in \mathcal{N}$ with the radius s. We will use the symbol $\widetilde{\mathcal{B}_s}$ to denote the family of balls $\{B_x(s)\}_{x \in \mathcal{N}}$. For such family we define a value

$$m(\widetilde{\mathcal{B}_s}) = \sup_x \sup_{y \in B_x} d(A_y, A_x).$$

Since $\mathcal{N}$ is at least of class C^1, matrix $A_x = A(x)$ depends continuously from the point x, which provides that the function $m(s) = m(\widetilde{\mathcal{B}_s})$ is decreasing as s is approaching 0. Therefore there exist a number $t > 0$ such that $m(\widetilde{\mathcal{B}_t}) < \epsilon/2$ for arbitrary chosen ϵ. Using the compactness of $\mathcal{N}$ from the family $\widetilde{\mathcal{B}_t}$ we can choose a finite subfamily of balls $\mathcal{B}$. The family $\mathcal{B}_t$ has the property that, for every point $y \in \mathcal{N}$, distance between projection matrices A_y and A_{x_i} such that $y \in B(x_i, t) \in \mathcal{B}$ is less than $\epsilon/2$.

1.5. Construction of almost perpendicular leaves

In this section we will introduce and explore the mapping γ which plays the main role in construction of the foliated nbd. It takes the point x from the manifold $\mathcal{N}$ and assigns the matrix of the projection onto a subspace which is close to the $N_x\mathcal{N}$. A disc at the point 0 inside the eigenspace of the matrix $\gamma(x)$ will be a leaf of the foliation passing through the point $x \in \mathcal{N}$.

If a k-dimensional manifold $\mathcal{N} \subset \mathbb{R}^n$ is compact and covered by the family of balls $\mathcal{B}$, then there exists a smooth partition of unity $\{\varphi_i\}$ subordinate to the cover $\mathcal{B}$. That is, a family of functions $\varphi_i : \mathbb{R}^n \longrightarrow [0, 1]$ such that:

i) $\operatorname{supp}(\varphi_i) \subset B_i$ and $\varphi_i \in C^\infty(\mathbb{R}^n)$;

ii) $0 \le \sum_i \varphi_i(x) \le 1$ for all $x \in \mathbb{R}^n$;

iii) $\sum_i \varphi_i(x) = 1$ for all $x \in \mathcal{N}$.

Consider the mapping $\gamma : \mathcal{N} \longrightarrow \mathcal{P}(p, n)$ defined by

$$\gamma(x) = \pi\left(\sum_i A_{x_i}\varphi_i(x)\right).$$

Three things are important considering γ; first that it is well defined, second that it fulfils the inequality

$$d(\gamma(x), A_x) < \epsilon$$

and lastly that it is of at least class C^1. Notice that the sum $\sum_{i=1}^k A_{x_i}\varphi_i(x)$ is contained in a convex hull of points A_{x_i}, denote this set by $\mathrm{conv}((A_{x_i})_{i\in I})$ where $I = \{i : \varphi_i(x) \neq 0\}$. We got to check if the sum is contained in the domain of π, defined in (1.3). We already know that the balls in the family $\mathcal{B}$ have radius such that normal spaces for every pair of points within each ball are closer than ϵ. Fix the point $x \in \mathcal{N}$, all non-zero factored A_{x_i} inside the definition of $\gamma(x)$ are in the ball $B(A_x, \epsilon/2) \subset M_{n\times n}(\mathbb{R})$. Thus convex hull of those points also lies in this ball, therefore we know that map γ is well defined because $B(A_x, \epsilon/2) \subset U_\epsilon$. Above argument is equivalent to

$$d\left(A_x\,, \sum_i A_{x_i}\varphi_i\,(x)\right) < \epsilon/2.$$

For simplicity denote $A_i := A_{x_i}$ and mark $\eta := \sum_i A_i\varphi_i\,(x)$, then

$$d\,(\gamma\,(x)\,, \eta) = d\,(\pi(\eta), \eta) = d\,(\eta, \mathcal{P}(p, n)) < \epsilon/2$$

holds due to the property of the mapping π, that is $d(A, \pi(A)) = d(A, \mathcal{P}(p, n))$ and inequality is a consequence of the observation that

$$d(\eta, A_i) \leq \epsilon/2 \tag{1.4}$$

for some index i. Suppose, contrary to (1.4), that $\forall_{i\in I}\; d(\eta, A_i) \geq \epsilon/2$. Thus all points A_i are in the set $W = B(A_x, \epsilon/2) \setminus B(\eta, \epsilon/2)$. It's obvious that $\mathrm{conv}(A_i) \subset \mathrm{conv}(W)$, but since both balls have the same radius $\eta \notin \mathrm{conv}(W)$, which contradicts definition of η.

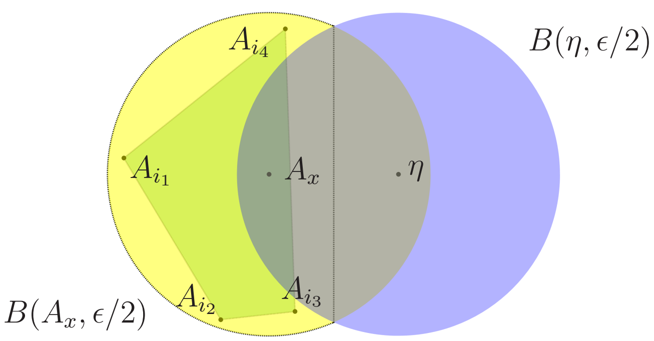

Figure 1.2: Sets $B(A_x, \epsilon/2)$(yellow), $B(\eta, \epsilon/2)$(blue) and convex hull of points A_{i_j}(green). Boundary of set $\mathrm{conv}(W)$ is marked by dashed line.

It is easy to show that the distance between the space $N_x\mathcal{N}$ and the eigenspace of $\gamma(x)$ is less than an arbitrarily chosen ϵ. Using the triangle inequality we obtain

$$\begin{aligned} d(A_x, \gamma(x)) &\leq d(A_x, \sum_i A_{x_i}\varphi_i(x)) + d(\sum_i A_{x_i}\varphi_i(x), \gamma(x)) \\ &\leq \epsilon/2 + \epsilon/2 \\ &\leq \epsilon. \end{aligned}$$

Last thing to prove is that mapping γ is of class C^l. Both φ_i and π are smooth(C^∞). Since the manifold is of class C^l, function γ is also of this class because the local parametrisation $h_i : \mathbb{R}^{\dim(\mathcal{N})} \longrightarrow \mathcal{N}$ are of class C^l.

1.6. TFU and the foliation of a neighbourhood around the C^1 manifold

Our goal is to show that there exist nbd. $V\mathcal{N}$ around the manifold $\mathcal{N}$ such that spaces given by $\gamma(x)$ do not cross but they really forms a foliation around $\mathcal{N}$. We will use the Implicit function theorem (TFU for short) to associate to every point $u \in \mathrm{V}\mathcal{N}$ a base point $y \in \mathcal{N}$ and a vector $z \in E_{\gamma(y)}$ such that $u = y + z$. First we remind statement, as in [I] and introduce notation which will be used later.

Theorem 1 (Implicit function theorem).
Let $\mathbf{x} = (x_1, \dots, x_k)$*,* $\mathbf{y} = (y_1, \dots, y_n)$ *and*

$$\begin{aligned} &\Phi : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ &\Phi(\mathbf{x}, \mathbf{y}) = (\phi_1(\mathbf{x}, \mathbf{y}), \dots, \phi_n(\mathbf{x}, \mathbf{y})) \end{aligned}$$

be a mapping of class C^k *with* $k \geq 1$*. Suppose that* $\Phi(0,0) = 0$ *and Jacobian determinant in* $\mathbf{y}$ *coordinates at the point* $(0,0)$ *is not equal to zero, i.e.*

$$0 \neq \frac{\partial \Phi}{\partial \mathbf{y}} = \frac{\partial(\phi_1 \dots, \phi_n)}{\partial(y_1, \dots, y_n)} = det \begin{bmatrix} \frac{\partial \phi_1}{\partial y_1} & \dots & \frac{\partial \phi_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial y_1} & \dots & \frac{\partial \phi_n}{\partial y_n} \end{bmatrix}.$$

Then there exists an open neighbourhood $U = U_{\mathbf{x}} \times U_{\mathbf{y}} \ni (0_{\mathbf{x}}, 0_{\mathbf{y}})$ *and functions* $f_1, \dots, f_n : U_{\mathbf{x}} \longrightarrow U_{\mathbf{y}}$ *of class* C^k *such that*

$$\Phi(\mathbf{x}, f_1(\mathbf{x}), \dots, f_n(\mathbf{x})) = 0 \quad \textit{for every } \mathbf{x} \in U_{\mathbf{x}}$$

and functions f_i *are unique satisfying the condition*

$$\{(\mathbf{x}, \mathbf{y}) \in U : \Phi(\mathbf{x}, \mathbf{y}) = 0\} = \{\mathbf{x} \in U_{\mathbf{x}},\ y_i = f_i(\mathbf{x}) \textit{ for } i = 1, \dots, n\}.$$

□

Let $\varphi : \mathbb{R}^k \longrightarrow \mathcal{N} \subset \mathbb{R}^n$ be a local parametrization of the open subset $U \subset \mathcal{N}$ and $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_{n-k})$. Denote also $\gamma_\mathbf{x} = \gamma(\varphi(\mathbf{x}))$. We shall consider the function

$$F : \mathbb{R}^n_{\mathbf{x},\mathbf{y}} \times \mathbb{R}^n_\mathbf{u} \longrightarrow \mathbb{R}^n$$

$$\begin{aligned} F(\mathbf{x}, \mathbf{y}, \mathbf{u}) &= \begin{bmatrix} \varphi_1(\mathbf{x}) \\ \vdots \\ \varphi_n(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} & \gamma_\mathbf{x} & \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \\ \vdots \\ y_p \end{bmatrix} - \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \\ &= \varphi(\mathbf{x}) + \gamma_\mathbf{x}(\mathbf{y}) - \mathbf{u}. \end{aligned} \tag{1.5}$$

Suppose that $\operatorname{span}\{y_1, \ldots, y_{n-k}\} \cap \ker \gamma_\mathbf{x} = 0$ for all $\mathbf{x} \in U$, which gives us a parametrization of the eigenspace $E_{\gamma_\mathbf{x}}$ by the coordinate $\mathbf{y}$. Check that assumptions of TFU are fulfilled, that is if F has a non zero Jacobian determinant

$$\begin{aligned} \frac{\partial(f_1, \ldots, f_n)}{\partial(\mathbf{x}, \mathbf{y})} &= \det \left[\begin{array}{ccc|ccc} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_k} & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_{n-k}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_k} & \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_{n-k}} \end{array} \right] \\ &= \det[\, D_\mathbf{x} F \mid D_\mathbf{y} F \,] \\ &= \det[\, D_\mathbf{x} \left(\varphi\left(\mathbf{x}\right) + \gamma_\mathbf{x}\left(\mathbf{y}\right)\right) \mid D_\mathbf{y} \gamma_\mathbf{x}(\mathbf{y}) \,] \\ &= \det[\, T_\mathbf{x}\mathcal{N} + D_\mathbf{x}\gamma_\mathbf{x}(\mathbf{y}) \mid E_{\gamma_\mathbf{x}} \,]. \end{aligned} \tag{1.6}$$

Where we denote by $E_{\gamma_\mathbf{x}}$ the $n \times (n-k)$ matrix which consists of $(n-k)$ vertical vectors spanning an eigenspace of $\gamma_\mathbf{x}$ and by $T_\mathbf{x}\mathcal{N}$ we mean the $n \times k$ matrix consisting of k vectors tangent to $\mathcal{N}$. Vectors in $E_{\gamma_\mathbf{x}}$ and $T_\mathbf{x}\mathcal{N}$ are linearly independent and the element $D_\mathbf{x}\gamma_\mathbf{x}(\mathbf{y})$ does not break the independence because

$$D_\mathbf{x}(\gamma_\mathbf{x}(\mathbf{y})) = (D_\mathbf{x}\gamma_\mathbf{x})(\mathbf{y})$$

is the matrix multiplication, therefore for sufficiently small $\mathbf{y}$ the Jacobian determinant is non-zero. Thus, TFU implies that there exists unique functions

$$\begin{aligned} (h_1, \ldots, h_k) &= g_1 : U_\mathbf{u} \longrightarrow U_\mathbf{x} \\ (h_{k+1}, \ldots, h_n) &= g_2 : U_\mathbf{u} \longrightarrow U_\mathbf{y} \end{aligned}$$

such that

$$F(g_1(\mathbf{u}), g_2(\mathbf{u}), \mathbf{u}) = 0.$$

in $V\mathcal{N}$.The second part of TFU implies that for every $\mathbf{u}$ in a sufficiently small $\epsilon-$neighbourhood of $\mathcal{N}$ the following equality of sets holds

$$\{(\mathbf{x}, \mathbf{y}, \mathbf{u}) : F(\mathbf{x}, \mathbf{y}, \mathbf{u}) = 0\} = \{\mathbf{u} \in U_u,\ \mathbf{x} = g_1(\mathbf{u}),\ \mathbf{y} = g_2(\mathbf{u})\}.$$

Writing explicitly F, one will obtain

$$\mathbf{u} = \varphi\left(g_1(\mathbf{u})\right) + \gamma_{g_1(\mathbf{u})}(g_2(\mathbf{u})),$$

which gives the foliation with leaves $\mathcal{L}_\mathbf{u} := \{$ eigenspace of $\gamma_{g_1(\mathbf{u})}\}$ passing though point $\varphi(g_1(\mathbf{u}))$.

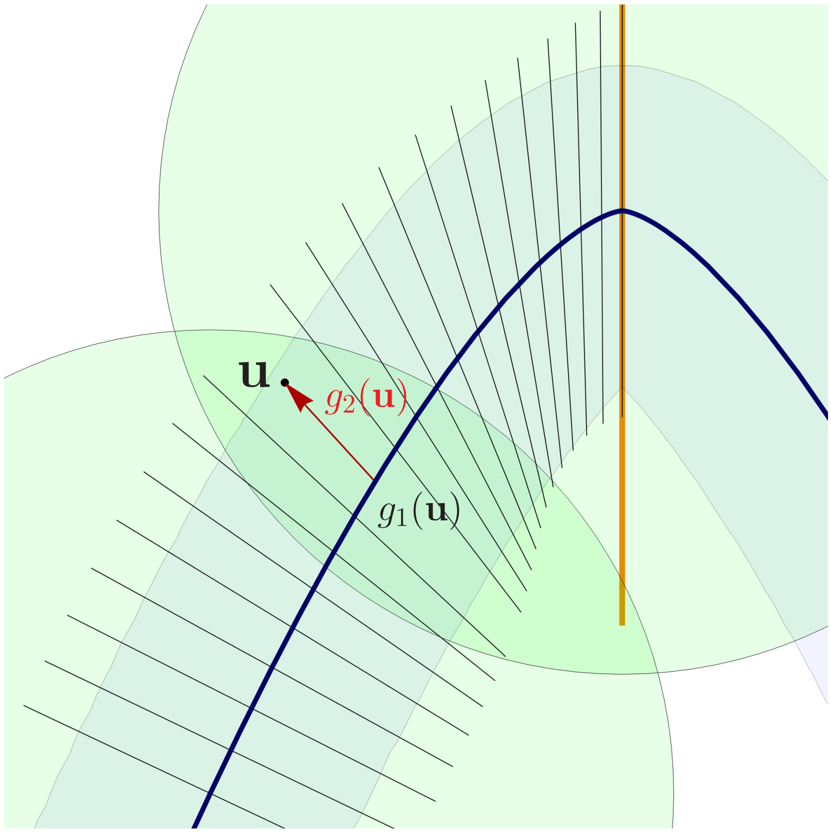

Figure 1.3: Graph of function $f(x) = -|x|^{3/2}$ with almost perpendicular foliation of neighbourhood. Blue shaded set is foliated nbd. of manifold and green filled balls come from covering $\mathcal{B}$. Point u (black dot) determines $\gamma_{g_1(\mathbf{u})}(g_2(\mathbf{u}))$ (red vector) and $\varphi\,(g_1(\mathbf{u}))$ (base point of red vector). Orange line represents coordinates $\{y_1, \ldots, y_{n-k}\}$.

Fact that it is a foliation follows from the uniqueness of functions g_i. Construction of the desired projection is immediate

$$P : V\mathcal{N} \longrightarrow \mathcal{N}$$
$$P\,(\mathbf{u}) = \varphi(g_1(\mathbf{u}))$$

which collapse $V\mathcal{N}$ along leaves $\mathcal{L}_u$ to the point $\varphi(g_1(u))$. Note that the function $g_2(\mathbf{u})$ inform us about the position of $\mathbf{u}$ inside a leaf $\mathcal{L}_{\mathbf{u}}$ while g_1 where $\mathcal{L}_u$ crosses the manifold $\mathcal{N}$.

The assumption about the continuity of differential is irremovable. Tt follows from the proof because we used TFU and it's one of assumptions there, but also can be seen on example. Consider the manifold $\mathcal{M} \subset \mathbb{R}^2$ which is a graph of the function $f(x) = x^2 \sin(\frac{1}{x})$ on the interval $[-1, 1] \setminus 0$ and $f(0) = 0$. This function is differentiable everywhere but df is not continuous at the point 0. In the neighbourhood of point zero, the normal space have infinitely many oscillations with big amplitude, therefore there is no way to define foliations with leaves ϵ-close to perpendicular in the neighbourhood of the zero.

Chapter 2

Volume of tubes around C^1 manifolds in $\mathbb{R}^n$

2.1. Tubes

Now we are interested in finding a limit of the volumes of the tube around C^1 manifolds divided by its diameter raised to the power equal to the codimension of $\mathcal{N}$. We will prove the equation

$$\lim_{\epsilon \to 0} \frac{|U_{\mathcal{N}}(\epsilon)|}{w_p \ \epsilon^p} = \mathcal{H}^k(\mathcal{N}) \overset{loc}{=} \int_V \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^k$$

where $1-$forms dx^i is a basis of a cotangent bundle T^*V for some open set $V \subset \mathcal{N}$. Define the tube $U_{\mathcal{N}}$ by

$$U_{\mathcal{N}}(\epsilon) = \{x \in \mathbb{R}^n | d(x, \mathcal{N}) < \epsilon\} = \bigcup_{x \in \mathcal{N}} B_n(x, \epsilon).$$

The strait bracket $|\cdot|$ means the $n-$dimensional Lebesgue measure and $\mathcal{H}^k(\mathcal{N})$ is the k^{th} Hausdorff measure and finally w_p is the volume of a $p-$dimensional ball $\{x \in \mathbb{R}^p : |x| < 1\}$.

The idea of the proof is to catch the tube $U_{\mathcal{N}}(\epsilon)$ between the two foliated neighbourhoods. Then calculate the same limit but with a volume of the foliated nbd. in the numerator and prove that above limit for the foliated nbds. is the area of the manifold.

In previous chapter the distance between the the normal space and the corresponding leaf at the given point, was small but fixed. From now, denote this distance by the symbol η. In order to obtain area of the manifold in procedure described above, we cannot fix this distance η and take limit with respect to the diameter of tube ϵ. Easy example to see why we need η approaching to the 0 is to take a parallelogram which represents our set $U_{\mathcal{N}}$ with fixed angle η close but not equal to $\pi/2$ and one of the side equal to 1, while other side has length of 2ϵ. Then $\mathrm{Vol}(\epsilon) = 1 \cdot \sin(\eta) \cdot 2\epsilon$ and the analogous limit $\lim\limits_{\epsilon \to 0} Vol(\epsilon)/2\epsilon = \sin(\eta) \neq 1$ which is proper *area* of the side remaining constant.

2.2. Approximation of $U_{\mathcal{N}}(\epsilon)$ by foliated neighbourhoods $\gamma(\epsilon)$

Here we will work with the function F_ϵ, similar to one introduced in (1.5). Denote the parametrisation of some open nbd. V in $\mathcal{N}$ by $\varphi : U_{\mathbf{x}} \longrightarrow V \subset \mathcal{N}$. Let F_ϵ denotes the mapping

$$F_\epsilon : U_{\mathbf{x}} \times B_p(0,1) \longrightarrow \mathbb{R}^n$$
$$F_\epsilon(\mathbf{x},\mathbf{y}) = \varphi(\mathbf{x}) + [\gamma_{\mathbf{x}}](0, \epsilon\,\mathbf{y}).$$

The main animal in the rest of the work will be an image of the above mapping and as it will appear constantly, we will write

$$\gamma(\epsilon) = F_\epsilon(U_{\mathbf{x}} \times B_p(0,1))$$

Lets denote the intersection $V = \mathcal{N} \cap B(x_i, t)$, where the ball $B(x_i, t)$ is taken from the covering $\mathcal{B}$ (1.4) and let $U_{\mathbf{x}}$ be the pre-image of V under the parametrisation φ. Change coordinates $(\mathbf{x},\mathbf{y})$ such that set of vectors $\{(e_i^x, 0)\}_{i=1,\dots,k}$ is an ONB of $T_{x_i}\mathcal{N}$ and $\{(0, e_j^{\mathbf{y}})\}_{j=k+1,\dots,n}$ is an ONB of $T_{x_i}\mathcal{N}^\perp = N_{x_i}\mathcal{N}$, where e_i^x means vector d/dx_i. Note that the leaf of $\gamma(\epsilon)$ is the image of the orthogonal projection $\gamma(x)$. Norm of that projection is the same as for any other orthogonal one, that is $\|\gamma(x)\| = 1$. Thus, leaf passing through the point $x \in \mathcal{N}$ let call it $\gamma_x(\epsilon)$, is contained inside the ball $B(x, \epsilon)$ and we obtain first inclusion

$$\gamma(\epsilon) \subseteq U_{\mathcal{N}}(\epsilon).$$

We will now show an opposite inclusion with the radius of a leaf slightly bigger than ϵ. Lets consider the set $\gamma(\epsilon)^c = \mathbb{R}^n \setminus \gamma(\epsilon)$ where $\gamma(\epsilon)$ means the foliated nbd. over all manifold $\mathcal{N}$. We can assume that the leaves are unique even after taking the closure of $gamma(\epsilon)$. For every $\eta > 0$ the manifold $\mathcal{N}$ and $\gamma(\epsilon)^c$ are separate sets and let us define the function m by the equation

$$m\epsilon = m(\eta)\cdot\epsilon(\eta) = d(\mathcal{N}, \gamma(\epsilon)^c) = \max\{\sup_{u\in\gamma(\epsilon)^c}\inf_{x\in\mathcal{N}} d(u,x), \sup_{x\in\mathcal{N}}\inf_{u\in\gamma(\epsilon)^c} d(x,y)\}$$

Take a point $u \in \partial\gamma(\epsilon)$, then there exist $x \in \mathcal{N}$ minimising the distance and we have $u \in x + N_x\mathcal{N}$. We have know that point x need not to be unique. However, from assumption on $\gamma(\epsilon)$ we know that the point u is contained in an unique leaf γ_y at the point $y \in \mathcal{N}$.

Lemma 1. *With notation as above*

$$\liminf_{\eta\longrightarrow 0} m(\eta) = 1.$$

Proof. By the contradiction, assume that there exist a sequence $\{\eta_i\}$ such that $m(\eta_i) \stackrel{i\to\infty}{\longrightarrow} m_1 < 1$. Let the number m be such that $m_1 < m < 1$. If $u(\eta_i)$ denotes the point from $\partial\gamma(\epsilon(\eta_i))^c$ minimising distance to the manifold $\mathcal{N}$ then for $i > i_0$

$$u(\eta_i) \in B(x, m\cdot\epsilon(\eta_i))$$

for any of the corresponding points $x \in \mathcal{N}$. Note that for coordinates $(\mathbf{x},\mathbf{y})$ there exist an open set $W \subset \mathcal{N}$ which is the graph of the function defined on some nbd. $W_0 \subset \mathbb{R}^k \times 0$. Lets call this function

$$h_{\mathcal{N}} : T_{\mathbf{x}}\mathcal{N} \supset W_0 \longrightarrow N_{\mathbf{x}}\mathcal{N}.$$

Note that the mapping $h_{\mathcal{N}}$ belongs to the class C^k, this means that $h'(\mathbf{x})$ is at least continuous and from the choice of coordinates we have $h'(0) = 0$. Now we want to trap some part of W in the cone

$$C_a = \{(\mathbf{x},\mathbf{y}) \;:\; |\mathbf{y}| \le a(m)|\mathbf{x}|\}. \tag{2.1}$$

Observe that the above cone does not depend on η and i. Hence we can cover the manifold with parts that are contained in such cones and solving problem locally gives the global result. The point u minimising the distance to $\mathcal{N}$ will be in the set $U_{C_a}(m\epsilon)$ that is a $m\,\epsilon-$nbd. around the cone C_a. Key part of the proof is to find such slope of the cone C_a that $S_{p-1}(0,\epsilon)$ denoting the $(p-1)$-dimensional sphere of radius ϵ in the normal space and $U_{C_a}(m\epsilon)$ are separate.

In order to find the proper slope a of a cone, that is such that it contains some part of the manifold and is separate from a sphere in the normal space. First observe that if we take the cone that is completely flat i.e. $a = 0$ the set $U_{C_0}(m\epsilon)$ is obviously separate from the sphere S_p because $m < 1$. Now if the number a is increasing the intersection is behaving according to the rule

$$U_{C_a}(m\epsilon) \cap N_y\mathcal{N} = B_p\left(0, m\,\epsilon\,\sqrt{1+a^2}\right).$$

Observe that the radius funtion is strictly increasing on $(0,\infty)$ and has a minimum $m\,\epsilon$ in the point 0. There is also such slope $a = \sqrt{1/m^2-1}$ such that ball in the intersection has radius ϵ. Restrict the domain W_0 of the mapping $h_{\mathcal{N}}$ to the ball $B_k(0,-) = W_x \subset T_x\mathcal{N}$ which has radius such that $\|h'(\mathbf{x})\| \leq a/2 = \sqrt{1/m^2-1}/2$ in every point $\mathbf{x} \in W_x$. Open sets $h_{\mathcal{N}}(W_x)$ cover the entire manifold independently from η and gives covering mentioned earlier.

Go back to points minimising "thickness" of the foliated tube $\gamma(\epsilon)$. For ϵ small enough, the point $u \in \gamma_y \cap \gamma(\epsilon)^c$ and $x \in \mathcal{N}$ such that $d(\gamma(\epsilon)^c, \mathcal{N}) = d(u,x)$, will be contained in sets $U_{C_{a/2}}(m\,\epsilon)$ and $h_{\mathcal{N}}(W_y)$ respectively where cone $C_{a/2}$ has its vertex at the point y. Observe that $U_{C_{a/2}}(m\,\epsilon)$ is separate from the sphere $S_p(\epsilon) \subset N_y\mathcal{N}$ with distance $l\cdot\epsilon$ for some constant l.

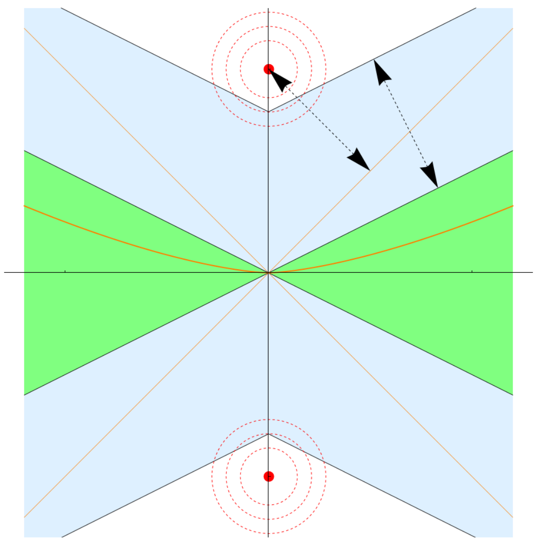

Figure 2.1: Two red dots represents the $p-1$ dimensional sphere in space normal to $\mathcal{N}$ - orange curve. Green cone is the one of cones constructed above containing some part of $\mathcal{N}$ and blue hourglass shape is set of points which distance to green cone is less than $m\epsilon$. Thus minimising distance point u is in blue area but from assumptions on $\gamma(\epsilon)$ we know that boundaries of leaves are in red circles around the sphere in the normal space and for η small enough those set are separate this produces the contradiction

Lets denote the orthogonal projection on a subspace A by $\Pi(A)$. Then from properties of the foliated nbd. we have the inequality

$$\|\Pi(N_y\mathcal{N}) - \Pi(\gamma_y)\|_{HS} \leq \eta_i \,. \tag{2.2}$$

For any square $n \times n$ matrix A the following norm inequalities

$$\|A\| \leq \|A\|_{HS} \leq n^{1/2}\|A\|$$

holds, where the norm without an index is the usual operator norm $\|A\| = \sup_{\|x\|=1} \|A(x)\|$. Because the point u is in the leaf $\gamma_y(\epsilon)$, it is an image of a point $t \in N_y\mathcal{N}$ under the orthogonal projection. Note that t belongs to the sphere $S(y,\epsilon) \subset N_y\mathcal{N}$ because u is a point from the boundary of a leaf. We want to know what is the distance between the sphere $S(y,\epsilon)$ and the point u. Note that

$$d(S(y,\epsilon),u) \leq \|t-u\| = \|(\Pi(N_y\mathcal{N}) - \Pi(\gamma_y))(t)\| \leq \eta_i \cdot \epsilon \,,$$

thus we know that every minimising distance point u is in the set $U_{S(y,\epsilon)}(\eta_i\epsilon) = \{z \in \mathcal{R}^n : d(z, S(y,\epsilon)) < \eta_i\epsilon\}$. In order to finish the proof we only need to notice that the intersection $U_C(m\epsilon) \cap U_{S(y,\epsilon)}(\eta_i\epsilon)$ is empty for η_i small enough. This is obvious because as we mentioned earlier distance $d(S(y,\eta_i\epsilon), U_C(m\epsilon)) \geq l\cdot\epsilon$ for some constant l, while nbd. containing the point u around the sphere $S_p(y,\epsilon)$ is arbitrarily small compared with $l \cdot \epsilon$. Therefore we obtained the contradiction because the point u is simultaneously in two separate sets. Q.E.D

Lemma implies that any tube around the manifold $\mathcal{N}$ smaller than ϵ will eventually fall into the foliated nbd. $\gamma(\epsilon)$ when $\eta \to 0$. Formally we can express it as

Corollary 1.

$$\underset{i\geq 2}{\forall} \ \underset{\eta>0}{\exists} \quad U_{\mathcal{N}}(\epsilon) \subset \gamma(\epsilon + \frac{\epsilon}{i}) \quad \textit{where} \quad \epsilon = \epsilon(\eta). \tag{2.3}$$

2.3. Volume of foliated tubes

To calculate the volume of the foliated $\epsilon-$nbd. we need to find the integral:

$$\frac{1}{w_p\,\epsilon^p} \int\limits_{F_\epsilon(U_x \times B_p(0,1))} 1\, d\lambda^n = \frac{1}{w_p\,\epsilon^p} \int\limits_{U_x \times B_p(0,1)} |DF_\epsilon| \; dx\, dy$$

Lets take a closer look on the Jacobian $|DF_\epsilon|$, note that, same as in (1.6), k first columns are differentials $\partial/\partial x_i$ and last $p = n-k$ columns are differentials over coordinates y_j and we shall denote the Jacobian matrix with a vertical line in between the k^{th} and the $(k+1)^{\text{st}}$ column. The multilinearity of the determinant in columns yields the following

$$\begin{aligned} \det\left[D_{xy}F_\epsilon\right] &= \det\left[D_xF_\epsilon \mid D_yF_\epsilon\right] && (2.4)\\ &= \det\left[D_xF_\epsilon \mid D_yF_\epsilon - \epsilon N_x\mathcal{N} + \epsilon N_x\mathcal{N}\right] && (2.5)\\ &= \det\left[D_xF_\epsilon \mid D_yF - \epsilon N_x\mathcal{N}\right] + \sum_\sigma |\ldots| + \det\left[D_xF_\epsilon \mid \epsilon N_x\mathcal{N}\right]. && (2.6) \end{aligned}$$

Where $\epsilon N_x\mathcal{N}$ denotes the $n\times p$ matrix of p vectors of the length ϵ which form the orthogonal base of space normal to $T_x\mathcal{N}$. Properties of the foliated nbd. and inequality (2.2) provides that we can assume that $\|\epsilon N_i - \epsilon\gamma_i\| \leq \eta\epsilon$ if ϵN_i is the i-th column of $\epsilon N_x \mathcal{N}$ and $\epsilon\gamma_i$ is the i-th column of $D_y F$. The sum $\sum\limits_\sigma |\ldots|$ is generated treating columns as a sum of two vectors: $(D_y F - \epsilon N_x \mathcal{N})$ and $\epsilon N_x \mathcal{N}$ by the multilinearity in columns again. Then in every component of this sum at least one of the last p columns is a difference, for example $(k+i)^{\text{th}}$, $D_{y_i}F - \epsilon N_i$.

Let see what happen to the integral with one such difference in i^{th} of last p vectors:

$$\frac{1}{w_p\,\epsilon^p} \int\limits_{U_x \times B_p(0,1)} \det\left[D_x F_\epsilon \mid D_y F_\epsilon - \epsilon\, N_x \mathcal{N}^i\right]\, dx\, dy = \tag{2.7}$$

$$= \frac{1}{w_p\,\epsilon^p} \int\limits_{U_x \times B_p(0,1)} \det\left[D_x F_\epsilon \mid \epsilon\gamma_1 \cdots \epsilon(\gamma_i - N_x \mathcal{N}^i) \cdots \epsilon\gamma_p\right]\, dx\, dy \tag{2.8}$$

$$\leq \frac{1}{w_p} \int\limits_{U_x \times B_p(0,1)} \prod_{j=1}^{k} \|D_{x_j} F_\epsilon\| \cdot \prod_{\substack{j=k+1 \\ j\neq i}}^{n} \|\gamma_j\| \cdot \eta\, dx\, dy \tag{2.9}$$

$$\overset{*}{\leq} C\,\eta. \tag{2.10}$$

To pass from the line (2.8) to the (2.9) we used linearity of the determinant to draw ϵ from each one of p columns and the Hadamard's inequality which states that if a matrix A consists of columns ν_i then $\det[A] \leq \prod \|\nu_i\|$.

The inequality (2.10) has a hidden difficulty, namely we don't control the differentials of the partition of unity which appears in $\|D_x F\| = \|D_x\varphi + D_x\,[\gamma_x]\,(0,\,\epsilon y)\,\|$ inside the definition of $gamma_x$. When η approach 0, balls (which are supports of the partition of unity) have radii tending to 0. If we imagine the easiest such family of hats with the value 1 in the center of the ball, then we see that the derivatives of functions may be arbitrarily large. Until now, ϵ was small enough to provide uniqueness of mapping F_ϵ. Now we modify $\epsilon(\eta)$ to be small enough that

$$\begin{aligned}\|D_x\varphi(x) + (D_x\gamma_x)(0,\,\epsilon\,y)\| &\leq \|D_x\varphi(x)\| + \|(D_x\gamma_x)(0,\,\epsilon\,y)\| \\ &\leq \|D_x\varphi(x)\| + \epsilon\sup\|D_x\gamma_x\| \leq \|D_x\varphi(x)\| + 1\end{aligned}$$

therefore we establish the new function ϵ

$$\epsilon(\eta) := \min\{\epsilon(\eta), \frac{\eta}{\sup_{x\in\mathcal{N}}\|D_x\gamma_x\|}\}. \tag{2.11}$$

Additional η in the numerator is not needed now, as we want any bound of the partition derivatives. This will be needed while integrating the last component in (2.6). With above radius of leaves in the foliated nbd. we have that there exists a constant C in (2.10) and it follows that:

$$\frac{1}{w_p\,\epsilon^p} \int\limits_{U_x \times B_p(0,1)} \left(\det\left[D_x F_\epsilon \mid D_y F - \epsilon N_x \mathcal{N}\right] + \sum_\sigma |\ldots|\right) dx\, dy \leq 2^p\, C\,\eta \overset{\eta\to 0}{\longrightarrow} 0.$$

Now, what happen if we integrate the last component in (2.6).

$$\frac{1}{w_p\,\epsilon^p}\int\limits_{U_x\times B_p(0,1)} \det\left[D_xF \mid \epsilon N_x\mathcal{N}\right]\,dx\,dy =$$

$$= \frac{1}{w_p\,\epsilon^p}\int\limits_{U_x\times B_p(0,1)} \det\left[D_x\varphi(x) + (D_x\gamma_x)\begin{bmatrix}0\\ \epsilon\,y_1\\ \vdots\\ \epsilon\,y_p\end{bmatrix} \middle|\ \epsilon N_x\mathcal{N}\right]\,dx\,dy \overset{\star}{=}$$

First, from the multilinearity of the determinant we can draw ϵ from every one of the last p columns, which cancels with the fraction before the integral. Then we continue :

$$\overset{\star}{=} \frac{1}{w_p}\int\limits_{U_x\times B_p(0,1)} \det\left[D_x\varphi(x) + (D_x\gamma_x)\begin{bmatrix}0\\ \epsilon\,y_1\\ \vdots\\ \epsilon\,y_p\end{bmatrix} \middle|\ N_x\mathcal{N}\right]\,dx\,dy \tag{2.12}$$

$$\Big\downarrow{\scriptstyle \eta\to 0}$$

$$\frac{1}{w_p}\int\limits_{U_x\times B_p(0,1)} \det\left[D_x\varphi(x) \mid N_x\mathcal{N}\right]\,dx\,dy$$

convergence results from the Lebesgue dominated convergence theorem and from new definition of $\epsilon(\eta)$ in (2.11). Dominant is easily obtained by the Hadamard inequality and fact that we have compact manifold, thus we can assume that the parametrisation have upper bound of their derivative $\|D\varphi\| \le C_1$ Therefore, the Hadamard and the triangle inequality yields

$$\overset{\star}{\le} \frac{1}{w_p}\int\limits_{U_x\times B_p(0,1)} \prod_{i=1}^{k}(C_1+1)\,dx\,dy \le \frac{1}{w_p}(C_1+1)^k|U_x| \le \frac{1}{w_p}(C_1+1)^k|B_k(0,1)|,$$

an integrable dominant.

Lets return to calculating the limit, observe that under integral sign (2.12) no dependence from the y coordinate, thus fraction $\frac{1}{w_p}$ cancels with the integration over y and we obtain that above limit is equal to

$$\int_U \det\left[D_x\varphi(x) \mid N\mathcal{N}\right]\,dx.$$

Recollect that the determinant of n vectors in $\mathbb{R}^n$ is the volume of the parallelogram spanned on those. We have chosen vectors in the matrix $N_x\mathcal{N}$ such that they form an ONB of the space normal to N. Therefore the determinant $\det\left[D_x\varphi(x) \mid N\mathcal{N}\right]$ is equal to the k-dimensional volume of the parallelogram spanned on columns of $D_x\varphi$ times length of altitudes, but those are equal to 1. Denoting by $|\cdot|_k$ k - dimensional measure and by $\mathrm{Par}(\partial_i\varphi)$ the parallelogram spanned by vectors $D_{x_i}\varphi$ with $i = 1,\ldots,k$ we obtain that

$$\int_U \det\left[D_x\varphi(x) \mid N\mathcal{N}\right]\,dx = \int_U |\mathrm{Par}(\partial_i\varphi)|_k\,dx = \int_U \sqrt{|g|}\,dx_1\ldots dx_k = Vol_k(\varphi(U))$$

where $|g|$ is the determinant of the metric tensor, when the parametrisation is provided and manifold is embedded in an euclidean space $\mathbb{R}^n$ like in our case, this can be easily calculated $g_{ij} = \langle \partial_i \varphi, \partial_j \varphi \rangle_{\mathbb{R}^n}$

2.4. Comparing neighbourhoods

We already know the following inclusions of tubes around the manifold $\mathcal{N}$

$$\gamma(\epsilon) \subset U_{\mathcal{N}}(\epsilon) \subset \gamma\left(\left(1+\frac{1}{i}\right)\epsilon\right).$$

Above translates into inequalities for their volumes

$$\frac{|\gamma(\epsilon)|}{w_p\,\epsilon^p} \leq \frac{|U_{\mathcal{N}}(\epsilon)|}{w_p\,\epsilon^p} \leq \frac{|\gamma\left(\left(1+\frac{1}{i}\right)\epsilon\right)|}{w_p\,\epsilon^p}. \tag{2.13}$$

we may assume that the set $\gamma(1+\frac{1}{i})$ is also foliated then it can be split into the two sets that will be $\gamma(\epsilon)$ and $\gamma(\epsilon,(1+\frac{1}{i})\epsilon) = \gamma((1+\frac{1}{i})\epsilon) \setminus \gamma(\epsilon)$. Only thing left to prove is that

$$\frac{|\gamma(\epsilon,(1+\frac{1}{i})\epsilon)|}{w_p\,\epsilon^p} \longrightarrow 0 \quad \text{when} \quad \eta \to 0$$

Observe that the limit is the same when we divide by the same factor, that is

$$\lim_{\epsilon\to 0} \frac{\left|\gamma\left(\left(1+\frac{1}{i}\right)\epsilon\right)\right|}{w_p\left(\left(1+\frac{1}{i}\right)\epsilon\right)^p} = \lim_{\epsilon\to 0} \frac{|\gamma(\epsilon)|}{w_p\,\epsilon^p}$$

$$\lim_{\epsilon\to 0} \frac{\left|\gamma\left(\left(1+\frac{1}{i}\right)\epsilon\right)\right|}{w_p\,\epsilon^p}\frac{\epsilon^p}{\left(\left(1+\frac{1}{i}\right)\epsilon\right)^p} = \lim_{\epsilon\to 0} \frac{|\gamma(\epsilon)|}{w_p\,\epsilon^p}$$

$$\lim_{\epsilon\to 0} \frac{\left|\gamma\left(\left(1+\frac{1}{i}\right)\epsilon\right)\right|}{w_p\,\epsilon^p} = \lim_{\epsilon\to 0} \frac{|\gamma(\epsilon)|}{w_p\,\epsilon^p}\frac{\left(1+\frac{1}{i}\right)^p\epsilon^p}{\epsilon^p} = \lim_{\epsilon\to 0} \frac{|\gamma(\epsilon)|}{w_p\,\epsilon^p}(1+i)^p$$

Thus, from corollary (2.3), inequalities (2.13) transform to

$$\underset{i\geq 2}{\forall}\ \underset{\eta>0}{\exists}\quad \frac{|\gamma(\epsilon)|}{w_p\,\epsilon^p} \leq \frac{|U_{\mathcal{N}}(\epsilon)|}{w_p\,\epsilon^p} \leq \frac{|\gamma\,(\epsilon)\,|}{w_p\,\epsilon^p}(1+\frac{1}{i})^p.$$

When $i \to \infty$ then there exist a sequence η_i for which we have inequalities for the volume of the middle set of thickness $\epsilon(\eta_i)$. Then from the squeeze theorem and from the knowledge about the limit for the foliated tube, we obtain the equality

$$\lim_{\epsilon\to 0} \frac{|U_{\mathcal{N}}(\epsilon)|}{w_p\epsilon^p} = \mathrm{Vol}_k(\mathcal{N}).$$

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