

Numerical Methods, project B, Number 32

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Chapter 1

Find all zeros of function

1.1 a) False position method

1.1.1 Problem

We have to find zeros of the function

$$f(x) = -2.1 + 0.3x - xe^{-x}$$

In the interval $[-5; 10]$ using false position method.

1.1.2 Theoretical Introduction

False position method also called *regula falsi* in fancier circles is similar to the bisection method, the difference is that the interval we use $[a_n, b_n]$ is divided into two subintervals. We have:

- α - The root
- a_n - 'left' interval
- b_n - 'right' interval
- $f(a_n)$ - Value at left interval
- $f(b_n)$ - Value at right interval

We get:

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = \frac{f(b_n) - 0}{b_n - c_n}$$

From which we get:

$$c_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)} = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$$

Then we choose next interval as in the bisection method so: we calculate products of function values at a_n and b_n and that subinterval is selected for the next iteration of *false position* method. This subinterval corresponds to the negative product value.

Properties of *false position method*

This method is always convergent, simillary to bisection method, since it will always choose and shorten the interval which contains the root. If the function is continous and differentiable the method is linearly convergent. That being said the convergence may become sluggish. It can happen if for example one of the endpoints of the intervals will remain the same and the iterating will not shorten the interval to 0. One of the examples of functions that lead to that are barrier functions used in constrained optimization methods.

Improvement to the method In order to improve the formula and avoid aforementioned situation we can take smaller value of the function for the value that does not change. For right end:

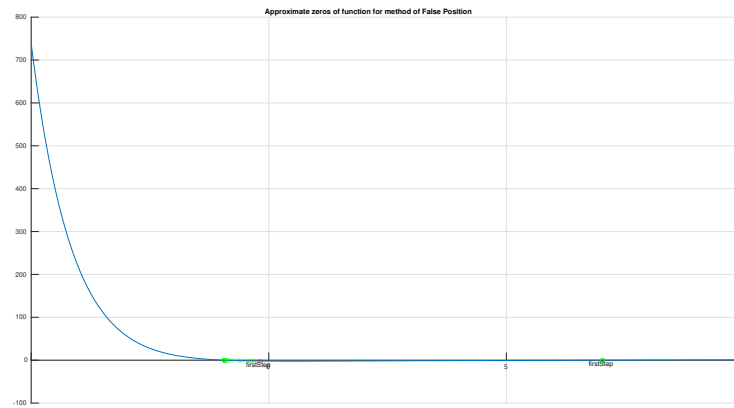
$$c_n = \frac{a_n \frac{f(b_n)}{2} - b_n f(a_n)}{\frac{f(b_n)}{2} - f(a_n)}$$

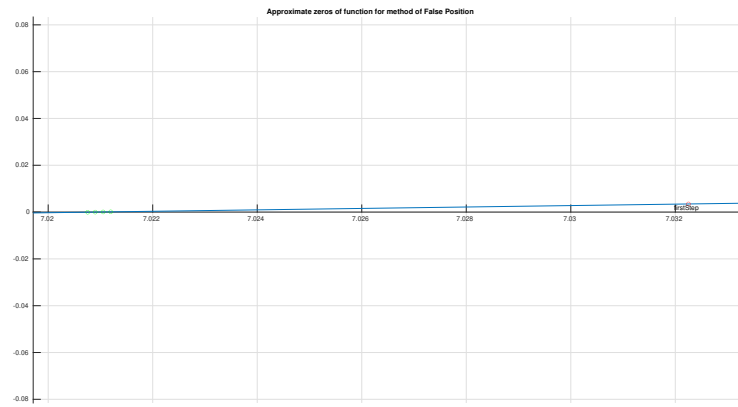
And for left end:

$$c_n = \frac{a_n f(b_n) - b_n \frac{f(a_n)}{2}}{f(b_n) - \frac{f(a_n)}{2}}$$

This is called *modified regula falsi* or *Illinois algorithm*. It is superlinearly convergent, globally convergent and length of intervals we get in each iterations converges to zero.

1.1.3 Results





Iteration	root	f(root)
1	-0.17673	-1.9421
2	-0.32943	-1.7409
3	-0.39578	-1.6308
4	-0.61812	-1.1386
5	-0.76152	-0.69765
6	-0.84516	-0.38573
7	-0.89015	-0.19911
8	-0.91304	-0.098733
9	-0.92431	-0.047922
10	-0.92976	-0.02301
11	-0.93237	-0.01099
12	-0.93362	-0.005236
13	-0.93421	-0.0024915
14	-0.9345	-0.0011848
15	-0.93463	-0.0005633
16	-0.93469	-0.00026777
17	-0.93473	-0.00012728
18	-0.93474	-6.0499e-05
19	-0.93475	-2.8756e-05
20	-0.93475	-1.3668e-05
21	-0.93475	-6.4964e-06
22	-0.93475	-3.0878e-06
23	-0.93475	-1.4676e-06
24	-0.93475	-6.9758e-07
25	-0.93475	-3.3157e-07
26	-0.93475	-1.5759e-07
27	-0.93475	-7.4906e-08
28	-0.93475	-3.5603e-08
29	-0.93475	-1.6922e-08
30	-0.93475	-8.0433e-09
31	-0.93475	-3.823e-09
32	-0.93475	-1.8171e-09
33	-0.93475	-8.6368e-10
34	-0.93475	-4.1051e-10
35	-0.93475	-1.9512e-10
36	-0.93475	-9.2741e-11

Iteration	root	f(root)
1	7.0323	0.0034663
2	7.0212	9.0344e-05
3	7.0211	4.6349e-05
4	7.0208	-4.3921e-05
5	7.0209	4.8939e-11

1.2 b) the Newton's method

1.2.1 Problem

We have to find zeros of the function

$$f(x) = -2.1 + 0.3x - xe^{-x}$$

In the interval $[-5; 10]$

using the Newton's method

1.2.2 Theoretical Introduction

The *Newton's method* also called *the tangent method* relies on first order part of its expansion into Taylor series for a given current approximation of root.

$$f(x) \approx f(x_n) + f'(x_n)(x - x_n)$$

Then we obtain the next point x_{n+1} by finding root of linear function:

$$f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0$$

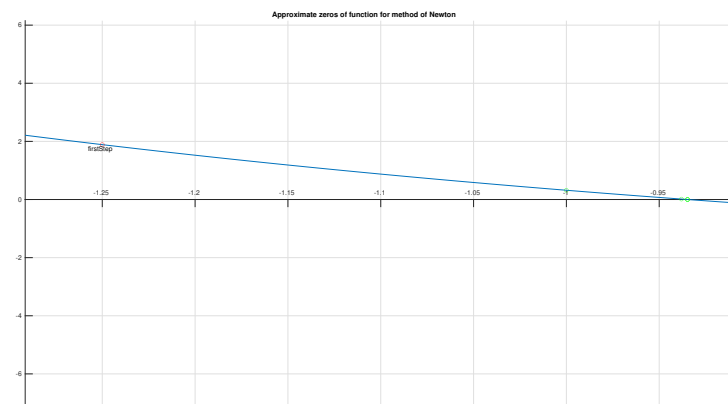
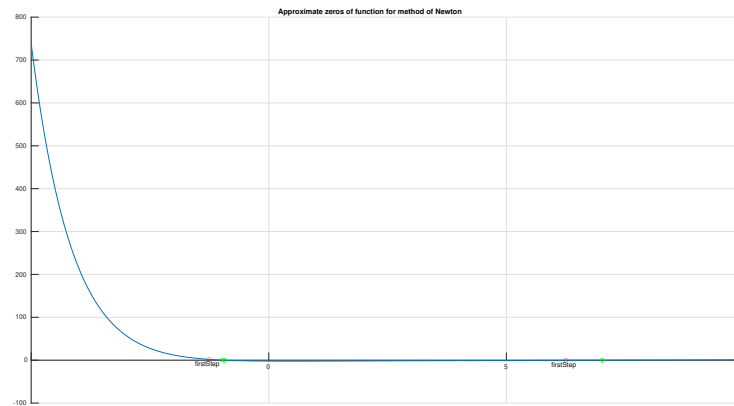
From this we get formula for x_{n+1} :

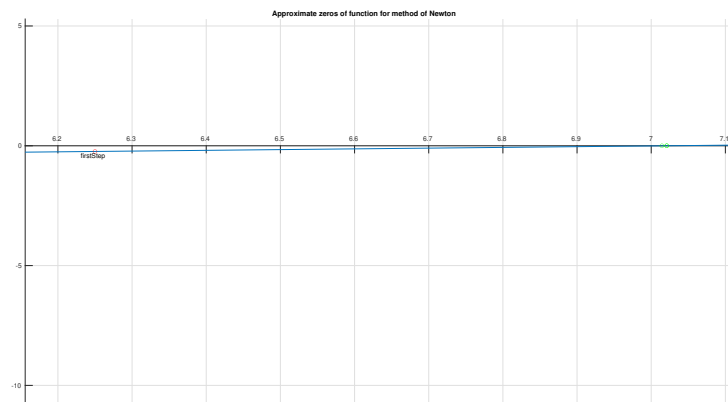
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This method as opposed to *regula falsi* method is locally convergent, should we choose initial point too far from the root (area which is close enough to root is called set of attraction) then we can get a divergence. On the other side if the Newton's method will converge then it is quite rapid with convergence of order $p = 2$ - quadratic convergence.

Newton's method is also effective if the function derivative is far from zero, so the slope of the function is steep, conversely if the derivative is close to zero the method is not recommended.

1.2.3 Results





Iteration	root	f(root)
1	-1.25	1.8879
2	-1.0001	0.31855
3	-0.93804	0.015256
4	-0.93476	4.0314e-05
5	-0.93475	2.8356e-10
6	-0.93475	0

Iteration	root	f(root)
1	6.25	-0.23707
2	7.0144	-0.0019866
3	7.0209	-9.517e-08
4	7.0209	7.468e-16

Chapter 2

Find real and complex roots of the polynomial

2.1 Problem

We have to find all real and complex roots of the polynomial:

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

where:

$$[a_4 \ a_3 \ a_2 \ a_1 \ a_0] = [-2 \ 12 \ 4 \ 1 \ 3]$$

So our polynomial looks like this:

$$f(x) = -2x^4 + 12x^3 + 4x^2 + 1x + 3$$

Using the Müller's method. We have to implement both MM1 and MM2 versions. We also need to find real roots using the Newton's method and compare these results with what we got from MM2 version of the Müller's method.

2.2 Theoretical Introduction

Müller's method revolves around the idea of approximating the polynomial locally close to the root by a quadratic function. Based on three different points we can use quadratic interpolation and develop our method. This means that we can treat it as a generalization of secant method. That being

said we can also realize it in an efficient way if we use just one point. We can use for this case values of polynomial, and its first and second derivative at current point.

Accordingly there are two versions of Müller's method: **MM1** and **MM2**.

2.2.1 MM1

Given three points: $x_0; x_1; x_2$ and their polynomial values: $f(x_0), f(x_1), f(x_2)$ we construct a (quadratic) function passing through these points. Then we find roots of this parabola and we choose one of these roots for the approximation of the result.

For example: Assume that x_2 is the approximation of the root. Let's introduce variable z such that:

$$z = x - x_2$$

And differences:

$$z_0 = x_0 - x_2$$

$$z_1 = x_1 - x_2$$

We have quadratic function:

$$y(z) = az^2 + bz + c$$

Using three points from above we get:

$$y(z_0) = az_0^2 + bz_0 + c = f(x_0)$$

$$y(z_1) = az_1^2 + bz_1 + c = f(x_1)$$

$$y(z_2) = c = f(x_2)$$

And then we get system of equation that we can solve to find a and b :

$$az_0^2 + bz_0 = f(x_0) - f(x_2)$$

$$az_1^2 + bz_1 = f(x_1) - f(x_2)$$

Roots are equal to:

$$z_+ = \frac{-2c}{b + \sqrt{b^2 - 4ac}}$$

$$z_- = \frac{-2c}{b - \sqrt{b^2 - 4ac}}$$

We choose a root with smaller absolute value for next iteration:

$$z_{min} = \min |z_+, z_-|$$

$$x_3 = x_2 + z_{min}$$

Then we choose new point x_3 and two selected from x_0, x_1, x_2 which were closer to x_3 .

This method should also work for $\Delta < 0$

2.2.2 MM2

This method being numerically more effective is usually recommended. We calculate values of a polynomial and its first and second derivatives at one point.

from definition of quadratic function:

$$y(z) = az^2 + bz + c$$

we can get:

$$z = x - x_k$$

If $z = 0$ then:

$$y(0) = c = f(x_k)$$

$$y'(0) = b = f'(x_k)$$

$$y''(0) = 2a = f''(x_k)$$

We can derive from that formula for roots:

$$z_{\pm} = \frac{-2f(x_k)}{f'(x_k) \pm \sqrt{(f'(x_k))^2 - 2f(x_k)f''(x_k)}}$$

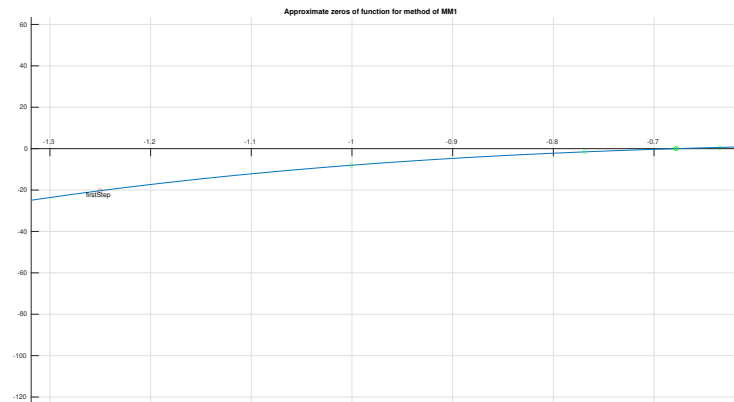
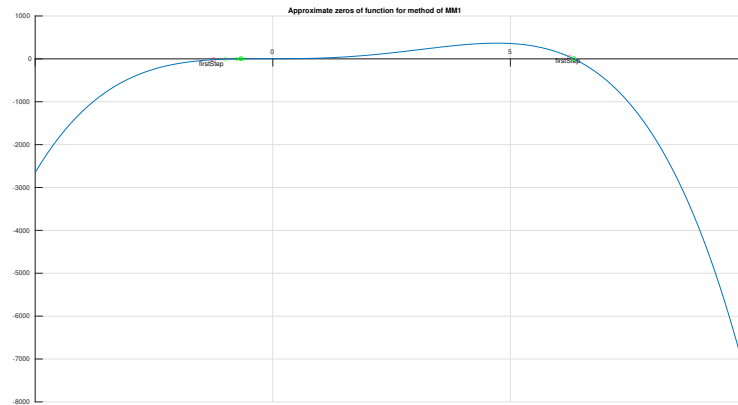
Then we choose root with smaller absolute value for next iteration:

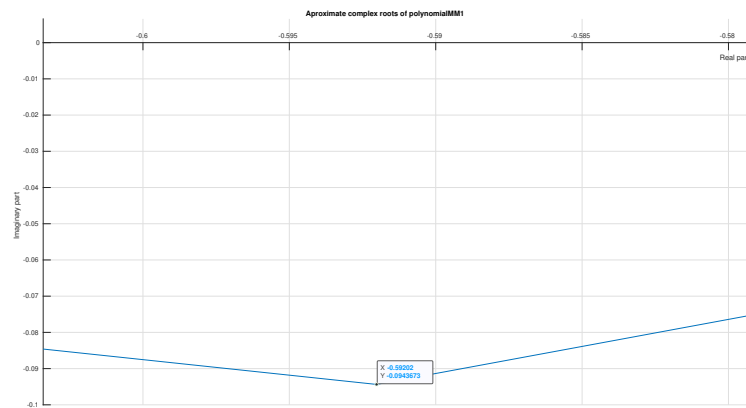
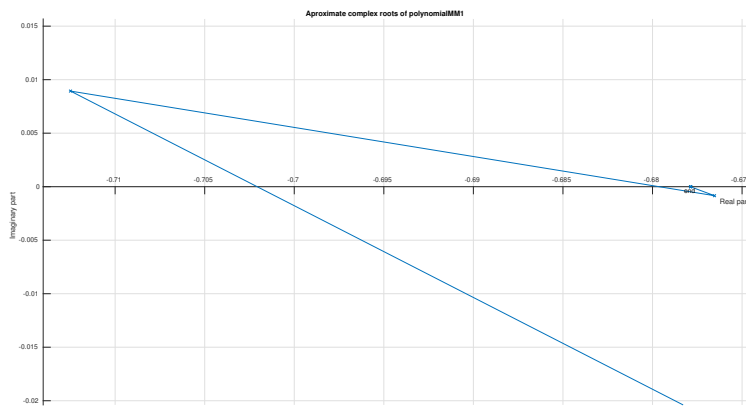
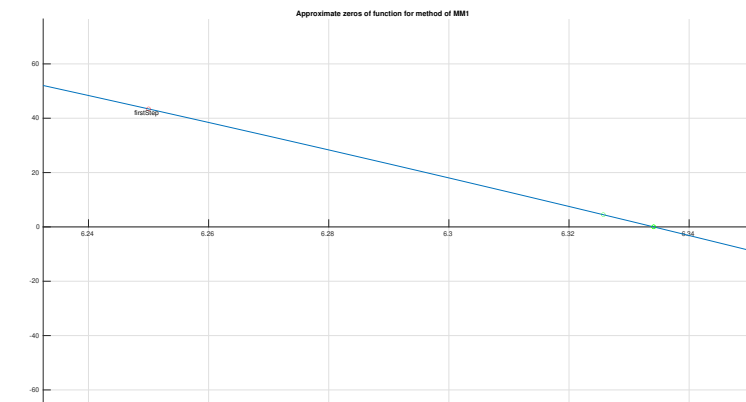
$$x_{k+1} = x_k + z_{min}$$

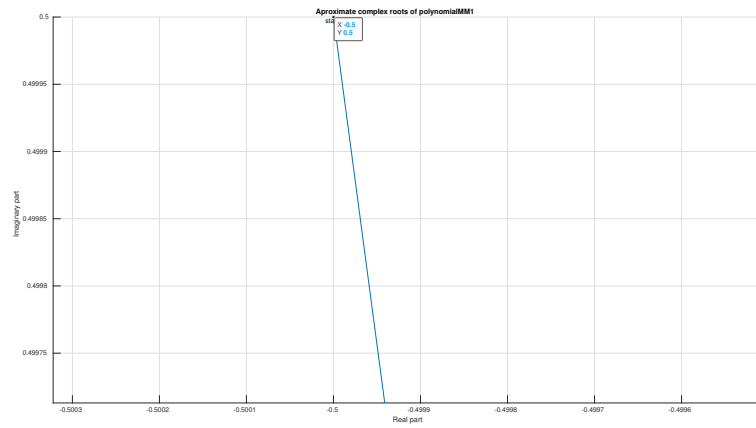
Again this method should be implemented in complex number arithmetic. This method is locally convergent with order of convergence equal to 1.84. It is locally more effective than secant method and it is almost as fast as Newton's method while being capable of finding complex roots. It can be used to find roots of polynomials or another nonlinear functions.

2.3 Results

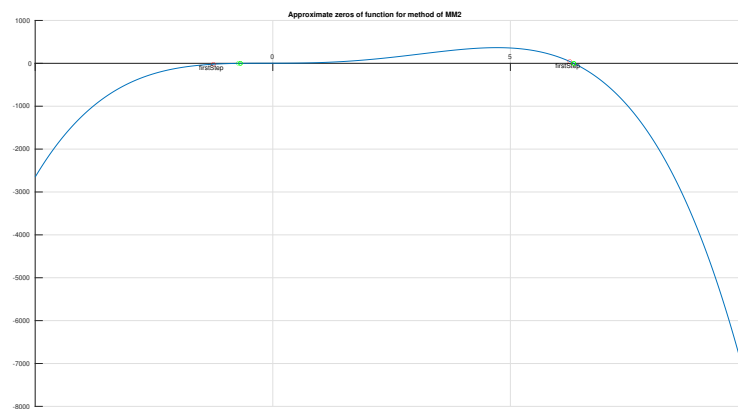
2.3.1 Graphs for MM1

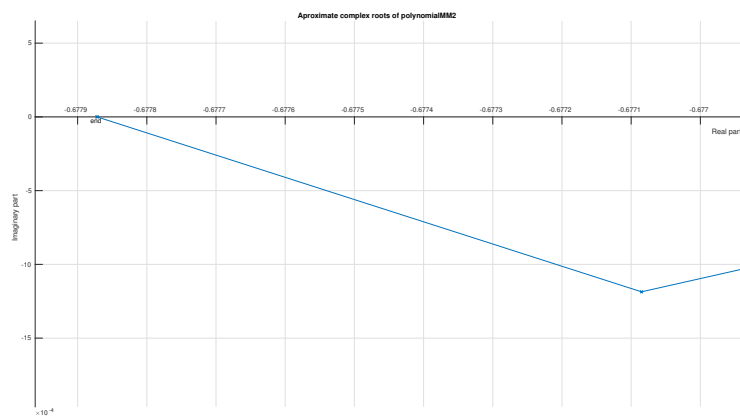
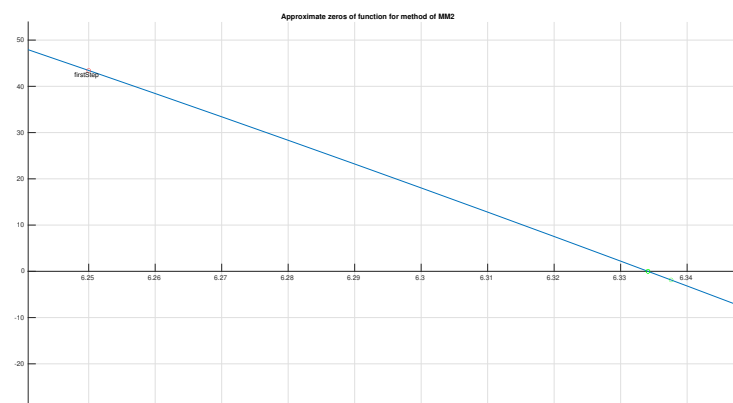
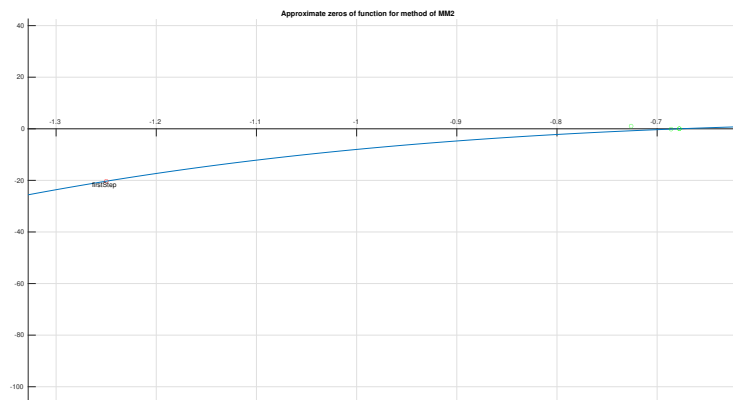


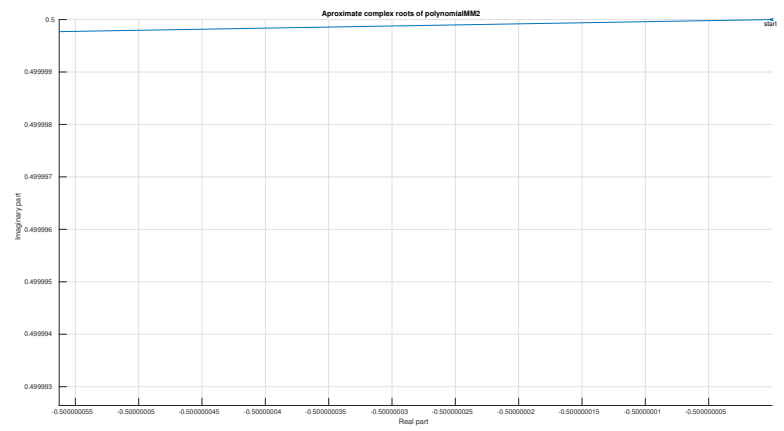
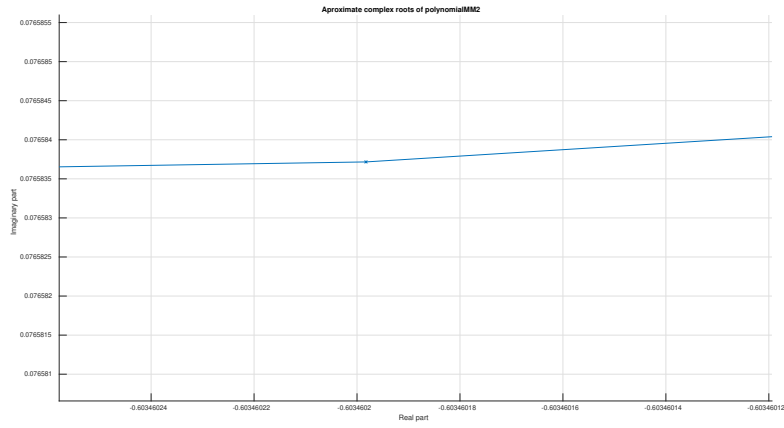




2.3.2 Graphs for MM2







2.3.3 Tables for MM1

Iteration	root	f(root)
1	-1.25	-20.32
2	-1.0009	-8.0351
3	-0.76855	-1.5512
4	-0.63496	0.58064
5	-0.67945	-0.023155
6	-0.67788	-0.00010725
7	-0.67787	-8.6306e-09
8	-0.67787	0

Iteration	root	f(root)
1	6.25	43.43
2	6.3257	4.4966
3	6.3341	0.033379
4	6.3341	-1.6916e-06
5	6.3341	-7.1942e-14

Iteration	root	f(root)
1	-0.5-0.5i	6.1847
2	-0.42823-0.15033i	2.6872
3	-0.59202+0.094367i	1.6012
4	-0.71249-0.0089336i	0.55524
5	-0.67654+0.00084164i	0.022962
6	-0.67787+8.4773e-06i	0.00012412
7	-0.67787+3.6458e-10i	8.3668e-09
8	-0.67787+9.1168e-18i	1.3321e-16

2.3.4 Tables for MM2

Iteration	root	f(root)
1	-1.25+0i	-20.32+0i
2	-0.72604-0.25326i	1.0515-4.0744i
3	-0.68613-0.01483i	-0.11667-0.22303i
4	-0.67788+2.5407e-06i	-0.00010163+3.7123e-05i
5	-0.67787-8.0819e-13i	1.4105e-11-1.1809e-11i

Iteration	root	f(root)
1	6.25	43.43
2	6.3376	-1.8647
3	6.3341	0.0029695
4	6.3341	1.6294e-08
5	6.3341	-4.6896e-13

Iteration	root	f(root)
1	-0.5-0.5i	6.1847
2	-0.60346-0.076584i	1.3673
3	-0.67709+0.001186i	0.020767
4	-0.67787+1.518e-08i	2.2673e-07
5	-0.67787-2.0556e-17i	3.0035e-16

- 2.3.5 Comparison of results between MM1 and MM2
- 2.3.6 Comparison of results between Newton's method and MM2

Chapter 3

Find real and complex roots of the polynomial using Laguerre's method

3.1 Problem

We have to find all (real and complex) roots of the polynomial from previous exercise:

$$f(x) = -2x^4 + 12x^3 + 4x^2 + 1x + 3$$

Using the Laguerre's method. Then we should compare those results with the MM2 version of the Müller's method.

3.2 Theoretical Introduction

Laguerre's method is defined by a single formula:

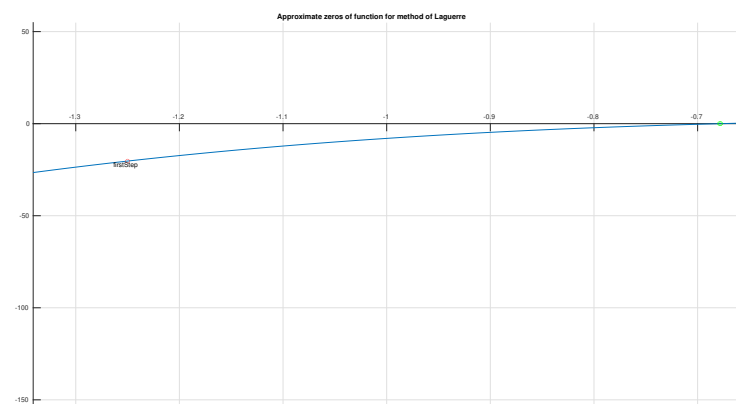
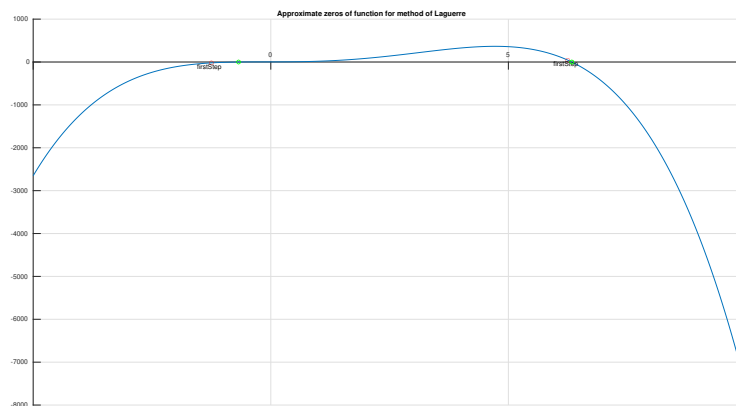
$$x_{k+1} = x_k - \frac{nf(x_k)}{f'(x_k) \pm \sqrt{(n-1)[(n-1)((f'(x_k))^2 - nf(x_k)f''(x_k))]}}$$

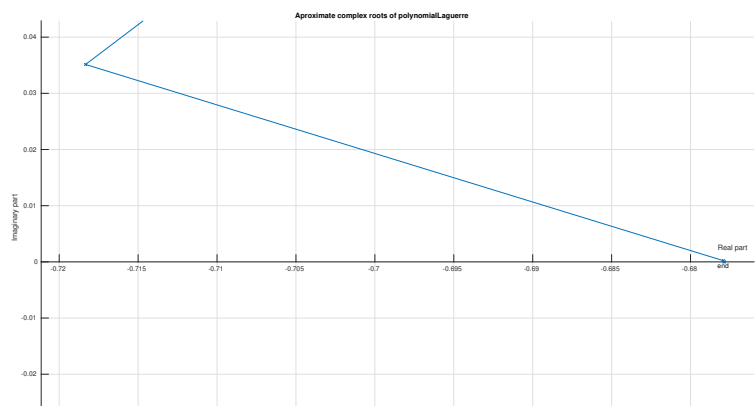
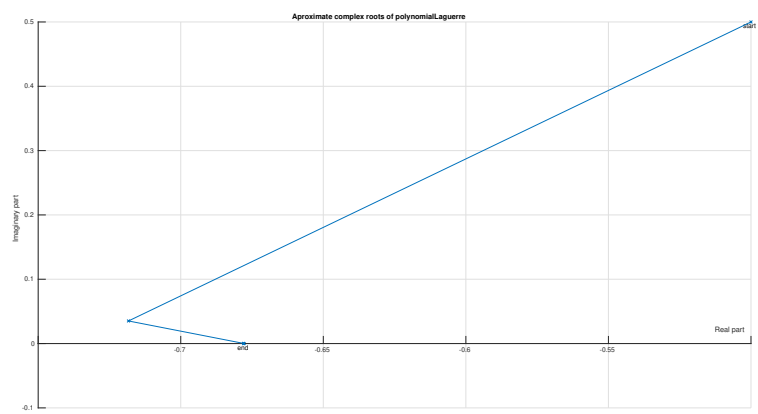
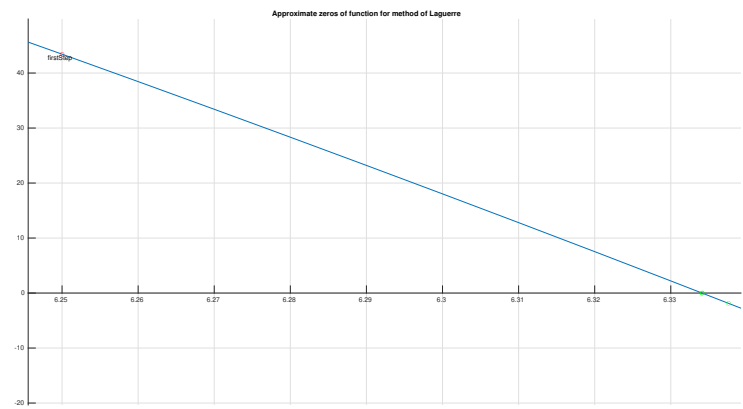
Where: n - order of the polynomial

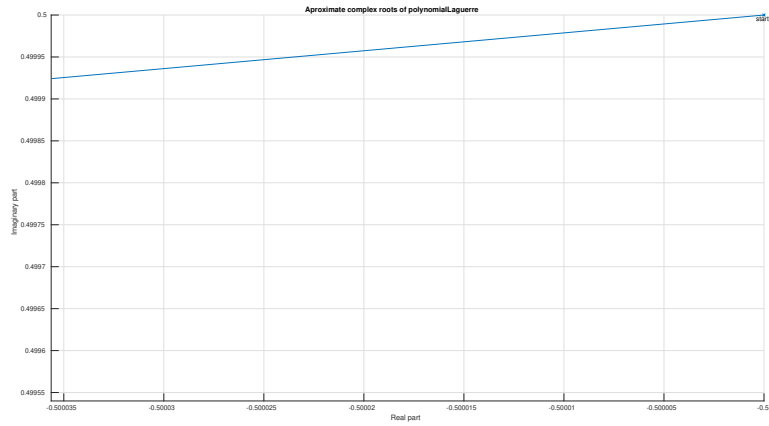
This formula is similar to the one from MM2 but also takes order of the polynomial into consideration. In general this method is better. For polynomials with real roots it is globally convergent. It does not have formal

analysis for complex roots but it usually shows good numerical properties, although divergence may happen.

3.3 Results







Iteration	root	f(root)
1	-1.25	-20.32
2	-0.67905	-0.017268
3	-0.67787	-3.1489e-07
4	-0.67787	0

Iteration	root	f(root)
1	-1.25	-20.32
2	-0.67905	-0.017268
3	-0.67787	-3.1489e-07
4	-0.67787	0

Iteration	root	f(root)
1	-1.25	-20.32
2	-0.67905	-0.017268
3	-0.67787	-3.1489e-07
4	-0.67787	0

3.3.1 Comparison of results between MM1 and MM2

Chapter 4

Code appendix

4.1 Task 1

4.1.1 task1Bisection.m

Top of task1Bisection.m

```
1 interval = [-5, 10];
2 rootBrackets = rootBracketing(@taskFunction, interval(1),
   interval(2));
3
4 printGraph(@taskFunction, 'False Position', @falsePosition,
   interval, rootBrackets, 'Approximate zeros of function for
   method of ');
```

taskFunction

```
1 function y = taskFunction(x)
2     y = -2.1 + 0.3*x - x*exp(1)^(-x);
3 end
```

falsePosition

```
1 function [zero, iterations] = falsePosition(taskFunction, a, b,  
    tolerance)  
2  
3     [iterations, lastTwoA, lastTwoB, i] = initialize();  
4     [zero, iterations, a, b, lastTwoA, lastTwoB] =  
        firstTwoIterations(a, b, taskFunction, iterations,  
            lastTwoA, lastTwoB);  
5     [zero, iterations] = falsePositionLoop(taskFunction, zero,  
        tolerance, lastTwoA, lastTwoB, i, a, b, iterations);  
6 end
```

initialize

```
1 function [iterations, lastTwoA, lastTwoB, i] = initialize()  
2     iterations = double.empty(2, 0);  
3     lastTwoA = double.empty(2, 0);  
4     lastTwoB = double.empty(2, 0);  
5     i = 0;  
6 end
```

taskFunction

```
1 function y = taskFunction(x)  
2     y = -2.1 + 0.3*x - x*exp(1)^(-x);  
3 end
```

firstTwoIterations

```
1 function [zero, iterations, a, b, lastTwoA, lastTwoB] =  
    firstTwoIterations(a, b, taskFunction, iterations, lastTwoA,  
        lastTwoB)  
2     for j = 1 : 2  
3         zero = (a*taskFunction(b) - b * taskFunction(a)) / (  
            taskFunction(b) - taskFunction(a));  
4         iterations(:, size(iterations, 2) + 1) = [zero,  
            taskFunction(zero)];  
5         if sign(taskFunction(a)) ~= sign(taskFunction(zero))  
6             b = zero;  
7         else  
8             a = zero;  
9         end  
10        lastTwoA(j) = a;  
11        lastTwoB(j) = b;  
12    end  
13 end
```

falsePositionLoop

```
1 function [zero, iterations] = falsePositionLoop(taskFunction,  
    zero, tolerance, lastTwoA, lastTwoB, i, a, b, iterations)  
2     while abs(taskFunction(zero)) > tolerance  
3         [lastTwoA, i, a, lastTwoB, b, tolerance, zero,  
            iterations] = insideLoop(lastTwoA, i, a, lastTwoB, b  
                , tolerance, taskFunction, iterations);  
4     end  
5 end
```

insideLoop

```
1 function [lastTwoA, i, a, lastTwoB, b, tolerance, zero,  
    iterations] = insideLoop(lastTwoA, i, a, lastTwoB, b,  
    tolerance, taskFunction, iterations)  
2     [lastTwoA, lastTwoB] = changeLastTwoAB(lastTwoA, lastTwoB,  
        i, a, b);  
3     zero = calculateZero(lastTwoB, tolerance, a, b, lastTwoA,  
        taskFunction);  
4     iterations(:, size(iterations, 2) + 1) = [zero,  
        taskFunction(zero)];  
5     [a, b] = newSubInterval(taskFunction, a, b, zero);  
6 end
```

changeLastTwoAB

```
1 function [lastTwoA, lastTwoB] = changeLastTwoAB(lastTwoA,  
    lastTwoB, i, a, b)  
2     lastTwoA(mod(i, 2) + 1) = a;  
3     lastTwoB(mod(i, 2) + 1) = b;  
4 end
```

calculateZero

```
1 function [zero] = calculateZero(lastTwoB, tolerance, a, b,  
    lastTwoA, taskFunction)  
2     if(abs(lastTwoB(1) - lastTwoB(2)) < tolerance)  
3         zero = (a*(taskFunction(b) / 2) - b * taskFunction(a))  
            / (taskFunction(b) / 2 - taskFunction(a));  
4     elseif (abs(lastTwoA(1) - lastTwoA(2)) < tolerance)  
5         zero = (a*taskFunction(b) - b * (taskFunction(a) / 2))  
            / (taskFunction(b) - (taskFunction(a) / 2));  
6     else  
7         zero = (a*taskFunction(b) - b * taskFunction(a)) / (  
            taskFunction(b) - taskFunction(a));  
8     end  
9 end
```

newSubInterval

```
1 function [a, b] = newSubInterval(taskFunction, a, b, zero)  
2     if sign(taskFunction(a)) ~= sign(taskFunction(zero))  
3         b = zero;  
4     else  
5         a = zero;  
6     end  
7 end
```

4.1.2 task1Newton.m

Top of task1Newton.m

```
1 interval = [-5, 10];  
2 rootBrackets = rootBracketing(@taskFunction, interval(1),  
    interval(2));  
3  
4 printGraph(@taskFunction, 'Newton', @newtonMethod, interval,  
    rootBrackets, 'Approximate zeros of function for method of '  
    ');
```

taskFunction

```
1 function y = taskFunction(x)
2     y = -2.1 + 0.3*x - x*exp(1)^(-x);
3 end
```

newtonMethod

```
1 function [zero, iterations] = newtonMethod(taskFunction, a, b,
    tolerance)
2     [iterations, iteration, zero] = initialize(a, b);
3     [zero, iterations] = newtonLoop(iterations, iteration, zero
        , a, b, tolerance, taskFunction);
4 end
```

initialize

```
1 function [iterations, step, zero] = initialize(a, b)
2     iterations = double.empty(2, 0);
3     step = sqrt(eps);
4     zero = (a + b) / 2;
5     iterations(:, size(iterations, 2) + 1) = [zero,
        taskFunction(zero)];
6 end
```

newtonLoop

```
1 function [zero, iterations] = newtonLoop(iterations, iteration,
    zero, a, b, tolerance, taskFunction)
2     while abs(taskFunction(zero)) > tolerance
3         [zero, iterations] = insideLoop(taskFunction, zero,
            iteration, iterations, a, b);
4     end
5 end
```

insideLoop

```
1 function [zero, iterations] = insideLoop(taskFunction, zero,  
    iteration, iterations, a, b)  
2     [zero, iterations] = calculateZeroIterations(taskFunction,  
        zero, iteration, iterations);  
3     checkForDivergence(zero, a, b);  
4 end
```

calculateZeroIterations

```
1 function [zero, iterations] = calculateZeroIterations(  
    taskFunction, zero, iteration, iterations)  
2     derivative = (taskFunction(zero + iteration) - taskFunction  
        (zero - iteration)) / (2 * iteration);  
3     zero = zero - taskFunction(zero) / derivative;  
4     iterations(:, size(iterations, 2) + 1) = [zero,  
        taskFunction(zero)];  
5 end
```

checkForDivergence

```
1 function checkForDivergence(zero, a, b)  
2     if zero < a || zero > b  
3         error('Divergent iteration');  
4     end  
5 end
```


4.2 Task 2

4.2.1 task2MM1.m

Top of task2MM1

```
1 interval = [-5, 10];
2 rootBrackets = rootBracketing(@polynomial, interval(1),
   interval(2));
3
4 printGraph(@polynomial, 'MM1', @mm1, interval, rootBrackets, '
   Approximate zeros of function for method of ');
5
6 printComplexGraph(@polynomial, 'MM1', @mm1, [-1 + i, 0], '
   Aproximate complex roots of polynomial');
```

polynomial

```
1 function y = polynomial(x)
2     y = -2 * x^4 + 12 * x^3 + 4 * x^2 + 1 * x + 3;
3 end
```

mm1

```
1 function [approximation, iterations] = mm1(polynomial, a, b,
   tolerance)
2     [approximation, approximationValue, iterations] =
   initialize(a, b, polynomial);
3     [approximation, iterations] = mm1Loop(approximation,
   tolerance, approximationValue, iterations, polynomial);
4 end
```

initialize

```
1 function [approximation, approximationValue, iterations] =  
    initialize(a, b, polynomial)  
2     approximation = [a, b, (a + b) / 2];  
3     approximationValue = arrayfun(polynomial, approximation);  
4     iterations = [approximation(3); polynomial(approximation(3)  
        )];  
5 end
```

mm1Loop

```
1 function [approximation, iterations] = mm1Loop(approximation,  
    tolerance, approximationValue, iterations, polynomial)  
2     while abs(polynomial(approximation(3))) > tolerance  
3         [approximation, approximationValue, iterations] =  
            insideLoop(approximation, approximationValue,  
                polynomial, iterations);  
4     end  
5 end
```

insideLoop

```
1 function [approximation, approximationValue, iterations] =  
    insideLoop(approximation, approximationValue, polynomial,  
        iterations)  
2     equationsSystem = createEquationSystem(approximation,  
        approximationValue);  
3     [zPlus, zMinus] = rootsOfQuadraticFormula(equationsSystem,  
        approximationValue);  
4     [approximation, approximationValue, iterations] =  
        updateApproximations(zPlus, zMinus, approximation,  
            iterations, polynomial);  
5 end
```

createEquationSystem

```
1 function equationsSystem = createEquationSystem(approximation,  
    approximationValue)  
2     [z0, z1, difference0, difference1] =  
        initializeEquationSystem(approximation,  
            approximationValue);  
3     equationsSystem = solveEquationSystem(z0, difference0, z1,  
        difference1);  
4 end
```

rootsOfQuadraticFormula

```
1 function [zPlus, zMinus] = rootsOfQuadraticFormula(  
    equationsSystem, approximationValue)  
2     [a, b, c] = createApproximatedQuadraticFormula(  
        equationsSystem, approximationValue);  
3     [zPlus, zMinus] = findRootsOfQuadraticFormula(a, b, c);  
4 end
```

updateApproximations

```
1 function [approximation, approximationValue, iterations] =  
    updateApproximations(zPlus, zMinus, approximation,  
        iterations, polynomial)  
2     newApproximation = chooseNewRoot(zPlus, zMinus,  
        approximation);  
3     iterations = addZeroToIterationVector(newApproximation,  
        iterations, polynomial);  
4     worstApproximationIndex = getWorstApproximationIndex(  
        approximation, newApproximation);  
5     [approximation, approximationValue] =  
        deleteWorstApproximation(worstApproximationIndex,  
            approximation, polynomial, newApproximation);  
6 end
```

initializeEquationSystem

```
1 function [z0, z1, difference0, difference1] =  
    initializeEquationSystem(approximation, approximationValue)  
2     z0 = approximation(1) - approximation(3);  
3     z1 = approximation(2) - approximation(3);  
4     difference0 = approximationValue(1) - approximationValue(3)  
        ;  
5     difference1 = approximationValue(2) - approximationValue(3)  
        ;  
6 end
```

solveEquationSystem

```
1 function equationsSystem = solveEquationSystem(z0, difference0,  
    z1, difference1)  
2     equationsSystem = [z0 ^ 2, z0, difference0; z1 ^ 2, z1,  
        difference1];  
3     reductor = equationsSystem(2, 1) / equationsSystem(1, 1);  
4     equationsSystem(2, :) = equationsSystem(2, :) - reductor *  
        equationsSystem(1, :);  
5     equationsSystem(2, 1) = 0;  
6     equationsSystem(2, :) = equationsSystem(2, :) ./  
        equationsSystem(2, 2);  
7     equationsSystem(1, :) = equationsSystem(1, :) -  
        equationsSystem(1, 2) * equationsSystem(2, :);  
8     equationsSystem(1, :) = equationsSystem(1, :) ./  
        equationsSystem(1, 1);  
9 end
```

createApproximatedQuadraticFormula

```
1 function [a, b, c] = createApproximatedQuadraticFormula(  
    equationsSystem, approximationValue)  
2     a = equationsSystem(1, 3);  
3     b = equationsSystem(2, 3);  
4     c = approximationValue(3);  
5 end
```

findRootsOfQuadraticFormula

```
1 function [zPlus, zMinus] = findRootsOfQuadraticFormula(a, b, c)  
2     zPlus = -2 * c / (b + sqrt(b ^ 2 - 4 * a * c));  
3     zMinus = -2 * c / (b - sqrt(b ^ 2 - 4 * a * c));  
4 end
```

chooseNewRoot

```
1 function newApproximation = chooseNewRoot(zPlus, zMinus,  
    approximation)  
2     if abs(zPlus) < abs(zMinus)  
3         newApproximation = approximation(3) + zPlus;  
4     else  
5         newApproximation = approximation(3) + zMinus;  
6     end  
7 end
```

addZeroToIterationVector

```
1 function iterations = addZeroToIterationVector(newApproximation  
    , iterations, polynomial)  
2     zero = newApproximation;  
3     iterations(:, size(iterations, 2) + 1) = [zero, polynomial(  
        zero)];  
4 end
```

getWorstApproximationIndex

```
1 function worstApproximationIndex = getWorstApproximationIndex(  
    approximation, newApproximation)  
2     worstApproximationIndex = -1;  
3     worstApproximationDifference = 0;  
4     for i = 1:size(approximation, 2)  
5         diff = abs(approximation(i) - newApproximation);  
6         if diff > worstApproximationDifference  
7             worstApproximationIndex = i;  
8             worstApproximationDifference = diff;  
9         end  
10    end  
11 end
```

deleteWorstApproximation

```
1 function [approximation, approximationValue] =  
    deleteWorstApproximation(worstApproximationIndex,  
    approximation, polynomial, newApproximation)  
2     approximation(worstApproximationIndex) = [];  
3     approximation(3) = newApproximation;  
4     approximationValue = arrayfun(polynomial, approximation);  
5 end
```

4.2.2 task2MM2.m

top of task2MM2.m

```
1 interval = [-5, 10];
2 rootBrackets = rootBracketing(@polynomial, interval(1),
   interval(2));
3
4 printGraph(@polynomial, 'MM2', @mm2, interval, rootBrackets, '
   Approximate zeros of function for method of ');
5
6 printComplexGraph(@polynomial, 'MM2', @mm2, [-1 + i, 0], '
   Aproximate complex roots of polynomial');
```

polynomial

```
1 function y = polynomial(x)
2     y = -2 * x^4 + 12 * x^3 + 4* x^2 + 1 * x + 3;
3 end
```

mm2

```
1 function [approximation, iterations] = mm2(polynomial, a, b,
   tolerance)
2     [approximation, iterations] = initialize(a, b, polynomial);
3     [approximation, iterations] = mm2Loop(approximation,
   iterations, polynomial, tolerance);
4 end
```

initialize

```
1 function [approximation, iterations] = initialize(a, b,
   polynomial)
2     approximation = (a + b) / 2;
3     iterations = [approximation; polynomial(approximation)];
4 end
```

mm2Loop

```
1 function [approximation, iterations] = mm2Loop(approximation,  
2     iterations, polynomial, tolerance)  
3     while abs(polynomial(approximation)) > tolerance  
4         [approximation, iterations] = insideLoop(approximation,  
5             polynomial, iterations);  
6     end  
7 end
```

insideLoop

```
1 function [approximation, iterations] = insideLoop(approximation  
2     , polynomial, iterations)  
3     [a, b, c] = getABC(approximation, polynomial);  
4     [zPlus, zMinus] = findRoots(a, b, c);  
5     newApproximation = chooseNewApproximation(zPlus, zMinus  
6         , approximation);  
7     [approximation, iterations] = updateApproximations(  
8         newApproximation, iterations, polynomial);  
9 end
```

getABC

```
1 function [a, b, c] = getABC(approximation, polynomial)  
2     c = polynomial(approximation);  
3     b = derivative(polynomial, approximation, 1);  
4     a = derivative(polynomial, approximation, 2) / 2;  
5 end
```

findRoots

```
1 function [zPlus, zMinus] = findRoots(a, b, c)  
2     zPlus = -2 * c / (b + sqrt(b ^ 2 - 4 * a * c));  
3     zMinus = -2 * c / (b - sqrt(b ^ 2 - 4 * a * c));  
4 end
```


chooseNewApproximation

```
1 function newApproximation = chooseNewApproximation(zPlus,  
2     zMinus, approximation)  
3     if abs(zPlus) < abs(zMinus)  
4         newApproximation = approximation + zPlus;  
5     else  
6         newApproximation = approximation + zMinus;  
7 end
```

updateApproximations

```
1 function [approximation, iterations] = updateApproximations(  
2     newApproximation, iterations, polynomial)  
3     approximation = newApproximation;  
4     iterations(:, size(iterations, 2) + 1) = [approximation,  
5         polynomial(approximation)];  
6 end
```

derivative

```
1 function y = derivative(function_, x, degree)  
2     if degree == 0  
3         y = function_(x);  
4         return  
5     end  
6  
7     step = sqrt(eps);  
8     y = (derivative(function_, x + step, degree - 1) -  
9         derivative(function_, x - step, degree - 1)) / (2 * step  
10 );  
11 end
```

4.3 Task 3

Top of Task 3

```
1 interval = [-5, 10];
2 rootBrackets = rootBracketing(@polynomial, interval(1),
   interval(2));
3
4 printGraph(@polynomial, 'Laguerre', @laguerre, interval,
   rootBrackets, 'Approximate zeros of function for method of '
   );
5
6 printComplexGraph(@polynomial, 'Laguerre', @laguerre, [-1 + i,
   0], 'Aproximate complex roots of polynomial');
```

polynomial

```
1 function y = polynomial(x)
2     y = -2 * x^4 + 12 * x^3 + 4* x^2 + 1 * x + 3;
3 end
```

laguerre

```
1 function [zero, iterations] = laguerre(polynomial, a, b,
   tolerance)
2     [degree, zero, iterations] = initialize(a, b, polynomial);
3     [zero, iterations] = laguerreLoop(polynomial, zero,
   tolerance, iterations, degree);
4
5 end
```

initialize

```
1 function [degree, zero, iterations] = initialize(a, b,  
    polynomial)  
2     degree = 4;  
3     zero = (a + b) / 2;  
4     iterations = [zero; polynomial(zero)];  
5 end
```

laguerreLoop

```
1 function [zero, iterations] = laguerreLoop(polynomial, zero,  
    tolerance, iterations, degree)  
2     while abs(polynomial(zero)) > tolerance  
3         [iterations, zero] = insideLoop(polynomial, zero,  
            degree, iterations);  
4     end  
5 end
```

insideLoop

```
1 function [iterations, zero] = insideLoop(polynomial, zero,  
    degree, iterations)  
2     [derrivative0, derrivative1, derrivative2] =  
        calculateDerrivatives(polynomial, zero);  
3     [zPlus, zMinus] = calculateZ(degree, derrivative0,  
        derrivative1, derrivative2);  
4     newZero = chooseNewZero(zPlus, zMinus, zero);  
5     [zero, iterations] = updateZeros(newZero, iterations,  
        polynomial);  
6 end
```

calculateDerrivatives

```
1 function [derrivative0, derrivative1, derrivative2] =  
    calculateDerrivatives(polynomial, zero)  
2     derrivative0 = polynomial(zero);  
3     derrivative1 = derivative(polynomial, zero, 1);  
4     derrivative2 = derivative(polynomial, zero, 2);  
5 end
```

calculateZ

```
1 function [zPlus, zMinus] = calculateZ(degree, derrivative0,  
    derrivative1, derrivative2)  
2     expressionUnderSquareRoot = (degree - 1) * ((degree - 1) *  
        derrivative1 ^ 2 - degree * derrivative0 * derrivative2)  
        ;  
3     lagsqrt = sqrt(expressionUnderSquareRoot);  
4  
5     zPlus = degree * derrivative0 / (derrivative1 + lagsqrt);  
6     zMinus = degree * derrivative0 / (derrivative1 - lagsqrt);  
7 end
```

chooseNewZero

```
1 function newZero = chooseNewZero(zPlus, zMinus, zero)  
2     if abs(zPlus) < abs(zMinus)  
3         newZero = zero - zPlus;  
4     else  
5         newZero = zero - zMinus;  
6     end  
7 end
```

updateZeros

```
1 function [zero, iterations] = updateZeros(newZero, iterations,  
    polynomial)  
2     zero = newZero;  
3     iterations(:, size(iterations, 2) + 1) = [zero, polynomial(  
        zero)];  
4 end
```

4.3.1 rootBracketing.m

updateZeros

```
1 % find the root brackets of a function within the given range  
2 function rootBrackets = rootBracketing(givenFunction,  
    intervalLeft, intervalRight)  
3     [a, b, rootBrackets, resolution] = initializeValues(  
        intervalLeft, intervalRight);  
4     rootBrackets = bracketingLoop(a, b, rootBrackets,  
        intervalRight, resolution, givenFunction);  
5 end
```

initializeValues

```
1 function [a, b, rootBrackets, resolution] = initializeValues(  
    intervalLeft, intervalRight)  
2     % define search resolution  
3     resolution = (intervalRight - intervalLeft) / 6;  
4     % The higher the value of denominator the less iterations  
        will it take  
5     % to reach the roots, however in order to have nice graph  
        showing those  
6     % brackets I will choose relatively small denominator – I  
        have choosen  
7     % the smallest natural number that still generates brackets  
        on a graph  
8  
9     % start search at the start of the range  
10    a = intervalLeft;  
11    b = intervalLeft + resolution;  
12    rootBrackets = double.empty(2, 0); % initialize empty  
        vector of size 2  
13 end
```

bracketingLoop

```
1 function rootBrackets = bracketingLoop(a, b, rootBrackets,  
   intervalRight, resolution, givenFunction)  
2     while b ~= intervalRight % if the bracket can't be expanded  
       end loop  
3       % if the function changes sign inside the interval that  
         means that we passed through a root that means that  
         a bracket has been found  
4       if sign(givenFunction(a)) ~= sign(givenFunction(b))  
5         % save bracket  
6         rootBrackets(:, size(rootBrackets, 2) + 1) = [a, b  
           ]; % Add the new bracket to existing ones  
7       end  
8       % check next bracket  
9       a = b;  
10      b = min(a + resolution, intervalRight);  
11      % Once a + resolution > intervalRight, then we will  
        know that we  
12      % reached beyond the interval and we must stop  
13      end  
14 end
```

4.3.2 printGraph.m

```
1 % graph the real roots of a function
2 function printGraph(taskFunction, algorithmName, algorithm,
   interval, rootBrackets, plotTitle)
3     figure()
4     grid on; % Get y values lines
5     hold on; % Retain current plot when adding new plots
6     title([plotTitle, algorithmName]);
7     set(gca, 'XAxisLocation', 'origin'); % Set properties of
   current axis
8     x = interval(1):0.01:interval(2);
9     % x is a vector of values between left and right interval
   with every value being higher by 0.01
10    y = arrayfun(taskFunction, x); % We sketch the function
   from task for each x
11    plot(x, y);
12
13    % iterate over rootBrackets and add them to the plot
14    for rootBracket = rootBrackets
15        % find all zeros within the bracket using the given
   algorithm
16        % Get all steps from the algorithm we use
17        [~, steps] = algorithm(taskFunction, rootBracket(1),
   rootBracket(2), 1e-10);
18
19        firstStepColor = [1 0 0]; % Red
20        otherStepsColor = [0 1 0]; % Green
21        % plot first steps
22        scatter(steps(1, 1), steps(2, 1), [], firstStepColor);
23        text(steps(1, 1), steps(2, 1), 'firstStep', '
   HorizontalAlignment', 'center', 'VerticalAlignment',
   'top'); % It makes text appear neatly
24        % plot other steps
25        scatter(steps(1, 2:end), steps(2, 2:end), [],
   otherStepsColor);
26
27
```



```
28
29     % print root table
30     disp([plotTitle, ' (' , algorithmName, ')']);
31     columns = {'step', 'root', 'value at root'};
32     disp(table((1:size(steps, 2))', steps(1, :)', steps(2, :)',
33               'VariableNames', columns));
34 end
```

4.3.3 printComplexGraph.m

```
1 % graph the complex roots of a function
2 function printComplexGraph(printComplexGraph, algorithmName,
   algorithm, rootBrackets, plottitle)
3     figure();
4     grid on; % Get y values lines
5     hold on; % Retain current plot when adding new plots
6     title([plottitle, algorithmName]);
7     xlabel(Real part);
8     ylabel(Imaginary part);
9     set(gca, 'XAxisLocation', 'origin'); % Set properties of
       current axis
10
11 % find all zeros within the bracket using the given
       algorithm
12 [~, steps] = algorithm(printComplexGraph, rootBrackets(1),
       rootBrackets(2), 1e-15);
13
14 % plot first step
15 text(real(steps(1, 1)), imag(steps(1, 1)), 'start', '
       HorizontalAlignment', 'center', 'VerticalAlignment', '
       top');
16
17 % plot steps on graph
18 plot(real(steps(1, :)), imag(steps(1, :)), '-x');
19
20 % plot last step
21 text(real(steps(1, end)), imag(steps(1, end)), 'end', '
       HorizontalAlignment', 'center', 'VerticalAlignment', '
       top');
22
23
24 % print root table
25 disp([plottitle, ' (' , algorithmName, ')']);
26 columns = {'step', 'root', 'abs value at root'};
27 disp(table([1:size(steps, 2)]', steps(1, :)', abs(steps(2,
       :))', 'VariableNames', columns));
```

28 | **end**

Bibliography

- [1] Piotr Tatjewski (2014) *Numerical Methods*, Oficyna Wydawnicza Politechniki Warszawskiej