Chapter 1

Metric spaces

1.1 Definitions and examples

Lemma 1.1.1. Let $(x_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be another real number. Then $(x_n)_{n=m}^{\infty}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

Exercise 1.1.1. Prove Lemma 1.1.1.

Solution 1.1.1. By definition of limit, for any $\varepsilon > 0$ there exists an $N \geq m$ such that $d(x_n, x) = |x_n - x| < \varepsilon$ for all $n \geq N$. This is exactly the definition of convergence in previous book.

Exercise 1.1.2. Show that the real line with the metric $d(x, y) := |x - y|$ is indeed a metric space. (Hint: you may wish to review your proof of Proposition 4.3.3.)

Solution 1.1.2. We verify the four properties listed in Definition 1.1.2

1. $d(x, x) = |x - x| = 0$.
2. For all $x \neq y$, $d(x, y) = |x - y| > 0$.
3. For all x and y, $d(x, y) = |x - y| = |y - x| = d(y, x)$.
4. For all x, y and z, $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$.

Exercise 1.1.3. Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function.

(a) Give an example of a pair (X, d) which obeys axioms (bcd) of Definition 1.1.2, but not (a). (Hint: modify the discrete metric.)

(b) Give an example of a pair (X, d) which obeys axioms (acd) of Definition 1.1.2, but not (b).

(c) Give an example of a pair (X, d) which obeys axioms (abd) of Definition 1.1.2, but not (c).

(d) Give an example of a pair (X, d) which obeys axioms (abc) of Definition 1.1.2, but not (d). (Hint: try examples where X is a finite set.)

Solution 1.1.3. In general, setting X to be a finite set and defining d by specifying values for all pairs of elements can produce a lot of metric spaces for each of the problem. For this exercise to be more challenging, we try to avoid abusing this approach.

(a) Let d be the metric such that for all $x, y \in X$, $d(x, y) = 1$. This is similar to the discrete metric except that $d(x, x) = 1$ so it does not obey (a).

(b) Similar to (a), we can as well set $d(x, y) = 0$ for all $x, y \in X$.

(c)

(d) Let $X = \{1, 2, 3\}$ and define $d(x, x) = 0$ for $x \in X$, $d(1, 2) = d(2, 1) = 1$, $d(2, 3) = d(3, 2) = 1$ and finally $d(1, 3) = d(3, 1) = 3$.

Exercise 1.1.4. Show that the pair $(Y, d|_{Y\times Y})$ defined in Example 1.1.5 is indeed a metric space.

Solution 1.1.4. Each of the four properties remains after d is restricted to $Y \subset X$ and is vacuous to verify.

Exercise 1.1.5. Let $N \geq 1$, and let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. Verify the identity

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right)$$

and conclude the *Cauchy-Schwarz inequality*

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}} \tag{(1.3)}$$

Then use the Cauchy-Schwarz inequality to prove the triangle inequality

$$\left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}}$$

Solution 1.1.5.

Exercise 1.1.6. Show that $(\mathbf{R}^n, d_{l^2})$ in Example 1.1.6 is indeed a metric space. (Hint: use Exercise 1.1.5.)

Exercise 1.1.7. Show that the pair $(\mathbf{R}^n, d_{l^1})$ in Example 1.1.7 is indeed a metric space.

Exercise 1.1.8. Prove the two inequalities in (1.1). (For the first inequality, square both sides. For the second inequality, use Exercise 1.1.5.)

Exercise 1.1.9. Show that the pair $(\mathbf{R}^n, d_{l^\infty})$ in Example 1.1.9 is indeed a metric space.

Exercise 1.1.10. Prove the two inequalities in (1.2).

Exercise 1.1.11. Show that the discrete metric $\mathbf{R}^n, d_{\text{disc}}$ in Example 1.1.11 is indeed a metric space.

Exercise 1.1.12. Prove Proposition 1.1.18.

Exercise 1.1.13. Prove Proposition 1.1.19.

Exercise 1.1.14. Prove Proposition 1.1.20. (Hint: modify the proof of Proposition 6.1.7.)

Exercise 1.1.15. Let

$$X := \left\{(a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty\right\}$$

Exercise 1.1.16.