§4

№1(4)

$f = x^y$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = y\,x^{y-1}dx + \ln x \cdot x^y dy$

$d(df) = \left(\frac{\partial^2 f}{\partial x^2}dx + \frac{\partial^2 f}{\partial x \partial y}dy\right)dx + \left(\frac{\partial^2 f}{\partial y \partial x}dx + \frac{\partial^2 f}{\partial y^2}dy\right)dy =$

$= y(y-1)x^{y-2}dx^2 + \frac{1}{x}\cdot x^y dy dx + \ln x \cdot y \cdot x^{y-1} dy dx +$

$+ x^{y-1}dx dy + y \cdot \ln x \cdot x^{y-1} dx dy + \ln^2 x\, x^y dy^2$

№4

$f = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$

$\frac{\partial f}{\partial x} = \lim_{x\to 0}\frac{\cancel{x}y\frac{x^2-y^2}{x^2+y^2} - 0}{\cancel{x}} = \lim_{x\to 0} y \cdot \frac{-y^2}{y^2} = -y$

$\frac{\partial^2 f}{\partial y \partial x} = (-y)' = \boxed{-1} = f''_{xy}$

$\frac{\partial f}{\partial y} = \lim_{y\to 0}\frac{xy\frac{x^2-y^2}{x^2+y^2}}{y} = x$

$\frac{\partial^2 f}{\partial x \partial y} = x' = 1 = f''_{yx}.$

№7(2)

$f = x^{y^z} \quad (e, 1, 1)$

$$\frac{\partial f}{\partial x} = y^z \cdot x^{y^z-1} \qquad \frac{\partial f}{\partial y} = \ln x \cdot x^{y^z} \cdot z \cdot y^{z-1} \qquad \frac{\partial f}{\partial z} = \ln x \cdot x^{y^z} \cdot \ln y \cdot y^z$$

$$\frac{\partial^2 f}{\partial x^2} = y^z \cdot (y^z-1) \cdot x^{y^z-2} = 1 \cdot 0 = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = z \cdot y^{z-1}(x^{y^z}-1) + y^z \ln x \cdot x^{y^z-1} \cdot z \cdot y^{(z-1)} = 1 \cdot 1(e-1) + 1 \cdot 1 \cdot e^0 \cdot 1 = e$$

$$\frac{\partial^2 f}{\partial z \partial x} = \underbrace{\ln y}_{0} \cdot y^z \cdot x^{y^z-1} + y^z \cdot \ln x \cdot x^{y^z-1} \cdot \underbrace{\ln y}_{0} \cdot y^z = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = z \cdot y^{z-1}\left(\frac{1}{x} \cdot x^{y^z} + \ln x \cdot y^z \cdot x^{y^z-1}\right) = 1 \cdot \left(\frac{1}{e} \cdot e + 1\right) = 2$$

$$\frac{\partial^2 f}{\partial y^2} = \ln x \cdot z \left(\ln x \cdot x^{y^z} \cdot z \cdot y^{z-1} + \underbrace{x^{y^z}(z-1)\, y^{z-2}}_{0}\right) = 1 \cdot 1 \cdot e \cdot 1 \cdot 1 = e$$

$$\frac{\partial^2 f}{\partial z \partial y} = \underbrace{\ln x}_{1}\left(\underbrace{x^{y^z} \cdot y^{z-1}}_{e} + \underbrace{x^{y^z} \cdot z \cdot \ln y \cdot y^{z-1}}_{0} + \underbrace{z \cdot y^{z-1} \cdot \ln x \cdot x^{y^z} \cdot \ln y}_{0} \cdot y^z\right) = e$$

$$\frac{\partial^2 f}{\partial x \partial z} = \underbrace{\ln y}_{0} \cdot y^z \left(\frac{1}{x} \cdot x^{y^z} + \ln x \cdot y^z \cdot x^{y^z-1}\right) = 0$$

$$\frac{\partial^2 f}{\partial y \partial z} = \ln x\left(\ln x \cdot x^{y^z} \cdot z \cdot y^{z-1} \cdot \underbrace{\ln y}_{0} \cdot y^z + x^{y^z} \cdot \frac{1}{y} \cdot y^z + x^{y^z} \cdot \underbrace{\ln y}_{0} \cdot z \cdot y^{z-1}\right) =$$

$$= e$$

$$\frac{\partial^2 f}{\partial z^2} = \ln x \cdot \ln y \cdot \left(\ln x \cdot x^{y^z} \ln y \cdot y^z \cdot y^z + x^{y^z} \cdot z \cdot y^{z-1}\right) = 0$$

№25(2)

$$df = \frac{1}{x+y+z}(dx+dy+dz)$$

$$d^2 f = -\frac{(dx+dy+dz)^2}{(x+y+z)^2} \qquad d^3 f = -\frac{2(dx+dy+dz)^3}{(x+y+z)^3}$$

$$\vdots$$

$$d^n f = (-1)^{n-1}\frac{(dx+dy+dz)^n}{(x+y+z)^n}\cdot(n-1)!$$

$$\Downarrow$$

$$d^{n+1} f = d\left((-1)^{n-1}\frac{(dx+dy+dz)^n(n-1)!}{(x+y+z)^n}\right) = \frac{(-1)^n n!\,(dx+dy+dz)^{n+1}}{(x+y+z)^{n+1}}$$

Чтд

№27(3)

$$\varphi(x,y,z) = f(u), \quad u = xyz$$

$$d\varphi = f'(u)(yz\,dx + xz\,dy + xy\,dz)$$

$$d^2 f = f''(u)(yz\,dx + xz\,dy + xy\,dz)^2 + f'(u)\cdot(z\,dx\,dy + y\,dx\,dz + z\,dy\,dx + x\,dy\,dz + y\,dz\,dx + x\,dz\,dy) =$$

$$= f''(u)(yz\,dx + xz\,dy + xy\,dz)^2 + 2f'(u)\cdot(z\,dx\,dy + y\,dx\,dz + x\,dy\,dz).$$

§ 3

№19(3,9)

3) $f = (\sin x + \sqrt[3]{xy})^2$

$$\frac{\partial f}{\partial x} = \lim_{x\to0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to0}\frac{\sin^2 x}{x} = 1\cdot\lim_{x\to0}\sin x = 0$$

$$\frac{\partial f}{\partial y} = \lim_{y\to0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to0}\frac{0}{y} = 0$$

$$\frac{(\sin x+\sqrt[3]{xy})^2}{\sqrt{x^2+y^2}} \le \frac{(x+\sqrt[3]{xy})^2}{\sqrt{x^2+y^2}} \le \frac{\left(\sqrt{x^2+y^2}+\sqrt[3]{(x^2+y^2)}\right)^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} + 2\sqrt[3]{x^2+y^2} + \sqrt[6]{x^2+y^2} \to 0$$

8) $f = \begin{cases} x\sin\frac{y}{\sqrt{x}} & x \neq 0 \\ 0, & x=0 \end{cases}$ $\quad \frac{\partial f(0,0)}{\partial x} = \lim\limits_{x\to 0} \frac{\cancel{x}\sin 0}{\cancel{x}} = 0$

$\frac{\partial f(0,0)}{\partial y} = \lim\limits_{y\to 0} 0 \cdot = 0$

$$\frac{x\sin\frac{y}{\sqrt{x}} - 0x - 0y - 0}{\sqrt{x^2+y^2}} \leq \frac{x\cdot\frac{y}{\sqrt{x}}}{\sqrt{x^2+y^2}} = \frac{\sqrt{x}\cdot y}{\sqrt{x^2+y^2}} \leq \frac{\sqrt[4]{x^2+y^2}\cdot\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2}}} \to 0$$

№ 20 (3, 6)

3) $f = \sqrt[3]{x^3+y^3}$ $\quad \frac{\partial f(0,0)}{\partial x} = \lim\limits_{x\to 0} \frac{\sqrt[3]{x^3+0}-0}{x} = 1 = \frac{\partial f}{\partial y}(0,0)$

$\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt[3]{x^3+y^3}-x-y}{\sqrt{x^2+y^2}}$ — ?

$x=y$: $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt[3]{x^3+y^3}-x-y}{\sqrt{x^2+y^2}} = \frac{\sqrt[3]{2}-2}{\sqrt{2}} \neq 0 \Rightarrow$ не дифф-ма

6) $f = \ln(3+\sqrt[3]{x^2y})$ $\quad \frac{\partial f}{\partial x}(0,0) = \lim\limits_{x\to 0} \frac{\ln(3+\sqrt[3]{0}) - \ln(3)}{x} = 0 = \frac{\partial f}{\partial y}(0,0)$

$$\frac{\ln(3+\sqrt[3]{x^2y}) - 0x - 0\cdot y - \ln(3)}{\sqrt{x^2+y^2}} = \frac{\ln\left(1+\frac{\sqrt[3]{x^2y}}{3}\right)}{\sqrt{x^2+y^2}}$$

$x=y$: $\lim\limits_{x\to 0} \frac{\ln(1+\frac{x}{3})}{\sqrt{2}x}$ — ? $\quad \frac{\ln(1+\frac{x}{3})}{\sqrt{2}x} \sim \frac{\frac{x}{3}}{\sqrt{2}x} \sim \frac{1}{\sqrt{2}\cdot 3} \neq 0 \Rightarrow$ недифф.

№ 23(1)

$f = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right), & x^2+y^2 \neq 0 \\ 0, & x^2+y^2=0 \end{cases}$

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy =$

в т. $(0,0)$:

$$\lim_{x\to 0} \left(\frac{x^2 \cdot \sin\frac{1}{x^2}}{\underset{\downarrow 0}{x}}\right) = \lim_{x\to 0} \overset{\nearrow 0}{x} \left(\sin\frac{1}{x^2}\right) = 0$$

огранич

$$\frac{\partial f}{\partial y} = \lim_{y\to 0} \frac{y^2 \cdot \sin\frac{1}{y^2}}{y} = \lim_{y\to 0} \underset{\downarrow 0}{y} \left(\sin\frac{1}{y^2}\right) = 0$$

огранич

$\Rightarrow df(0,0) = 0$

$\Rightarrow f$ - дифф-ма в т. $(0,0)$, в ост. точках

также дифф-ма. и равна:

$$df = (2x\,dx + 2y\,dy)\sin\frac{1}{(x^2+y^2)} + (x^2+y^2)\cos\frac{1}{(x^2+y^2)}\left(-\frac{1}{(x^2+y^2)^2}\right)$$

$$\cdot (2x\,dx + 2y\,dy) =$$

$$= (2x\,dx + 2y\,dy)\left(\sin\frac{1}{x^2+y^2} - \cos\frac{1}{(x^2+y^2)} \cdot \frac{1}{x^2+y^2}\right).$$

Тогда рассм. посл-ть,

где $x = y$, $\frac{1}{x^2+y^2} = 2\pi + 2\pi k$

$$2x\,dx + 2y\,dy = \frac{1}{\sqrt{\pi + \pi k}}(dx + dy)$$

$$\sin\frac{1}{x^2+y^2} - \cos\frac{1}{x^2+y^2} \cdot \frac{1}{x^2+y^2} = -(2\pi + 2\pi k).$$

$$\Rightarrow (2x\,dx + 2y\,dy)\left(\sin\frac{1}{x^2+y^2} - \cos\frac{1}{x^2+y^2} \cdot \frac{1}{x^2+y^2}\right) =$$

$$= -2\sqrt{\pi+\pi k}\,\underbrace{(dx + dy)}_{const}$$

$k \to \infty \Rightarrow$ $-2\sqrt{\pi+\pi k}(dx+dy) \to \infty$,

при этом $x, y \to 0$.

но $df(0,0) = 0$ $\Rightarrow$ ф-ция не явл непр. дифф-мой.

§4

N71 (3)

$$f(x,y) = \sqrt{\frac{(1+x)^{\alpha}+(1+y)^{\beta}}{2}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{2\cdot 2}\frac{\sqrt{2}\cdot\alpha(1+x)^{\alpha-1}}{\sqrt{(1+x)^{\alpha}+(1+y)^{\beta}}} \qquad \frac{\partial f}{\partial y} = \frac{1\sqrt{2}\,\beta(1+y)^{\beta-1}}{2\cdot 2\sqrt{(1+x)^{\alpha}+(1+y)^{\beta}}}$$

$$\frac{\partial f}{\partial x}(0,0) = \frac{\alpha}{4} \qquad \frac{\partial f}{\partial y}(0,0) = \frac{\beta}{4}$$

$$\frac{\partial f}{\partial x^2} = \frac{\alpha}{4}\frac{\left((\alpha-1)(1+x)^{\alpha-2}\sqrt{\frac{(1+x)^{\alpha}+(1+y)^{\beta}}{2}} - (1+x)^{\alpha-1}\cdot\frac{\sqrt{2}\cdot\alpha(1+x)^{\alpha-1}}{4\sqrt{(1+x)^{\alpha}+(1+y)^{\beta}}}\right)}{\frac{(1+x)^{\alpha}+(1+y)^{\beta}}{2}} =$$

$$= \frac{\alpha(1+x)^{\alpha-2}\left((\alpha-1)\cdot\frac{(1+x)^{\alpha}+(1+y)^{\beta}}{2} - \frac{(1+x)^{\alpha}\cdot\alpha}{4}\right)}{4\left(\frac{(1+x)^{\alpha}+(1+y)^{\beta}}{2}\right)^{3/2}}$$

$$\frac{\partial f}{\partial x^2}(0,0) = \frac{\alpha\cdot(\alpha-1-\alpha/4)}{4} = \frac{\alpha(3\alpha-4)}{16}.$$

ан-но $\frac{\partial f}{\partial y^2}(0,0) = \frac{\beta(3\beta-4)}{16}$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\sqrt{2}}{4}\beta(1+y)^{\beta-1}\cdot\left(-\frac{1}{2}\right)\frac{\alpha(1+x)^{\alpha-1}}{\left(\sqrt{(1+x)^{\alpha}+(1+y)^{\beta}}\right)^{3/2}}$$

$$\frac{\partial^2 f}{\partial x\partial y} = -\frac{\sqrt{2}\alpha\,\beta}{4\cdot 2\cdot 2\sqrt{2}} = -\frac{\alpha\beta}{16}$$

$$\Rightarrow \sqrt{\frac{(1+x)^{\alpha}+(1+y)^{\beta}}{2}} = 1+\frac{\alpha x}{4}+\frac{\beta y}{4}+\frac{C_2^2}{2!}\cdot\frac{\alpha(3\alpha-4)}{16}x^2+\frac{C_2^1}{2!}\cdot\left(\frac{-\alpha\beta}{16}\right)xy$$

$$+\frac{C_2^2}{2!}\frac{\beta(3\beta-4)y^2}{16}+o(\rho^2)=1+\frac{\alpha x+\beta y}{4}+\frac{\alpha(3\alpha-4)+\beta(3\beta-4)+2\alpha\beta xy}{32}+o(\rho^2)$$

№74(4)

$$f=\frac{\sin x}{\cos y}=\frac{x-\frac{x^3}{6}+o(x^4)}{1-\frac{y^2}{2}+o(y^3)}=$$

$$=\left(x-\frac{x^3}{6}+o(x^4)\right)\left(1+\frac{y^2}{2}+o(y^3)\right)=$$

$$=x+\frac{xy^2}{2}+o(\rho^4)-\frac{x^3}{6}-\frac{x^3y^2}{12}+o(\rho^4)=$$

уходит в $o(\rho^4)$

$$=x-\frac{x^3}{6}+\frac{xy^2}{2}+o(\rho^4).$$