Домашняя работа
по вычислительной физике №2.

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SVD разложение: $A = U\Sigma V^T$ (2)

а) $A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$

1) $AA^T = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$

$$\begin{vmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = 0$$

$\lambda_1 = 9 \quad \lambda_2 = 4$

$\sigma_1 = 3 \quad \sigma_2 = 2 \quad \Rightarrow \quad \Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

$\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 9\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = 4\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2) $\hat{A}\vec{v}_k = \sigma_k \vec{u}_k$

$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$u_1 = 1$
$u_2 = 0 \quad \Rightarrow \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $\vec{u}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow \hat{U} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

б) $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

1) $BB^T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$-(4-\lambda)\lambda^2 = 0$

$\lambda = 0 \quad \lambda = 4 \quad \Rightarrow \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$

$\sigma = 0 \quad \sigma = 2$

2) $\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \hat{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; V^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3) $AV = U\Sigma$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2u_{12} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$u_{12} = 1 \Rightarrow U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Ответ: $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

1) $AA^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$\det\begin{pmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = 0$

$(2-\lambda)(2-\lambda) - 4 = 0$

$\lambda_1 = 4 \qquad \lambda_2 = 0 \qquad \Rightarrow \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

$\sigma_1 = 2 \qquad \sigma_2 = 0$

2) $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\begin{cases} 2v_1 + 2v_2 = 4v_1 \\ 2v_1 + 2v_2 = 4v_2 \end{cases}$ $\quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $2v_1 + 2v_2 = 0$

$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}; V^T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Нормировка (т.к. $V \cdot V^T = I$) $V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

3) $\hat{A}\vec{v}_k = \sigma_k \vec{u}_k$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$\begin{cases} 2 = 2u_1 \\ 2 = 2u_2 \end{cases}$ $\quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $\vec{u}_2 = \begin{pmatrix} \lambda \\ \beta \end{pmatrix}, \quad \lambda, \beta \in \mathbb{R}$ $\qquad$ $\vec{u}_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Нормировка: (т.к. $U \cdot U^T = \mathbb{I}$)

$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

a) Докажем, что

$$\|X\|_2 \leq \sqrt{m}\,\|X\|_\infty$$

$$\left(x_1^2 + \dots + x_m^2\right)^{\frac{1}{2}} \leq \sqrt{m \cdot \max_{1<j<m} |x_j^2|}$$