ত্রিকোণমিতি

Domain and Range:

y_1 = asinx ∴ f(x) = a sinx ; y → x x → (radian)
x = 0^0 , y = 0 ; x =$\pm\frac{\pi}{6}$, y =$\pm\frac{1}{2}$a ; x = $\pm\frac{\pi}{4}$, y = $\pm\frac{1}{\sqrt{2}}a$; x = $\pm\frac{\pi}{3}$, y = $\pm\frac{\sqrt{3}}{2}a$; x = $\pm\frac{\pi}{2}$, y = $\pm a$
∴ Range: $|y| \leq a \Rightarrow -a \leq y \leq a \Rightarrow -a \leq asinx \leq a \Rightarrow -1 \leq sinx \leq 1 \rightarrow sinx$ এর ব্যাবধি
$\Rightarrow -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \rightarrow x$ এর ব্যাবধি $\Rightarrow 0 \leq x \leq \pi$ (n$\in Z$), $0 \rightarrow 2\pi$ এর মধ্যে একটি পর্যায় সম্পন্ন হয়েছে।
∴ sinx এর পর্যায় $2\pi, 4\pi, 6\pi, 8\pi$ ইত্যাদি বৃত্তের সংখ্যা 4টি

y_2 = a cosx ; x = 0°, y = a ; x = π, y = − a .
এখানে ঘটনা একই ,
Range: $|y| \leq a \Rightarrow -a \leq a\ cox \leq a \Rightarrow -1 \leq cosx \leq 1 \Rightarrow \pi \leq x \leq 0$ ∴x = 2n$\pi, 2n\pi \pm \theta$ (n ∈ z)
যুগ্নভাবে দুটি ক্ষেত্রঃ $y_1^2 + y_2^2 = a^2 (\cos^2\theta + \sin^2\theta)$ $y_1 = a^2(1)$∴ $\cos^2 x + \sin^2 x = 1$
here, x = $\pi/4$ কারণ দুই বাহু সমান ।

$y_1 = y_2$ এর জন্য, sinx = cosx = cos($\pi/2 - x$)বা, sinx = sinx ($\pi/2 + x$) , দশা পার্থক্য= $\pi/2$
tanx = $\frac{sinx}{cosx}$; where cosx$\neq$ 0 i.e, x $\neq \pi/2$
 প্রতি n$\frac{\pi}{2}$ (n $= \pm 1, \pm 3, \pm 5 \ldots\ldots\ldots\ldots\ldots$) পয়েন্টে tanx cut হয়।
সুতরাং $]-\frac{\pi}{2}, \frac{\pi}{2}[$; $]\frac{\pi}{2}, \frac{3\pi}{2}[$; $]-\frac{3\pi}{2}, -\frac{\pi}{2}[$ tanx defined

∴ tanx এর পর্যায় π ; cosx এবং sinx এর পর্যায় 2π ∴sinx ± cosx এর পর্যায় 2π

Type-01: সূত্রগুলোর প্রয়োগ সংক্রান্ত সমস্যাবলী

$\sin^2\theta + \cos^2\theta = 1$; $\sec^2\theta + \tan^2\theta = 1$; $\mathrm{cosec}^2\theta + \cot^2\theta = 1$
sin(A ± B) = sinA. cosB ± cosA.sinB; cos (A ±B) = cosA. cosB ∓ sinA. SinB.
tan (A ± B) = $\frac{tanA \pm tanB}{1 \pm tanA\ tanB}$; cot (A ± B) = $\frac{cotB \pm cotA}{cotA.cotB \mp 1}$
sin(A + B) sin(A− B) = $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
cos (A + B) cos (A−B) = $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

EXAMPLE-01: প্রমান কর যে, tan$\alpha -$ tanβ = a , cotβ −cotα = b এবং তবে cot $(\alpha - \beta) = \frac{1}{a} + \frac{1}{b}$.
প্রমাণঃ tan$\alpha -$ tanβ = a $\Rightarrow \frac{sin\alpha}{cos\alpha} - \frac{sin\beta}{cos\beta}$ = a $\Rightarrow \frac{sin\alpha cos\beta - cos\alpha sin\beta}{cos\alpha cos\beta} = a$ ∴ $\frac{1}{a} = \frac{cos\alpha.cos\beta}{\sin(\alpha-\beta)}$

এবং cot$\beta -$ cotα = b ∴$\frac{1}{b} = \frac{sin\alpha sin\beta}{sin(\alpha-\beta)}$ ∴$\frac{1}{a} + \frac{1}{b} = \frac{\cos(\alpha-\beta)}{sin(\alpha-\beta)}$ = cot$(\alpha - \beta)$

$$[(a+b)/ab = cot(\alpha - \beta) \Rightarrow (a + b)\ tan\ (\alpha - \beta) = ab]$$

EXAMPLE-02: যদি $x = r\sin(\theta + 45^0)$ এবং $y = r\sin(\theta - 45^0)$ হলে প্রমান কর যে, $x^2 + y^2 = r^2$

প্রমান: $x = r\cos\{90^\circ - (\theta + 45^\circ)\} = r\cos(\theta - 45^0)$ এবং $y = r\sin(\theta - 45^0)$

$\therefore x^2 + y^2 = r^2$

EXAMPLE-03: $4n\theta = \pi$ হলে, প্রমান কর যে, $\tan.\theta\ \tan2\theta, \tan3\theta$ .... $\tan(2n-3)\theta\ \tan(2n-2)\theta\ \tan(2n-1)\theta = 1$.

$2n\theta = \frac{\pi}{2} \Rightarrow 2n\theta - \theta = \frac{\pi}{2} - \theta \Rightarrow (2n-1)\theta = \frac{\pi}{2} - \theta . \therefore \tan(2n-1)\theta = \cot\theta . \therefore \tan\theta = \cot\theta$

$\tan3\theta = \cot\theta \therefore \tan5\theta = \cot\theta$

$\therefore (2n-2)\theta = \frac{\pi}{2} - 2\theta . \therefore \tan(2n-2)\theta = \cot2\theta$.

$\tan2\theta = \cot2\theta, \tan4\theta = \cot2\theta, \tan\theta . \tan(2n-1)\theta = 1, \tan2\theta\ \tan(2n-2)\theta = 1$

$\tan3\theta\ \tan(2n-3)\theta = 1$

$\therefore \tan.\theta\ \tan2\theta, \tan3\theta$ .... $\tan(2n-3)\theta\ \tan(2n-2)\theta\ \tan(2n-1)\theta = 1$.

EXAMPLE-04: $\sin\theta = k\cos(\theta - \alpha)$ হলে দেখাও যে $\cot\theta = \frac{1-k\sin\alpha}{k\cos\alpha}$.

$\frac{1}{k} = \frac{\cos(\alpha - \theta)}{\sin\theta} = \cot\theta\cos\alpha + \sin\alpha . \therefore \cot\theta = \frac{1-k\sin\alpha}{k\cos\alpha}$.

EXAMPLE-05: $\tan\beta = \frac{n\sin\alpha.\cos\alpha}{1-n\sin^2\alpha}$ হলে প্রমান কর যে, $\tan(\alpha - \beta) = (1-n)\tan\alpha$

$\tan\beta = \frac{n\sin\alpha.\cos\alpha}{1-n\sin^2\alpha} = \frac{n\tan\theta}{\sin^2\alpha}$

$\therefore \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha.\tan\beta} = \frac{\tan\alpha - \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}{1+\tan\alpha\frac{b\sin\alpha\cos x\alpha}{1-n\sin^2\alpha}} = \frac{\tan\alpha - n\tan\alpha\sin^2\alpha - n\sin\alpha\cos\alpha + \sin^2\alpha - \sin^2\alpha}{1-n\sin^2\alpha + n\tan\alpha.\sin\alpha\cos\alpha}$

$\tan(\alpha - B) = \tan\alpha - n\tan\alpha + n\sin\alpha.\cos\alpha - n\sin\alpha . \cos x\alpha = (1-n)\tan\alpha$ (proved)

$(1-\cos^2\alpha = \sin^2\alpha$ এবং $\tan\alpha.\sin\alpha.\cos\alpha = \sin^2\alpha)$

EXAMPLE-06: $\tan\alpha = \frac{a}{b}$ প্রমান কর যে, $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta - \alpha)$

$b = r\sin\alpha$, $a = r\cos\alpha$ ধরা যায়, যেখানে, $r = \sqrt{a^2+b^2}$.

$\therefore a\cos\theta + b\sin\theta = r\cos(\theta - \alpha) = \sqrt{a^2+b^2}\cos(\theta - \alpha)$.

EXAMPLE-07: $\tan{}^{\theta}/_2 = \tan^3\frac{\varphi}{2}$ এবং $\tan\varphi = 2\tan\alpha$ হলে প্রমান কর, $\theta + \varphi = 2\alpha$

প্রমান : $\tan({}^{\theta}/_2 + {}^{\varphi}/_2) = \frac{\tan{}^{\theta}/_2 + \tan{}^{\varphi}/_2}{1-\tan{}^{\theta}/_2.\tan{}^{\varphi}/_2} = \frac{\tan^3\frac{\varphi}{2} + \tan{}^{\varphi}/_2}{1-\tan^4\frac{\varphi}{2}} = \frac{\tan{}^{\varphi}/_2(1+\tan^2{}^{\varphi}/_2)}{(1-\tan^2{}^{\varphi}/_2)(1+\tan^2{}^{\varphi}/_2)}$

$= \frac{1}{2}\ \frac{2\tan{}^{\varphi}/_2}{1-\tan^2{}^{\varphi}/_2} = \frac{1}{2}\tan\varphi = \frac{1}{2} \times 2\tan\alpha = \tan\alpha \Rightarrow {}^{\theta}/_2 + {}^{\varphi}/_2 = \alpha \therefore \theta + \varphi = 2\alpha$

EXAMPLE-08: $\sqrt{2}\cos A = \cos B + \cos^3 B$; $\sqrt{2}\sin A = \sin B - \sin^3 B$ হলে প্রমান কর: $\sin(A - B) = \pm\frac{1}{3}$

Solve: বর্গকরে : $\cos^2 B + \sin^2 B - 2\sin^4 B + 2\cos^4 B + \cos^6 B + \sin^6 B = 2$

$\Rightarrow 1 - 2(\sin^2 B - \cos^2 B)(\sin^2 B + \cos^2 B) + (cos^2B + sin^2B)^3 - 3sin^2B.cos^2B(cos^2B + sin^2B) = 2 \Rightarrow 1 - 2(\sin^2 B - \cos^2 B) + 1 - 3sin^2B.cos^2B = 2$

$\Rightarrow -2(2\sin^2 B - 1) - 3sin^2B + 3sin^4B = 0$

$\Rightarrow 3\sin^4 B - 7\sin^2 B + 2 = 0 \therefore \sin B = \pm\frac{1}{\sqrt{3}}$; $\cos B = \pm\sqrt{{}^2/_3}$

$\sin A = \pm\frac{4}{3\sqrt{6}}; cosA = \pm\frac{1}{3\sqrt{3}}$ or, $\sin A = \pm\frac{\sqrt{2}}{3\sqrt{3}}, cosA = \frac{5}{3\sqrt{3}}$

$\therefore \sin(A - B) = \sin A.\cos B - \cos A.\sin B = \pm\left(\frac{4}{3\sqrt{6}} \times \frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{3\sqrt{3}}.\frac{1}{\sqrt{3}}\right) = \pm(\frac{4}{9} - \frac{1}{9}) = \pm\frac{1}{3}$ <u>proved.</u>

again: $\sin(A - B) = \sin A.\cos B - \cos A.\sin B = \pm(\frac{\sqrt{2}}{3\sqrt{3}} \times \frac{\sqrt{2}}{\sqrt{3}} - \frac{5}{3\sqrt{5}}.\frac{1}{\sqrt{3}}) = \pm(\frac{2}{9} - \frac{5}{9}) = \mp\frac{1}{3}$

Try yourselfঃ

01. যদি $\cot\alpha + \cot\beta = a, \tan\alpha + \tan\beta = b$ এবং $\alpha + \beta = \theta$ হয় তবে
প্রমান করঃ $(a - b)\tan\theta = ab$

02. $A + B + C = \pi$ এবং $\cos A = \cos B.\cos C$ হয়,তবে প্রমান কর যে,$\tan A = \tan B + \tan C$

03. $\tan\alpha - \tan\beta = x$ এবং $\cot\beta - \cot\alpha = y$ হলে প্রমান কর যে, $\cot(\alpha - \beta) = \frac{1}{x} + \frac{1}{y}$.

04. $\tan\beta = \frac{sin2\alpha}{5+cos2\alpha}$ হলে, প্রমান কর যে,$3\tan(\alpha - \beta) = 2\tan\alpha$

05. $\theta - \alpha$ সুক্ষ্মকোণ এবং $\sin\theta + \sin\varphi = \sqrt{3}(cos\varphi - cos\theta)$ হলে দেখাও যে, $sin3\theta - sin3\varphi = 0$

06. $\cos(\alpha - \beta)\, cos\gamma = \cos(\alpha - \gamma + \beta)$ হলে দেখাও যে, $\cot\alpha + \cot\beta = 2\cos\gamma$.
অথবা,$\cot\alpha, \cot\gamma, \cot\beta$ সমান্তর প্রগমনভুক্ত।

07. $(a^2 - b^2)\sin\theta + 2ab\cos\theta = a^2 + b^2$ হলে, θ সুক্ষ্ম ও ঋণাত্মক কোন হলে ,$\tan\theta$ এর মান নির্ণয় কর।
Ans.$tan\theta = \frac{2ab}{a^2-b^2}$

08. $\tan\beta.\sin(\alpha + \gamma) = 2sin\alpha.sin\gamma$ হলে দেখাও যে, $\tan\alpha, \tan\beta, \tan\gamma$ হারমনিক প্রগ্রেশনভূক্ত।

09.$sin\alpha + cosec\alpha = 2$ হলে দেখাও যে,$sin^n\alpha + cosec^n\alpha = 2$

10.$xsin^3x + ycos^3x = sin\alpha cos\alpha$ও $xsin\alpha - ycos\alpha = 0$ হলে দেখাও যে,$x^2 + y^2 = 1$

11.$\frac{cos^4y}{cos^2x} + \frac{sin^4y}{sin^2x} = 1$ হলে দেখাও যে, $\frac{sin^4x}{sin^2y} + \frac{cos^4x}{cos^2y} = 1$

12. ACB ও AC^1B 1 ব্যাসার্ধ বিশিষ্ট দুটি বৃত্ত । পরস্পরকে অন্ত:স্থভাবে A ও B বিন্দুতে ছেদ করেছে। APBQ অংশের ক্ষেত্রফল কত? Ans: $\frac{2\pi-3\sqrt{3}}{3}$ sq.unit.

13. $cos\alpha + sec\alpha = \frac{2}{5}$ হলে দেখাও যে,$cos^n\alpha + sec^n\alpha = 2^n + 2^{-n}$

Type-02: $\sum T$ সংক্রান্ত সমস্যাবলী

EXAMPLE-01: $\cot(\alpha-\beta) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$ হলে দেখাও যে, $\sum cos\alpha = 0, \sum sin\alpha = 0$

$\cos\beta.\cos\gamma + \sin\beta.\sin\gamma + \cos\gamma\cos\alpha + \sin\gamma.\sin\alpha + \cos\alpha.\cos\beta + \sin\alpha.\sin\beta = -{}^{3}/_{2}$

$\Rightarrow 2\cos\beta.\cos\gamma + 2\sin\beta.\sin\gamma + 2\cos\gamma.\cos\alpha + 2\sin\gamma.\sin\alpha + 2\cos\alpha.\cos\beta + 2\sin\alpha.\sin\beta + 3 = 0$

$[\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 3]$

$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a+b+c)^2$

$\therefore (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0 \Rightarrow (\sum cos\alpha)^2 + (\sum sin\alpha)^2 = 0$

বর্গের সমষ্টি শূণ্য হলে তারা আলাদা ভাবে শূণ্য হতে হবে। $\sum cos\alpha = 0, \sum sin\alpha = 0$

এখন তোমরা উল্টাভাবে প্রমান কর

Try yourselfঃ 01.$\sum cosec\alpha = 0$ হলে দেখাও যে, $(\sum sin\alpha)^2 = \sum sin^2\alpha$.

02. $\sum cot\alpha = 0$ হলে দেখাও যে $(\sum tan\alpha)^2 = \sum tan^2\alpha$.

Type-03: সূত্রগুলোর প্রয়োগ সংক্রান্ত সমস্যাবলী

*2sinA. cosB = sin(A +B) + sin (A − B), * 2 cosA.sinB = sin (A + B) − sin(A −B)

2cosA. cosB = cos(A + B) + cos(A −B),*2sinA. sinB = cos(A − B) − cos(A + B)

A + B = C A − B = D প্রয়োগ করি,

* $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$,* $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

$\cos C + \cos D = 2\cos\frac{C+D}{2}.\cos\frac{C-D}{2}$,$\cos D - \cos C = 2\sin\frac{C+D}{2}.sin\frac{C-D}{2}$.

EXAMPLE-01: প্রমান কর,$\tan 20^0 . \tan 40^0 . \tan 80^0 = \sqrt{3}$

L.H.S = $\frac{sin80^0.sin40^0.sin20^0}{cos80^0.cos40^0.cos20^0} = \frac{(cos40^0 - cos120^0).sin20^0}{(cos120^0 + cos40^0)cos20^0} = \frac{\frac{1}{2}\times 2cos40^0.sin20^0 + \frac{1}{2}sin20^0}{-\frac{1}{2}cos20^0 + \frac{1}{2}\times 2\ cos40^0.cos20^0}$

$= \frac{\frac{1}{2}sin60^0 - {}^{1}/_{2}sin20^0 + \frac{1}{2}\ sin20^0}{-\frac{1}{2}cos20^0 + \frac{1}{2}cos60^0 + \frac{1}{2}cos20^0} = tan60^0 = \sqrt{3}$ Ans:

EXAMPLE-02: প্রমান কর,$\cos 20^0. \cos 40^0. \cos 60^0. \cos 80^0 = \frac{1}{16}$

L.H.S= $cos20^o cos40^o cos60^o cos80^o = \frac{1}{2}\left[\frac{1}{2}(cos60^o + cos20^o)\right] cos80^o$

$= \frac{1}{4}\left(\frac{1}{2} + cos20^o\right) cos80^o = \frac{1}{8}cos80^o + \frac{1}{4}cos80^o cos20^o$

$= \frac{1}{8}cos80^o + \frac{1}{4}\left[\frac{1}{2}(cos100^o + cos60^o)\right] = \frac{1}{8}cos80^o + \frac{1}{4}\left[\frac{1}{2}(cos100^o + cos60^o)\right]$

$= \frac{1}{8}cos80^o + \frac{1}{8}cos100^o + \frac{1}{8}.\frac{1}{2} = \frac{1}{8}(cos80^o + cos100^o) + \frac{1}{16}$

$= \frac{1}{8}\times 2cos90^o.cos10^o + \frac{1}{16} = \frac{1}{16}$

EXAMPLE-03: প্রমান কর, $\tan 70^0 = \tan 20^0 + 2\tan 50^0$

প্রমাণ: $\tan 70^0 = \tan(20^0 + 50^0) = \frac{\tan 20^0 + \tan 50^0}{1 - \tan 20^0 . \tan 50^0}$

$\Rightarrow \tan 70^0 - \tan(90^0 - 20^0).\tan 20^0 + \tan 50^0 = \tan 20^0 + \tan 50^0$

$\Rightarrow \tan 70^0 - \tan 50^0 = \tan 20^0 + 2\tan 50^0$

$\therefore \tan 70^0 = \tan 20^0 + 2\tan 50^0$

EXAMPLE-04: যদি $sinx + siny = a$এবং$cosx + cosy = b$ হয়, তবে প্রমাণ কর যে, $tan\frac{1}{2}(x-y) = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

প্রমাণ:$2sin\frac{1}{2}(x+y)cos\frac{1}{2}(x-y) = a$..........(i), $2cos\frac{1}{2}(x+y)cos\frac{1}{2}(x-y) = b$..........(ii)

(i) এবং (ii)কে বর্গ করে যোগ করে পাই,$4cos^2\frac{1}{2}(x-y)\left\{sin^2\frac{1}{2}(x+y) + cos^2\frac{1}{2}(x+y)\right\}=a^2 + b^2$

$\Rightarrow 4\left\{1 - sin^2\frac{1}{2}(x-y)\right\}=a^2+b^2 \Rightarrow sin\frac{1}{2}(x-y) = \pm\frac{1}{2}\sqrt{4-a^2-b^2}; cos\frac{1}{2}(x-y) = \pm\frac{1}{2}\sqrt{a^2+b^2} \therefore tan\frac{1}{2}(x-y) = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

Try yourself :01. প্রমাণ কর যে, $\sin 20^0 \sin 40^0 \sin 60^0 \sin 80^0 = \frac{3}{16}$

02. $x\cos\alpha + y\sin\alpha = k = x\cos\beta + y\sin\beta$ এবং $\alpha + \beta = 2\theta$ হলে,প্রমান কর যে, $\tan\theta = \frac{y}{x}$

03. $\sin x = m\sin y$ হলে প্রমাণ করঃ $\tan\frac{1}{2}(x-y) = \frac{m-1}{m+1}\tan\frac{1}{2}(x+y)$

04. $(\theta - \varphi) < 90^0$ এবং $\sin\theta + \sin\varphi = \sqrt{3}(cos\varphi - cos\theta)$ হলে প্রমাণ করঃ $\sin 3\theta + \sin 3\varphi = 0$

05.$sin\frac{\pi}{16}sin\frac{3\pi}{16}sin\frac{5\pi}{16}sin\frac{7\pi}{16}$ =?Ans: $\frac{\sqrt{2}}{16}$

06.$\sin 65^o + \cos 65^o = \sqrt{2}\cos 20^o$

07. প্রমাণ কর যে,$\sin x\sin(x+30^o) + \cos x\sin(x+120^o) = \frac{\sqrt{3}}{2}$

08.$\sin\alpha = k\sin(\alpha+\beta)$ হলে প্রমাণ করঃ$\tan(\alpha+\beta) = \frac{\sin\alpha}{\cos\beta - k}$

09.$\tan\alpha = \frac{b}{a}$ হলে প্রমাণ করঃ$a\cos\theta + \mathrm{b}\sin\beta = \sqrt{a^2+b^2}\cos(\theta-\alpha)$

10.$\cos\alpha + \cos\beta = a$ $\sin\alpha + \sin\beta = b$ হলে প্রমাণ করঃ$\cos(\alpha-\beta) = \frac{1}{2}(a^2+b^2-2)$

11.$\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$ হলে প্রমাণ করঃ$1 + \cot\alpha\tan\beta = 0$

Type-04: গুণিতক কোণের ত্রিকোণমিতিক অনুপাত সংক্রান্ত সমস্যাবলী

sin2A = 2sinA. cosA = 2tanA . $\cos^2$A = $\frac{2tanA}{sec^2A} = \frac{2tanA}{1+tan^2A}$
cos2A = $\cos^2$A − $\sin^2$A=1-2 $\sin^2$A=2 $\cos^2$A-1
2$\sin^2$A = 1 - cos2A; 2$\cos^2$A = 1 + cos2A $\cos^2$a = $\frac{1}{2}$ (1+ cos2A)
tan2A = $\frac{2tanA}{1-tan^2A}$; sin2A = $\frac{2tanA}{1+tan^2}$; cos2A = $\frac{1-tan^2A}{1+tan^2A}$
sin3A = 3sinA − 4$\sin^3$A ; $\sin^3$A = $\frac{1}{4}$ (3sinA − sin3A)
cos3A = 4$\cos^3$A − 3cosA; $\cos^3$A $\frac{1}{2}$ (3sinA + sin3A)
$\therefore$ tan3A = $\frac{3tanA-tan^3A}{1-3tan^2A}$,

EXAMPLE-01: $a\cos\alpha + b\sin\alpha = a\cos\beta + b\sin\beta$ হলে প্রমান কর যে, $\cos(\alpha+\beta) = \frac{a^2-b^2}{a^2+b^2}$

প্রমান:$a(\cos\alpha - \cos\beta) = b(\sin\beta - \sin\alpha)$

$\frac{a}{b} = \frac{sin\beta-sin\alpha}{cos\alpha-cos\beta} \therefore \frac{a^2}{b^2} = \frac{sin^2\alpha+sin^2\beta-2\sin\alpha\sin\beta}{cos^2\alpha+cos^2\beta-2\cos\alpha\cos\beta}$

$\frac{a^2-b^2}{a^2+b^2} = \frac{sin^2\alpha+sin^2\beta-2\sin\alpha\sin\beta-cos^2\alpha-cos^2\beta+2\cos\alpha\cos\beta}{sin^2\alpha+sin^2\beta-2\sin\alpha\sin\beta+cos^2\alpha+cos^2\beta-2\cos\alpha\cos\beta}$

$= \frac{-\cos 2\beta-cos2\alpha+2\cos(\alpha+\beta)}{1+1-2\cos(\alpha-\beta)} = \frac{2\cos(\alpha+\beta)-2\cos(\alpha-\beta).\cos(\alpha+\beta)}{2\{1-\cos(\alpha+\beta)\}} = \frac{2\cos(\alpha+\beta)\{1-\cos(\alpha-\beta\}}{2(1-\cos(\alpha-\beta)\}}$

$\therefore \cos(\alpha+\beta) = \frac{a^2-b^2}{a^2+b^2}$

EXAMPLE-02: $a\cos\theta + b\sin\theta = c$ সমীকরনটি θ এর ভিন্ন দুটি মান α ও β দ্বারা সিদ্ধ হলে,দেখাও যে, $sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$

সমাধান:ধরি, $acos\theta = c - bsin\theta \Rightarrow a^2cos^2\theta = c^2 + b^2sin^2\theta - 2bcsin\theta \Rightarrow (a^2 + b^2)sin^2\theta - 2bcsin\theta + c^2 - a^2 = 0$

$\sin\alpha$ এবং $\sin\beta$ $\sin\theta$ এর দুটি মূল, $\sin\alpha$. $\sin\beta = \frac{c^2-a^2}{a^2+b^2}$

$bsin\theta = c - acos\theta \Rightarrow b^2sin^2\theta = c^2 + a^2cos^2\theta - 2accos\theta \Rightarrow (a^2+b^2)cos^2\theta - 2accos\theta + c^2 - b^2 = 0$

$\cos\alpha$ এবং $\cos\beta$ $\cos\theta$ এর দুটি মূল, $\cos\alpha$. $\cos\beta = \frac{c^2-b^2}{a^2+b^2}$

$\cos\alpha$. $\cos\beta$ -$\sin\alpha$. $\sin\beta$=$\frac{c^2-b^2}{a^2+b^2}$ - $\frac{c^2-a^2}{a^2+b^2} \Rightarrow \cos(\alpha+\beta) = \frac{a^2-b^2}{a^2+b^2} \Rightarrow 1 - cos^2(\alpha+\beta) = 1 - \left(\frac{a^2-b^2}{a^2+b^2}\right)^2$

$\Rightarrow sin^2(\alpha+\beta) = \frac{4a^2b^2}{(a^2+b^2)^2} \therefore sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$

EXAMPLE-03: প্রমান কর যে,$\frac{2\cos 2^n\theta}{2\cos\theta+1} = (2cos\theta-1)(2cos2\theta-1)(2cos2^2\theta-1)\ldots(2cos2^{n-1}\theta)$

প্রমান:$(2cos\theta+1)(2cos\theta-1) = 4cos^2\theta-1 = 4\times\frac{1}{2}(1+cos2\theta)-1 = 2\cos 2\theta+1$

$(2\cos 2\theta+1)(2\cos 2\theta-1) = 2\cos 4\theta+1 = 2\cos 2^2\theta+1.$

$(2\cos 2^2\theta+1)(2\cos 2^2\theta-1) = (2\cos 2^3\theta+1)$.................

$(2\cos 2^3\theta+1)..(2\cos 2^{n-1}\theta+1)(2\cos 2^{n-1}\theta-1) = 2\cos 2^n\theta+1.$

সমীকৃত করে পাই, $(2cos\theta+1)(2cos\theta-1)(2\cos 2\theta+1)(2\cos 2\theta-1)(2\cos 2^2\theta+1)(2\cos 2^2\theta-1)(2\cos 2^3\theta+1)$............$(2\cos 2^{n-1}\theta+1)(2\cos 2^{n-1}\theta-1) =$
$(2\cos 2\theta+1)(2\cos 2^2\theta+1)(2\cos 2^3\theta+1)(2\cos 2^{n-1}\theta+1)(2\cos 2^n\theta+1).$

$\Rightarrow(2cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)$............$(2\cos 2^{n-1}\theta-1) = \frac{2\cos 2^n\theta+1}{(2cos\theta+1)}$

EXAMPLE-04 : $\theta = \frac{\pi}{2^n+1}$ হলে প্রমান কর যে, $\cos\theta\cos 2\theta\cos 2^2\theta\ldots.\cos 2^{n-1}\theta = \frac{1}{2^n}$

প্রমান: $\theta = \frac{\pi}{2^n+1} \Longrightarrow 2^n\theta = \pi-\theta \Longrightarrow 2.2^{n-1}\theta = \pi-\theta \Longrightarrow 2^{n-1}\theta = \frac{\pi}{2}-\frac{\theta}{2}$

$\cos 2^{n-1}\theta = sin\theta, 2cos\theta\cos 2^{n-1}\theta = 2cos\theta sin\theta = sin2\theta$.....(i)

$(i)\times 2cos2\theta \Rightarrow 2.2\cos 2\theta\, cos\theta\cos 2^{n-1}\theta = 2cos2\theta sin2\theta = sin4\theta = sin2^2\theta$......(ii)

(ii)$\times$2cos4θ $\Rightarrow 2.2.2cos4\theta cos2\theta cos\theta cos2^{n-1}\theta = 2cos4\theta sin4\theta = sin8\theta = sin2^3\theta.$

$\Rightarrow 2^3cos2^2\theta cos2\theta cos\theta cos2^{n-1}\theta = 2cos4\theta sin4\theta = sin8\theta = sin2^3\theta.$

....

$\Rightarrow 2^ncos2^{n-1}cos2^2\theta cos2\theta cos\theta cos2^{n-1}\theta = sin2^n\theta = \sin(\pi-\theta) = sin\theta = cos2^{n-1}\theta$

$\therefore \cos\theta\cos 2\theta\cos 2^2\theta\ldots.\cos 2^{n-1}\theta = \frac{1}{2^n}$

EXAMPLE-05: 2tanα = 3 tanβ হলে প্রমান কর যে tan$(\alpha-\beta)$ $= \frac{sin2\beta}{5-cos2\beta}$

L.H.S : tan $(\alpha-\beta)$ $= \frac{tan\alpha-tan\beta}{1+tan\alpha.tan\beta} = \frac{\frac{3}{2}tan\beta-tan\beta}{1+\frac{3}{2}tan^2\beta} = \frac{\frac{1}{2}tan\beta}{2+3tan^2\beta} = \frac{\frac{1}{2}tan\beta}{5-3+3\,tan^2\beta} = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2tan\beta}{1+tan^2\beta}}{\frac{5-3(1-tan^2\beta)}{1+tan^2\beta}}$

$= \frac{\frac{1}{4}sin2\beta}{\frac{5}{2}-\frac{1}{2}cos2\beta} = \frac{\frac{1}{2}sin2\beta}{5-cos^2\beta}$

EXAMPLE-06: প্রমান কর: cot3A $= \frac{cot^3A-3cotA}{3cot^2A-1}$

$\frac{cot^3A-3cotA}{3cot^2A-1} = \frac{\frac{cos^3A}{sin^3A}-3\frac{cosA}{sinA}}{3.\frac{cos^2A}{sin^2A}-1} = \frac{\cos^3A-3\cos A\sin^2A}{3\cos^2A.\sin A-\sin^3A} = \frac{4\cos^3A-3\cos A}{3\sin A-4\sin^3A} = \frac{\cos 3A}{\sin 3A} = \cot 3A$

EXAMPLE-07: প্রমান কর: (i)$\cos^4 x = \frac{3}{8} + \frac{1}{2}cos2x + \frac{1}{8}cos4x$; (ii) $\cos^5 x = \frac{17}{16}\cos x + \frac{19}{16}\cos 3x + \frac{1}{2}\cos 5x$

প্রমান: (i)$\cos^4 x = \cos^3 x.\cos x = \frac{1}{4}(3cosx + cos3x)cosx = \frac{3}{4}cos^2x + \frac{1}{4}\ cos3x.cosx.$

$= \frac{3}{8}(1+\cos 2x) + \frac{1}{8}[\cos 2x + \cos 4x] = \frac{3}{8} + \frac{1}{2}cos2x + \frac{1}{8}cos4x$. proved.

(ii) $\cos^5 x = \cos^3 x \cdot \cos^2 x = \frac{1}{4}[3\cos x + \cos 3x].\frac{1}{2}(1 + \cos 2x)$; $= \frac{1}{8}[3cosx + 3cos2x.cosx + cos3x + cos3x.cos2x]$

$= \frac{3}{8}\cos x + \frac{3}{16}(\cos 3x + \cos x) + \cos 3x + \frac{1}{2}[\cos 5x + \cos x]$; $= \frac{17}{16}\cos x + \frac{19}{16}\cos 3x + \frac{1}{2}\cos 5x$

Try yourself: 01.প্রমান কর যে, $\frac{tan2^n\theta}{tan\theta} = (1 + sec2\theta)(1 + sec2^2\theta)(1 + sec2^3\theta)\ldots.(1 + sec2^n\theta)$

02.$13\theta = \pi$ হলে দেখাও যে, $\cos\theta\cos 2\theta\cos 3\theta\cos 4\theta\cos 5\theta\cos 6\theta = \frac{1}{2^n}$

03. $c\sec\theta - b\tan\theta = a$ সমীকরনটি θ এর ভিন্ন দুটি মান α ও β দ্বারা সিদ্ধ হলে,দেখাও যে, $tan(\alpha + \beta) = \frac{2ab}{a^2 - b^2}$

04. x $= \sin\frac{\pi}{18}$ হলে দেখাও যে, $16x^4 + 8x^3 - 12x^2 - 4x$.

05.প্রমাণ কর যে, $\frac{1}{sin10^0} - \frac{\sqrt{3}}{cos10^0} = \frac{\sqrt{3}}{sin20^0} - \frac{1}{cos20^0} = 4$

06. $\tan\theta = {}^{y}/_{x}$ হলে প্রমাণ কর, $x\cos 2\theta + y\sin 2\theta = x$

07. প্রমাণ কর যে, $\frac{son\theta + sin5\theta + sin9\theta + sin13\theta}{cos\theta + cos5\theta + cos9\theta + cos13\theta} = tan7\theta$

08. দেখাও যে,$16cos\frac{2\pi}{15}cos\frac{4\pi}{15}cos\frac{8\pi}{15}cos\frac{14\pi}{15} = 1$

09. দেখাও যে, $sin^3x + sin^3(120^0 + x) + sin^3(240^0 + x) = -\frac{3}{4}\sin 3x$

Type-05: উপগুনিতক কোণের ত্রিকোণমিতিক অনুপাত সংক্রান্ত সমস্যাবলী

উপগুনিতক কোণ($\frac{\theta}{2},\frac{\theta}{3},\frac{\theta}{4}$..): sinA = sin2($\frac{A}{2}$) = 2 sin$\frac{A}{2}$. cos$\frac{A}{2}$ = $\frac{2tan^{A}/_{2}}{1+tan^{2A}/_{2}}$;$\sin\theta = 3\sin\frac{\theta}{3} - 4\sin^3\frac{\theta}{3}$

cosA = cos2($\frac{A}{2}$) = $\cos^2 {A}/_{2} - \sin^2 {A}/_{2}$ = $\frac{1-tan^{2A}/_{2}}{1+tan^{2A}/_{2}}$; $\cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$

tanA = $\frac{2tan^{A}/_{2}}{1-tan^{2A}/_{2}}$; $\tan\theta = \frac{3\tan\frac{\theta}{3}-\tan^3\frac{\theta}{3}}{1-3\tan^3\frac{\theta}{3}}$

$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1}{2}(1-\cos\theta)} = \pm\sqrt{\frac{1}{2}(1+\sin\theta)} \mp \sqrt{\frac{1}{2}(1-\sin\theta)}$;

$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1}{2}(1+\cos\theta)}$. $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1}{2}(1+\sin\theta)} \pm\sqrt{\frac{1}{2}(1-\sin\theta)}$

$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}. = \frac{-1\pm\sqrt{1+tan^2\theta}}{tan\theta}$

$\sin 3^o = \frac{1}{16}(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - \frac{1}{8}\sqrt{5+\sqrt{5}}(\sqrt{3}-1)$. $\cos 3^o = \frac{1}{16}(\sqrt{5}-1)(\sqrt{6}-\sqrt{2}) + \frac{1}{8}\sqrt{5+\sqrt{5}}(\sqrt{3}+1)$

$\sin 15^o = \frac{\sqrt{3}-1}{2\sqrt{2}}$. $\cos 15^o = \frac{\sqrt{3}+1}{2\sqrt{2}}$, $\sin 18^0 = \frac{1}{4}(\sqrt{5}-1)$. $\cos 18^0 = \frac{1}{4}\left(\sqrt{10+2\sqrt{5}}\right)$

$\sin 36^o = \frac{1}{4}\sqrt{10-2\sqrt{5}}$, $\cos 36^o = \frac{1}{4}(\sqrt{5}-1)$.

$2sin\frac{\pi}{2^n} = \sqrt{2-\sqrt{2+\sqrt{2-\cdots(n-1)}}}$সংখ্যক পদ ; $2cos\frac{\pi}{2^n} = \sqrt{2+\sqrt{2+\sqrt{2+\cdots(n-1)}}}$সংখ্যক পদ

EXAMPLE-01: প্রমান কর: $2\sin\frac{\pi}{16} = \sqrt{2-\sqrt{2+\sqrt{2}}}$

L. H. S= $\sqrt{4sin^2\frac{\pi}{16}} = \sqrt{2\left(1-cos\frac{\pi}{8}\right)} = \sqrt{2-2cos\frac{\pi}{8}} = \sqrt{2-\sqrt{4cos^2\frac{\pi}{8}}}$

$= \sqrt{2-\sqrt{2\left(1+\cos\frac{\pi}{4}\right)}} = \sqrt{2-\sqrt{2+\sqrt{2}}}$

EXAMPLE-02: দেখাও যে: $\sin 18^0 = \frac{1}{4}(\sqrt{5}-1)$

ধরি, $18^0 = \theta \Rightarrow 5\theta = \Rightarrow 3\theta + 2\theta = 90^0 \Rightarrow 3\theta = 90^0 - 2\theta \Rightarrow \cos 3\theta = \sin 2\theta$

$\Rightarrow 4\cos^3\theta - 3\cos\theta = 2\sin\theta.\cos\theta \Rightarrow 4 - 4\sin^2\theta - 3 - 2\sin\theta = 0$

$\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0 \Rightarrow \sin^2\theta + \frac{1}{2}\sin\theta - \frac{1}{4} = 0$

$\Rightarrow \sin^2\theta + 2\sin\theta.\frac{1}{4} + \frac{1}{16} - \frac{1}{16} - \frac{1}{4} = 0$

$\Rightarrow \left(\sin\theta + \frac{1}{4}\right)^2 = \frac{5}{16} \Rightarrow \sin\theta + \frac{1}{4} = \pm\frac{\sqrt{5}}{4} \Rightarrow \sin\theta = \pm\frac{\sqrt{5}-1}{4}$

$\therefore \sin 18^o = \frac{\sqrt{5}-1}{4}$ ($\theta \to$সূক্ষ্মকোণ বলে) ; $\cos 18^0 = \frac{1}{4}\left(\sqrt{10+2\sqrt{5}}\right)$

$\sin 15^0 = (60^0 - 45^0) = \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} \therefore \sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}}\ \cos 15^o = \frac{\sqrt{5}-1}{4}$

$\sin 3^0 = \cos(18^0 - 15^0) = \sin 18^0.\cos 15^0 - \cos 18^0.\sin 15^0 = \frac{\sqrt{5}-1}{4}.\frac{\sqrt{3}+1}{2\sqrt{2}} -$
$\frac{1}{4}\left(\sqrt{10+2\sqrt{5}}\right)\frac{\sqrt{3}-1}{2\sqrt{2}}$

$\sin 3^o = \frac{1}{16}(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - \frac{1}{8}\sqrt{5+\sqrt{5}}(\sqrt{3}-1).$

similarly, $\cos 3^o = \frac{1}{16}(\sqrt{5}-1)(\sqrt{6}-\sqrt{2}) + \frac{1}{8}\sqrt{5+\sqrt{5}}(\sqrt{3}+1)$

উক্ত কোণগুলির সম্পূরক কোণ 72^0, 54^0, 75^0, 77^0 এর ত্রিকোনমিতিক অনুপাত এবং 4এর উপগুনিতক কোনের ত্রিকোনমিতিক অনুপাত
বের করা শিখবে।

EXAMPLE-03: $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\varphi}{2}$ হলে প্রমান কর $\cos\varphi = \frac{cos\theta - e}{1-ecos\theta}$

সমাধান: $\tan^2\frac{\theta}{2} = \frac{l-e}{l+e}\tan^2\frac{\varphi}{2}$

আমরা জানি , $\cos\varphi = \frac{1+tan^2\frac{\varphi}{2}}{1-tan^2\varphi/_2} = \frac{1+\frac{1+e}{1-e}tan^2\frac{\theta}{2}}{1-\frac{1+e}{1-e}tan^2\frac{\theta}{2}} = \frac{1-e+tan^2\frac{\theta}{2}+etan^2\frac{\theta}{2}}{1-e-tan^2\frac{\theta}{2}-etan^2\frac{\theta}{2}} = \frac{\left(1+tan^2\frac{\theta}{2}\right)-e\left(1-tan^2\frac{\theta}{2}\right)}{\left(1-tan^2\frac{\theta}{2}\right)-e\left(1+tan^2\frac{\theta}{2}\right)}$

লব ও হরকে $\left(1-tan^2\frac{\theta}{2}\right)$ ভাগ করে পাই, $\cos\varphi = \frac{cos\theta-e}{1-ecos\theta}$

EXAMPLE-04: $a\sin\theta + bsin\varphi = c = acos\theta + bcos\varphi$ হলে দেখাও যে, $cos\frac{1}{2}(\theta-\varphi) =$
$\pm\sqrt{\frac{2c^2-(a-b)^2}{4ab}}$

সমাধান: $a\sin\theta + bsin\varphi = c$..........(i) , বর্গ করে , $a^2sin^2\theta + b^2sin^2\varphi + 2ab\sin\theta\sin\varphi = c^2$

$acos\theta + bcos\varphi = c$.........(ii) , বর্গ করে , $a^2cos^2\theta + b^2cos^2\varphi + 2ab\cos\theta\cos\varphi = c^2$

যোগ করে, $a^2sin^2\theta + b^2sin^2\varphi + 2ab\sin\theta\sin\varphi + a^2cos^2\theta + b^2cos^2\varphi +$
$2ab\cos\theta\cos\varphi = c^2 + c^2$

$\Rightarrow a^2 + b^2 + 2ab\cos(\theta-\varphi) = 2c^2 \Rightarrow a^2 + b^2 + 2ab\left\{2\cos^2\left(\frac{\theta-\varphi}{2}\right) - 1\right\} = 2c^2$

$\Rightarrow a^2 + b^2 - 2ab + 4ab\cos^2\left(\frac{\theta-\varphi}{2}\right) = 2c^2 \Rightarrow (a-b)^2 + 4ab\cos^2\left(\frac{\theta-\varphi}{2}\right) = 2c^2$

$\Rightarrow cos\frac{1}{2}(\theta-\varphi) = \pm\sqrt{\frac{2c^2-(a-b)^2}{4ab}}$

Try yourself: 01.A + B $\neq$ 0এবং sinA + sinB = 2sin (A + B) হলে প্রমান কর যে, $\tan\frac{A}{2}, tan\frac{B}{2} = \frac{1}{3}$

02. $\sin\theta = \frac{a-b}{a+b}$ হলে প্রমান কর $\tan\left(\frac{\pi}{2} - {}^{\theta}/_{2}\right) = \sqrt{\frac{a}{b}}$

03. $\sec(\theta + \alpha) + \sec(\theta - \alpha) = 2\sec\theta$ হলে দেখাও যে, $\cos\theta = \pm\sqrt{2}cos\frac{\alpha}{2}$.

04. sinA = $\frac{1}{\sqrt{2}}, sinB = \frac{1}{\sqrt{3}}$ হলে দেখাও যে, $\tan\frac{A+B}{2}cot\frac{A-B}{2} = 5 + 2\sqrt{6}$

05. প্রমান কর যে, $\tan 7\frac{1}{2}^{o} = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$

06. $\sin\alpha + \sin\beta = a$, $\cos\alpha + \cos\beta = b$ হলে দেখাও যে,$\cos\left(\frac{\alpha-\beta}{2}\right) = \pm\frac{1}{2}\sqrt{a^2 + b^2}$

Type-06: ত্রিকোণমিতিক অভেদাবলী সংক্রান্ত সমস্যাবলী

EXAMPLE-01: A + B + C = π হলে প্রমান কর যে, (i) $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

সমাধান: $\frac{\mathrm{A+B}}{2} = \frac{\pi}{2} - \frac{C}{2}$ $\therefore \sin\frac{\mathrm{A+B}}{2} = \cos\frac{C}{2}$, $\frac{C}{2} = \frac{\pi}{2} - \frac{\mathrm{A+B}}{2}$ $\therefore \cos\frac{C}{2} = \sin\frac{\mathrm{A+B}}{2}$

L.S = $\sin A + \sin B - \sin C = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} - \sin C = 2\cos\frac{C}{2}\cos\frac{A-B}{2} - 2\sin\frac{C}{2}\cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) = 2\cos\frac{C}{2}.2\sin\frac{A-B+A+B}{2}\sin\frac{A+B-A+B}{2} = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

(ii) $\cos^2B + \cos^2C - \cos^2A = 1 - 2\cos A\sin B\sin C$

সমাধান: $B + C = \pi - A$ $\therefore \cos(B + C) = -\cos A$

L.S = $\cos^2B + \cos^2C - \cos^2A = \frac{1}{2}(1 + \cos 2B) + \frac{1}{2}(1 + \cos 2C) - cos^2A$

$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2}(\cos 2B + \cos 2C) - cos^2A = 1 + \frac{1}{2}\times 2\cos\frac{2B+2C}{2}\cos\frac{2B-2C}{2} - cos^2A$

$= 1 + \cos(B + C)\cos(B - C) - cos^2A = 1 - \cos A\cos(B - C) - cos^2A$

$= 1 - \cos A\{\cos(B - C) - cosA\} = 1 - \cos A\{\cos(B - C) - cos(B + C)\}$

$= 1 - \cos A\, 2\sin B\sin C$

$\therefore$ $\cos^2B + \cos^2C - \cos^2A = 1 - 2\cos A\sin B\sin C$

(iii) $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4\sin A\sin B\sin C$

L.S = $\sin(\pi - 2A) + \sin(\pi - 2B) + \sin(\pi - 2C) = \sin 2A + \sin 2B + \sin 2C =$

$2\sin\frac{2A+2B}{2}\cos\frac{2A-2B}{2} + \sin 2C$

$= 2\sin(A + B)\cos(A - B) + 2\sin C\cos C = 2\sin(\pi - C)\cos(A - B) + 2\sin C\cos\{\pi - (A + B)\}$

$= 2\sin C\cos(A - B) - 2\sin C\cos(A + B) = 2\sin C\{\cos(A - B) - \cos(A + B)\}$

$= 4\sin A\sin B\sin C$

EXAMPLE-02: A + B + C = 2π হলে প্রমান কর যে, (i) $\cos^2 A + \cos^2 B + \cos^2 B = 1 + 2\cos A \cos B \cos C$

সমাধান: $2\cos A \cos B \cos C = \{\cos(A+B) + \cos(A-B)\}\cos C = \{\cos C + \cos(A-B)\}\cos C$

$=cos^2C + \cos(A-B)\cos\{2\pi - (A+B)\} = cos^2C + \cos(A+B)\cos(A-B)$

$=cos^2C + \cos^2 A - sin^2B = cos^2C + \cos^2 A - 1 + cos^2B$

$\therefore \cos^2 A + \cos^2 B + \cos^2 C = 1 + 2\cos A \cos B \cos C$

EXAMPLE-03: A + B + C = $\frac{\pi}{2}$ হলে প্রমান কর যে, $sin^2A + sin^2B + sin^2C = 1 - 2\sin A \sin B \sin C$

সমাধান: $2\sin A \sin B \sin C = \{\cos(A-B) - \cos(A+B)\}\sin C = \cos(A-B)\sin C - \cos\left(\frac{\pi}{2} - C)\right)\sin C$

$= \cos(A-B)\sin\left\{\frac{\pi}{2} - (A+B)\right\} - \sin C.\sin C = \cos(A+B)\cos(A-B) - sin^2C = \cos^2 A - sin^2B - sin^2C$

$= 1 - \sin^2 A - sin^2B - sin^2C$; $\therefore sin^2A + sin^2B + sin^2C = 1 - 2\sin A \sin B \sin C$

Try yourself: (a) A + B + C = π হলে প্রমান কর যে,

(i) cosA + cosB- cosC = 4cos$\frac{A}{2}$cos$\frac{B}{2}$sin$\frac{C}{2} - 1$

(ii) cos2A − cos2B + cos2C = 1- 4sinA.cosB.sinC

(iii) $\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1$

(iv) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(v) $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$

(vi) $\sin(B+2C) + \sin(C+2A) + \sin(A+2B) = 4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$

(vii) $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$

(viii) $\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2A \tan 2C$;

(ix) $\tan A + \tan B + \tan C = \tan A \tan A \tan C.$

(x) $cos^2 2A + cos^2 2B + cos^2 2C = 1 + 2\cos 2A \cos 2B \cos 2C$

(b) A + B + C = $\frac{\pi}{2}$ হলে প্রমান কর যে,(i) $\cos^2 A + \cos^2 B - \cos^2 C = 2\cos A \cos B \sin C$

(ii) $\cot A + \cot B + \cot C = \cot A \cot B \cot C$

Type-07: ত্রিকোণমিতিক অভেদাবলীর গুণাবলী সংক্রান্ত সমস্যাবলী

EXAMPLE-01: A + B + C $=(2n+1)\frac{\pi}{2}$ হলে প্রমান কর যে, $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$

সমাধান: $\cot(\mathrm{A}+\mathrm{B}+\mathrm{C})=\cot\left\{(2n+1)\ \frac{\pi}{2}\right\}=0$

$$\cot(\mathrm{A}+\mathrm{B}+\mathrm{C})=\frac{1}{\tan(\mathrm{A}+\mathrm{B}+\mathrm{C})}=\frac{1}{\frac{\tan A+\tan B+\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}}$$

$$=\frac{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}{\tan A+\tan B+\tan C}=0$$

Since , $\tan A+\tan B+\tan C\neq 0.\therefore\ 1-\tan A\tan B-\tan B\tan C-\tan C\tan A=0$
$\Rightarrow \tan A\tan B+\tan B\tan C+\tan C\tan A=1.$

Try yourself:

(a) A + B + C $=(2n+1)\frac{\pi}{2}$ হলে প্রমান কর যে, sin2A + sin2B + sin2C=$\pm 4\cos A\cos B\, cosC$

(b) A + B + C = π এবং cotA + cotB + cotC = = $\sqrt{3}$ হলে প্রমান কর যে, A = B = C

(c) A + B + C = π এবং $sin^2A+sin^2B+sin^2C=\sin A\sin B\sin C$ হলে প্রমান কর যে, A = B = C

(d) $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$ হলে প্রমান কর যে, A $\pm$B $\pm$C =(2n+1) π ; $n\in Z$

(e) $\tan A+\tan B+\tan C=\tan A\tan A\tan C$. হলে প্রমান কর যে, A + B + C = n π ; $n\in Z$

(f) xy+yz+zx=1. হলে প্রমান কর যে,$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}.\frac{2y}{1-y^2}.\frac{2z}{1-z^2}$

(g) x+y+z=xyz . হলে প্রমান কর যে, $\frac{(x^2-1)(y^2-1)}{xy}+\frac{(y^2-1)(z^2-1)}{yz}+\frac{(z^2-1)(x^2-1)}{zx}=4$

ত্রিভুজের গুণাবলী

সুত্রঃ (i) $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2R$ [R→পরিবৃত্তের ব্যাসার্ধ]

→এটা sinerule নামে পরিচিত।

→ (a,A), (b,B), (c,C) এগুলো ত্রিভুজের উপাদান যা দ্বারা ত্রিভূজটি আকার আকৃতি প্রকৃতি ঘুর্ণন পরিমাপ করা যায়।

(iii) **অভিক্ষেপ সুত্রঃ** a = cosB + bcosc ; b = c cosA + a cosc ; c = a cosB + bcosA

(ii) **cosine সুত্রঃ** cosA = $\frac{b^2+c^2-a^2}{2bc}$, cosB $\frac{c^2+a^2-b^2}{2ca}$, cosC = $\frac{a^2+b^2-c^2}{2ab}$

(iii) ক্ষেত্রফলঃ $\Delta = \frac{1}{2}absinC = \frac{1}{2}bcsinA = \frac{1}{2}casinB = \frac{abc}{4R}$= rs ; rs = যেখানে r হলো অন্তঃবৃত্তের ব্যাসার্ধ.

(iv) $\sin A/_2 = \frac{\sqrt{(s-b)(s-c)}}{bc}$,$\cos A/_2 = \frac{\sqrt{s(s-a)}}{bc}$ [$\therefore$ A$\leq 90^0$] ; $\tan\frac{A}{2} = \frac{(s-b)(s-c)}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{s(s-b)(s-c)}{\Delta}$

similarly: $\tan B/_2 = \frac{(s-a)(s-c)}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{s(s-a)(s-c)}{\Delta}$; $\tan C/_2 = \frac{(s-a)(s-b)}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{s(s-a)(s-b)}{\Delta}$

Type-08: উক্ত সুত্র নির্ভর সমাধান

EXAMPLE-01: প্রমান কর যে, $\tan\frac{B-C}{2} = \frac{b-c}{b+c}cot\, A/_2$

সমাধানঃ $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} \Rightarrow \frac{b}{c} = \frac{sinB}{sinC} \Rightarrow \frac{b-c}{b+c} = \frac{sinB-sinC}{sinB+sinC} = \frac{2cos\frac{B+C}{2}.sin\frac{B-C}{2}}{2sin\frac{B+C}{2}.cos\frac{B-C}{2}}$

$= \cot(\pi/_2 - A/_2).tan\frac{B-C}{2} = \tan A/_2 . \tan\frac{B-C}{2} \therefore \tan\frac{B-C}{2} = \frac{b-c}{b+c}tan\, A/_2.$

EXAMPLE-02: প্রমান কর যে, $\frac{b^2-c^2}{a^2}$ sin2A + $\frac{c^2-a^2}{b^2}$ sin2B + $\frac{a^2-a^2}{c^2}$ sin2C = 0

সমাধানঃ lst term = $\frac{b^2-c^2}{a^2}$ sin2A = $\frac{4R^2sin^2B-4R^2sin^2c}{4R^2sin^2A}$. 2sinA.cosA

$= \frac{\sin(B+C).\sin(B-C)}{\sin(\pi-\overline{B+C})} \times 2\cos(\pi-(B+C)) = -2\sin(B-C).\cos(B+C) = \sin 2C - \sin 2B$

similarly: 2nd term = sin2A − sin2C ; 3rd term = sin2B − sin2A; $\sum_1^3 terms = 0.$

EXAMPLE-03: প্রমান কর যে, $c^2 = (a-b)^2\cos^2\frac{c}{2} + (a+b)^2\sin^2\frac{c}{2}$

R.S = $(a^2+b^2)(\cos 2\, C/_2 + sin^2\, c/_2) - 2ab(\cos^2 C/_2 - sin^2\, c/_2) = a^2 + b^2 - 2ab$ cosC.= c^2 [$\therefore$ cos = $\frac{a^2+b^2-c^2}{2ab}$]

EXAMPLE-04: প্রমান কর যে, $b^2\sin 2C + c^2\sin 2B = 2bc\sin A = 4\Delta$

L. S = $4R^2 \sin^2 B.\sin 2C + 4R^2\sin^2 C \sin 2B = 4R^2 (\sin^2 B.2\sin C.\cos C + \sin^2 C. 2\sin B.\cos B)$
$= 4R^2 [2\sin B.\sin C [\sin B\cos C + \sin C.\cos B) = 4R^2. 2\sin A.\sin B.\sin(B+C)$
$= 4R^2. 2\sin A.\sin B.\sin C = \frac{4R^2\times abc}{2R\times 2R\times 2R} = \frac{abc}{R} = 4\Delta = \frac{4R^2\times 2bc\sin A}{4R^2} = 2bc\sin A$

Try yourself:

(i) $a\sin(A/_2 + B) = (b + c)\sin A/_2$; (ii) $a\sin B.\sin C + b\sin C.\sin A + c\sin A.\sin A.\sin B = \frac{3\Delta}{R}$.

(iii) $\frac{1}{a}\sin A + \frac{1}{b}\sin B + \frac{1}{C}\sin C = \frac{6\Delta}{abc} = \frac{3\Delta}{2R}$; (iv) $a(\cos C - \cos B) = 2(b - c)\cos^2 A/_2$

(v) $(b^2 - c^2)\cot A + (c^2 - b^2)\cot B + (a^2 - b^2)\cot c = 0$; (vi) $\sin A + \sin B + \sin C = \frac{S}{R}$

(vii) $\frac{1}{a}cos^2 A/_2 + \frac{1}{b}cos^2 + \frac{1}{c}cos^2 C/_2 = \frac{S^2}{abc}$;

Type-09: ত্রিভুজের সমাধান [বৃহত্তম বাহুর বিপরীত কোন বৃহত্তম]

EXAMPLE-01: $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$ হলে দেখাও যে $\angle C = 60^0$

সমাধানঃ $\Rightarrow (a + b + c + c)(a + b + c) = 3(ab + bc + ca + c^2) \Rightarrow (a + b + c)^2 + c(a + b + c) = 3(ab + bc + ca + c^2)$

$\Rightarrow a^2 + b^2 - ab - c^2 = 0 \Rightarrow \frac{a^2+b^2+c^2}{2ab} = \frac{1}{2} \Rightarrow cos = 60^0 \therefore C = 60^0$ proved

EXAMPLE-02: $2x + 3$, $x^2 + 3x + 3$ এবং $x^2 + 2x$ যথাক্রমে একটি ত্রিভুজের তিনটি বাহু হলে, বৃহত্তম কোনটি নির্ণয় কর।

সমাধানঃ x এর ধণাত্বক পূর্ণ মানের জন্য $x^2 + 3x + 3$ বৃহত্তম বাহু । ধরি এই বাহুর বিপরীত কোন A.

$\cos A = \frac{(2x+3)^2+(x^2+2x)^2-(x^2+3x+3)^2}{2(2x+3)(x^2+2x)} = \frac{4x^2+12x+9+x^4+4x^3+4x^2-x^4-9x^2-9-6x^3-18x-6x^2}{2(2x+3)(x^2+2x)} = \frac{-2x^3-7x^2-6x}{2x(2x+3)(x+2)}$

$= -\frac{1}{2}\frac{2x^2+7x+6}{2x^2+7x+6} = -\frac{1}{2} = \cos 120^0 \therefore A = 120^0$

EXAMPLE-03: $a = \sqrt{3} + 1, b = \sqrt{3} - 1$ এবং $C = 60^0$ হলে ত্রিভূজটি সমাধান কর

সমাধানঃ $\cos 15^0 = \frac{\sqrt{3}+1}{2\sqrt{2}}$, $\sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}}$, $\frac{a}{sinA} = \frac{b}{sinB}$

$\therefore \frac{a}{2\sqrt{2}} = \cos 15^0 = \sin 75^0 \quad \frac{b}{2\sqrt{2}} = \sin 15^0 = \cos 75^0$

$\therefore A = 15^0$ হলে $B = 105^0$ কারণ $C = 60^0$

আবার $A = 105^0$ হলে $B = 15^0$ কারণ $C = 60^0$

$c^2 = a^2 + b^2 - 2ab\cos C = 3 + 1 + 2\sqrt{3} + 1 - 2\sqrt{3} - 2(3-1).\frac{1}{2} = 6$

$\therefore$ c = $\sqrt{6}$, again, $\frac{b}{sin15^0} = \frac{\sqrt{6}}{\sqrt{3/2}}$ $\therefore$b = $\frac{\sqrt{3}.\sqrt{2}.2}{\sqrt{3}}.\frac{\sqrt{3}-1}{2\sqrt{2}} = \sqrt{3} - 1$, a = $2\sqrt{2}.\ \ \frac{\sqrt{3}+1}{2\sqrt{2}} = \sqrt{3} + 1$ (satisfied

$\therefore$ b ও c এর fixed value এর জন্যঃ A =105^0, B = 15^0 এবং c = $\sqrt{6}$ unit

EXAMPLE-04: $\frac{y}{z} + \frac{z}{x}, \frac{z}{x} + \frac{x}{y}$ এবং $\frac{x}{y} + \frac{y}{z}$ এই তিনটি বাহু দ্বারা আবদ্ধ ত্রিভুজের ক্ষেত্রফল কত?

সমাধানঃ 2s = $\frac{y}{z} + \frac{z}{x} + \frac{z}{x} + \frac{x}{y} + \frac{x}{y} + \frac{y}{z}$ $\therefore$ s = $\frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ $\therefore$s − a = $\frac{x}{y}, s - b = \frac{y}{x}, s - c = \frac{z}{x}$

$\therefore \Delta = \sqrt{(\frac{x}{y} + \frac{y}{z} + \frac{z}{x})\frac{xyz}{xyz}} = \sqrt{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}$ sq.unit Ans:

Try yourself:

(i) $c^4 - 2(a^2 + b^2) c^2 + a^4 + a^2b^2 + b^4 = 0$হলে দেখাও C = 60^0 or 120^0;(ii) A = 45^0, B = 75^0 হলে দেখাও যে, a + $\sqrt{2}c$ = 2b

(iii)$a^2+c^2=2b^2$ হলে দেখাও যে, cotA + cotC = 2cotB; (iv) (a + b + c) (b + c − a) = 3abc হলে দেখাও যে, <A = ? (60^0)

(v) m, n, $\sqrt{m^2 + mn + n^2}$ হলে বৃহত্তম কোন কত? (120^0);(vi) 3, 5, 7 বাহু বিশিষ্ট ত্রিভূজটির স্থুলকোনটি নির্ণয় কর। (120^0)

(vii) <A = 60^0 হলে দেখাও যে, b + c = 2acos $\frac{B-C}{2}$;(viii) $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ হলে প্রমান কর, $\frac{cosA}{7} = \frac{cosB}{19} = \frac{cosC}{25}$

(ix) a = 2b এবং A = 3B হলে ত্রিজটি সমাধান কর। A = 90^0, B = 30^0, C = 60^0

(x) a = 2, b = $\sqrt{3} + 1$এবং C = 60^0 হলে ত্রিভূজটির অপর বাহু এবং কোনদ্বয় নির্ণয় কর। A = 45^0, B = 75^0, C = $\sqrt{6}$.