Implementation of a BJT based jerk circuit: route to chaos with multiple attractors

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***Abstract*—Recently researchers have done much work on simple chaotic circuits based on jerk equations. In this paper, a jerk circuit was simulated and implemented using a Bipolar Junction Transistor (BJT) as a nonlinear switch. This nonlinear electronic circuit showed chaotic dynamics under different conditions. A jerk circuit constitutes a third order differential equation having hypothetical factors x, $\dot{x}$, $\ddot{x}$ and $\dddot{x}$. The *stretching and folding* phenomenon by changing control parameter of the dynamical system and its transition into chaos was observed. The system displayed a period-1, period-2, and period-4 responses (attractors) and then the period-doubling cascade led the system into chaos with multiple attractors. In addition, a bifurcation diagram was also drawn by numerically solving the nonlinear differential equation of the system.**

***Index Terms*—Bifurcations; chaos; nonlinear electronic circuits; jerk function; multiple attractors; nonlinear dynamics**

I. Introduction

Deterministic Chaos has been observed in wide range of dynamical systems related to all fields including biology, chemistry, computer sciences, among others. Chaos is defined as,"*aperiodic bounded dynamics of a deterministic system with sensitive dependence on initial conditions*" [1]. The discovery of chaos in mathematical models and laboratory research is an achievement of science. Dynamical systems theory also called *chaos theory* is changing the scientific opinion about physical systems [2], [3]. It emphasizes that complicated solutions are not necessarily due to noise and the erratic dynamics appear due to interaction of fundamental dynamical components.

Different research groups have implemented numerous nonlinear electronic circuits using different electronic components based on sets of ordinary differential equations (ODEs) [4]. These chaotic electronic circuits include Jerk circuit [5], Chua circuit [6], Colpitts oscillator [7], and power electronic converters [8]. Chaotic electronic circuits have been applied in different fields of electrical and computer engineering such as speech encryption [9], video encryption [10], secure image encryption [11], text encryption schemes [12], random bit generation [13], and secure communication [14].

In 1994, famous chaos theorist J. C. Sprott presented several sets of differential equations that may exhibit chaotic behavior [15]. These dynamical systems can be represented by 3D vector fields with five terms, including two nonlinear terms or six terms with a single quadratic or cubic nonlinearity. Afterward, Australian researcher Gottlieb [16] described that several models of Sprott may be redefined in an explicit third order form $x = J(x, \dot{x}, \ddot{x})$, which is termed as *Jerk function*.

American mathematician Schot [17] derived the term *Jerk*, which involved the third derivative of the displacement, x. Fig. 1 shows the schematic diagram of jerk, acceleration, velocity and position. Hence, jerk is the time rate of change of acceleration and it is used in different applications of mechanics and acoustics [18]. Jerk is a common daily experience. It should be minimized for smooth driving of a car or bus and the design of roller coasters and other rides. This subject is called the *Jerk dynamics* [19], [20].

An autonomous jerk circuit can be modeled by the differential equation [5]: $\dddot{x} + A\ddot{x} + x + f(x) = 0$. Which is the Jerk equation with Jerk function $J = -A\ddot{x} - x - f(x)$. Function $f(x)$ can vary from simple quadratic, piece-wise linear, to any nonlinear trigonometric, exponential or hyperbolic functions. Many chaotic jerk functions have been proposed in literature [21], [22].

In this paper, we consider the dynamics of a simple autonomous jerk circuit in which the nonlinear component necessary for generating chaotic oscillations is a transistor working as a switch. Due to exponential current-voltage (i–v) characteristics of the BJT, this novel Jerk circuit is highly

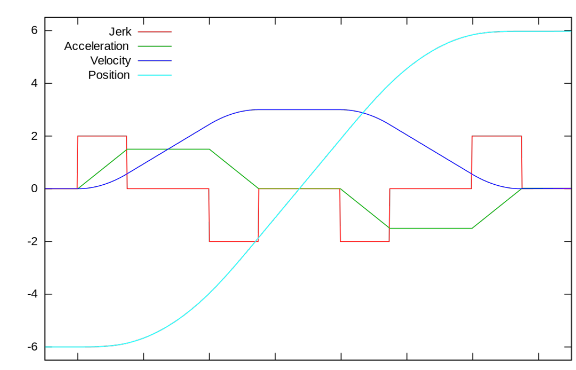


Fig. 1: Schematic diagram explains higher derivatives of motion [18]

nonlinear and thus can exhibit chaotic dynamics with multiple attractors for different values of control parameter.

II. Compact Literature Review

In nonlinear electronic circuits, various jerk functions have been proposed and implemented using electronic components with different ranges of complexity, stability and sensitivity to tolerance levels. Most of the jerk functions have been simulated and implemented using electronic components like diodes [5] and op-amps [23]. Theses circuits can display periodic and chaotic responses for different values of control parameters.

In literature, some of the autonomous jerk circuits were also presented and their transitions to chaos were observed using bifurcation diagrams. A bifurcation diagram provides enormous information about chaotic systems. It also identifies a route to chaos and by that *roadmap*, researchers have optimized and simplified the autonomous dynamic jerk systems [24]. Some of the Jerk systems can exhibit multiple attractors and different routes to chaos which depends on the nonlinear element [25]. Kengne et al. demonstrated that a jerk system with simple cubical nonlinear jerk function [26] or a diode bridge rectifier [27] can exhibit multiple attractors.

Different research groups are trying to optimize chaotic electronic circuits. This optimization is done by calculating parameter regions in jerk functions with precision and making the electronic circuit more stable [28], by simplifying the hardware design; by modeling the control parameter with simple electronic components to achieve accuracy and precision [5]; or by choosing simple nonlinear functions which includes the use of nonlinear elements [29].

In this paper, nonlinear function $f(\dot{x})$ has been chosen to be piece-wise linear function that can be modeled by a transistor working as a switch and the route to chaos of that electronic circuit is determined by varying its control parameter A. The simulation and experimental results are also compared with bifurcation diagram of the jerk function.

III. Mathematical Modeling

In this section, the implementation of the proposed jerk circuit is described. As discussed in the previous section, the jerk circuit has the Jerk function which can have any nonlinear term. The nonlinear function is added using transistor as switch. An ideal switch is the circuit that gives a zero voltage across it when it is ON while it will behave as open circuit in OFF state. Thus, a switch can be modeled as a piece-wise linear function that can be written as:

$$f(x) = \begin{cases} 1 \ when \ v_{be} > 0V \\ 0 \ when \ v_{be} < 0V \end{cases} \quad (1)$$

But as the transistor is a nonlinear device, therefore, its ON state characteristics are exponential which can be written as:

$$f(x) = \begin{cases} I_s e^{v_{be}/V_t} \ when \ v_{be} > 0V \\ \quad 0 \ when \ v_{be} < 0V \end{cases} \quad (2)$$

Using this nonlinear function, the equation (1) of jerk circuit under ON state of the system will become:

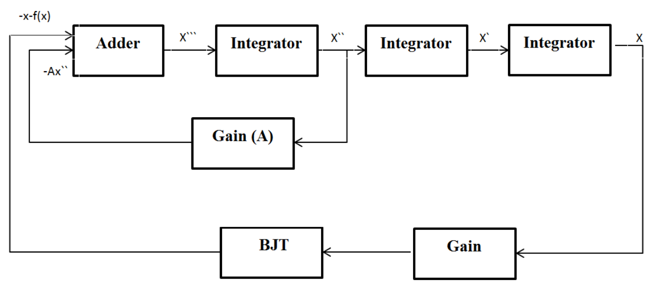


Fig. 2: Block diagram of the Jerk circuit

$$\dddot{x} + A\ddot{x} + x + I_s e^{v_{be}/V_t} = 0 \quad (3)$$

Here A is the control parameter of the system. This equation is implemented in the form of an electronic circuit using op-amp as an integrator and an adder in the block diagram illustrated in fig. 2.

To implement the adder and integrator circuit, resistor and capacitors with op-amp are used. To introduce the nonlinear term, BJT is used as switch. Fig. 3 shows the complete circuit diagram. Op-amps $U2B, U1A$ and $U2A$ are acting as integrators. $U1B$ is acting as the inverter circuit. The output of op-amp $U1A$ is $-\dot{x}$, while the output of $U2A$ is x. $U2B$ is giving the output $\ddot{x}$, which is integrating $\dddot{x}$ at is input. Mathematically,

$$\dddot{x} = -A\ddot{x} - x - I_s e^{v_{be}/V_t} \quad (4)$$

Transistor, $Q1$, $2N3904$ is acting as a switch while $U1B$ is acting as an inverter and providing the output x. Considering all capacitors to be equal $(C1 = C2 = C3 = C)$ complete output of the circuit can be written as:

$$\dddot{x} = \frac{1}{RC}(-\frac{R_7}{R_6}\ddot{x} - \frac{R_5}{R_4}x - I_s e^{v_{be}/V_t}) \quad (5)$$

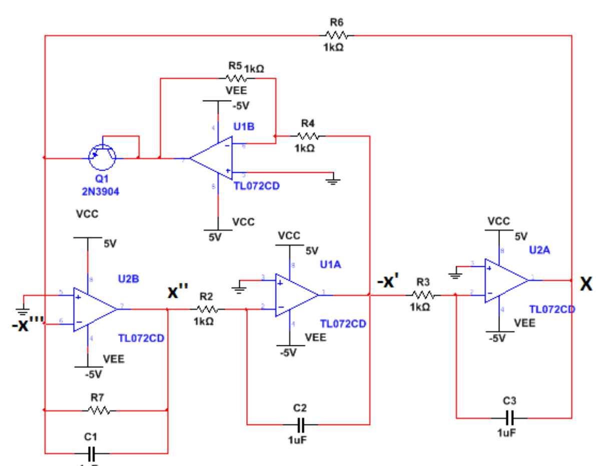
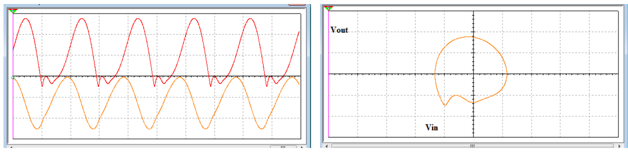
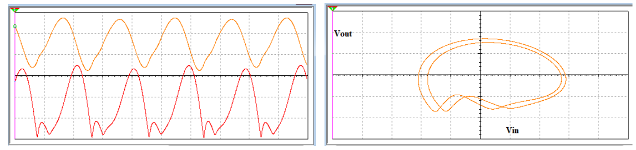


Fig. 3: Circuit diagram of the proposed jerk circuit



(a) Input and output waveforms (b) Phase portrait (Vin vs. Vout) (X Scale 0.5V/div, Y Scale 0.5V/div)

Fig. 4: Periodic behavior with period-1 at $R_7 = 600\Omega$ (v_{in} above 1V/div, v_{out} below 0.5V/div, Time 5ms/div)



(a) Input and output waveforms (b) Phase portrait (X Scale 0.5V/div, Y Scale 0.5V/div)

Fig. 5: Period-2 waveform and attractor at $R_7 = 660\Omega$ (input above 1V/div, output below 0.5V/div, Time 5ms/div)

If all the resistors are chosen to be equal except R_7 and taking $R_2C = 1$ then the eq (6) will be simplified as:

$$\dddot{x} = -\frac{R_7}{R_6}\ddot{x} - x - I_s e^{v_{be}/V_t} \tag{6}$$

Comparing eq (7) with eq (4), the control parameter A of the system can be written in terms of resistors as:

$$A = \frac{R_7}{R_6} \tag{7}$$

All the resistors connected in the circuit are of $1K\Omega$ and all the capacitors are of 1μF. R_7 was changed from 500Ω to $1K\Omega$ to observe the behavior of the system under different values of control parameter. While operating this circuit, the switch was turned ON. Until capacitors achieve their steady states, the circuit was kept on oscillating at certain frequency. After the transients, the output gave period-1, period-2, period-4 and chaotic response for different values of R_7.

This simple circuit can be controlled by only one parameter which can be changed easily by varying the value of resistance R_7. The natural frequency of the system can be written in terms of the values of resistors and capacitors attached to the integrator as:

$$f_r = \frac{1}{2\pi R_7 C} \tag{8}$$

IV. Simulation Results

After designing the values of resistor and capacitor, the proposed jerk circuit was simulated in NI MULTISIM and the route to chaos was determined by varying the value of resistor R_7.

Figs. 4a and 4b present the system response at $R_7 = 600, (A = 0.6)$ which corresponds to period-1 attractor and showing periodic response of the system. The system showed a stable limit cycle behavior with sustained oscillation with same frequency as that of natural frequency of the dynamical system. At lower values of A, no chaos was found in the system response. As the control parameter was increased, the period-doubling phenomenon occurred, the circuit showed period-2 and period-4 responses at $R_7 = 660$ and $R_7 = 700$ respectively before approaching chaotic response after $R_7 = 800\Omega$.

Figs. 5a and 5b show the time response and the attractor at $R_7 = 660\Omega$ which is a period-2 attractor. The period-doubling started just after 740Ω. Both time response and phase portrait are indicating persistent oscillations with no damping and frequencies with the multiples of natural frequency with few harmonics in them.

Similarly, fig. 6 present the time response and the attractor at $R_7 = 700$ which is period-4 attractor. So the number of period increases as the control parameter has been increased from $660\Omega - 700\Omega$. This dynamical system demonstrated chaotic behavior as it is depicted in fig. 7 at $R_7 = 1000, (A = 1.00)$. It followed the bifurcation route to chaos and showed strange attractor as control parameter A increases beyond 1.00. Table I summarizes the simulation results.

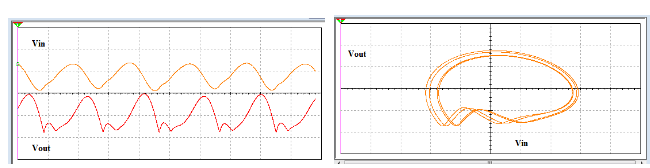
TABLE I: Summary of Simulation Results

Sr. #	Range of Resistance	Attractor Type
1	R_7 <100	Point Attractor
2	100 < R_7 <600	Period-1 Attractor (Limit Cycle)
3	601 < R_7 <660	Period-2 Attractor
4	661 < R_7 <700	Period-4 Attractor
5	R_7 >700	Chaotic (Strange Attractor)

V. Hardware Results

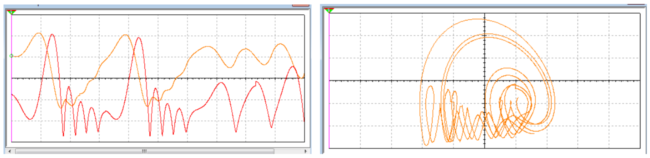
Similar approach was used to observe the route of chaos as was discussed in the simulation section. The value of resistor was varied linearly using a potentiometer and bifurcation was observed on Digital Storage Oscilloscope (DSO) SIGLENT $SDS1072CML$. It showed periodic and chaotic responses for different values of resistor R_7. The system showed period-1 response (*limit cycle attractor*) at $R_7 = 600\Omega$ as shown in fig. 7a and fig. 7b. Fig. 7a presents the input and output waveforms on 600Ω while in fig. 7b, the phase portrait is also showing limit cycle attractor.

By increasing the resistance, the bifurcation (branching) started after $R_7 = 650\Omega$. But bifurcation was so sensitive to the value of resistor that it was only observed during simulations. On hardware, due to tolerance of resistance values,



(a) Input and output waveshapes (b) Attractor

Fig. 6: Period-4 response at $R_7 = 700\Omega$ (input above 1V/div, output below 0.5V/div, Time 5ms/div)



(a) Input and output waveforms (b) Phase-plane figure (X Scale 0.5V/div, Y Scale 0.5V/div)

Fig. 7: Aperiodic behavior at $R_7 = 1K\Omega$

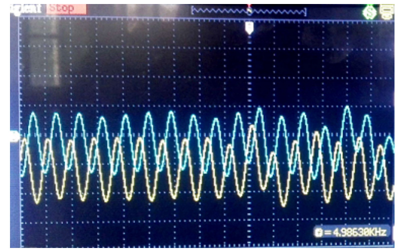


Fig. 9: Experimental input and output voltages at $R_7 = 700\Omega$ (Period-4 response)

only period-4 was observed at $R_7 = 700\Omega$. At $R_7 = 700\Omega$, input and output waveforms are shown in fig. 9.

At 820Ω, the system demonstrated chaotic dynamics. The system behavior at $R_7 = 820\Omega$ is shown in fig. 10. Fig. 10a shows the input and output waveforms of the system and fig. 10b illustrates corresponding phase portrait which is showing a strange attractor. The shape of the attractor changes by increasing the value of the resistor further but the system exhibited chaotic behavior. Fig. 11 presents the attractor at $R_7 = 1000\Omega$.

When the value of resistor (control parameter) was increased further, the shape of attractor changed into double lobe. The shape of the attractor at $R_7 = 2000\Omega$ is depicted in fig. 12. This change in attractor shape can also be observed from the bifurcation diagram. As the number of exponential curves keeps on increasing, the shape of attractor will also be changing and system will become more and more chaotic. Table II presents the experimental results.

VI. Discussion

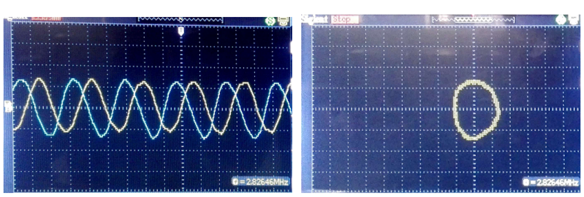
To understand the dynamics of the system and to get in-depth knowledge of the vector field solutions, the bifurcation diagram has been plotted, shown in fig. 12, on MATLAB using the differential equation of the system. The bifurcation diagram is a graph between one of the control parameters and the input which provides the models of transitions from one phase to other and also provides the range of control parameters in which the system will remain stable. The graph is between A and x. From the given bifurcation diagram, it can be seen that until $A = 0.60, (R_7 = 600\Omega)$, the system shows period-1 characteristics. After $A = 0.60$, the bifurcation starts, thus causing period-2 system. Afterward, new bifurcation occurs and it keeps on increasing and shooting away from the main axis, thus creating more and more periods, and leading the system into chaos. This bifurcation is instantaneous from $A = 0.60$ to $A = 0.80$. Period-4 was obtained from simulations at $R_7 = 700\Omega, (A = 0.7)$ at which system has four asymptotic curves, which represents period-4 characteristics.

Through bifurcation diagram shown in fig. 12, it is seen that the proposed chaotic jerk circuit has a unique bifurcation route due to the presence of BJT as nonlinear element which has exponential trans-conductance characteristics. Although BJT is used as a switch is the circuit, but still at higher values of control input, its exponential characteristics are changing the overall response of the system. Due to exponential nature of nonlinearity, the bifurcation diagram is indicating many positive Lyapunov exponents as the control element increases and system is becoming more and more chaotic and unstable with higher values of control element.

TABLE II: Summary of Hardware Results

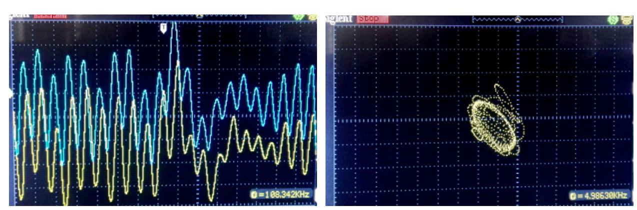
Sr. #	Range of Resistance	Attractor Type
1	R_7 <100	Point Attractor
2	100 <R_7 <600	Period-1 Attractor (Limit Cycle)
3	670 <R_7 <700	Period-4 Attractor
4	R_7 = 820	Chaotic (Strange Attractor), fig. 10b
5	R_7 = 1K	Strange Attractor, fig. 11
6	R_7 = 2K	Chaotic Attractor) fig. 12

The system is highly sensitive to the component values and the initial charge on the capacitors which provide the initial conditions for the system. This sensitivity was observable in the experimental results. This system is third order system with one exponential nonlinear term. Thus, it requires minimum of three precise initial conditions to estimate the response of the system. While ignoring the parasitic capacitances of a BJT and their effects on set of frequencies, this analysis is still imperfect. The critical and sub-critical bifurcation routes of



(a) Input and Output voltages (b) Phase-plane diagram

Fig. 8: Period-1 response, periodic behavior at $R_7 = 600\Omega$



(a) Input and output waveforms (b) Strange (chaotic) attractor

Fig. 10: Aperiodic (chaotic) behavior at $R_7 = 820\Omega$

(a) Phase Portrait at $R_7 = 1000\Omega$ (b) Phase Portrait at $R_7 = 2000\Omega$

Fig. 11: Change in shape of strange attractor

the system can be the part of future research. The route of chaos was observed in simulations with zero initial conditions.

With the diverse variety of attractors in a simple electronic circuit, this circuit can be very useful in chaotic encryption in communication as it has a completely unique response on even slightly different control input. This circuit is easy to implement and can be used to analyze chaos theory in the domain of chaotic electronics. It can be used to investigate the erratic dynamics of electronic circuit at undergraduate or graduate level.

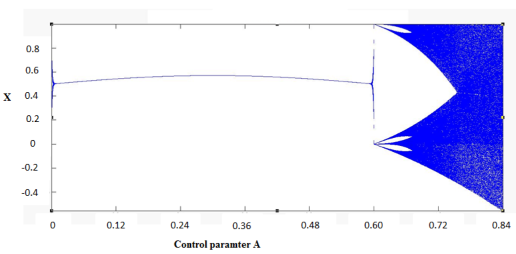


Fig. 12: Bifurcation diagram of the proposed Jerk circuit

VII. Conclusion

In this paper, a novel jerk circuit was proposed using a nonlinear switch as Jerk function. Its period bifurcation and change in shape of chaotic attractor was observed. The system equations were simulated and implemented on hardware using op-amp as integrator and as adder. The gain of the integrator was chosen as control parameter and by changing the control parameter, the bifurcation was observed. This study presented a simple technique to observe and study chaotic response of a circuit using op-amps and nonlinear elements like BJTs. In future, researchers can use this circuit and analyze the strange attractor and its shape changings could be explained.

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