Chapter 1

Axiomatics, the Social Sciences, and the Gödel Phenomenon: A Toolkit*

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1. Introduction

Gödel published his remarkable incompleteness theorems in 1931 (see Gödel, 1931). Gödel's reasoning was immediately recognized as correct, even if surprising, and several researchers then asked for its scope: since Gödel's argument exhibited an undecidable sentence that didn't quite reflect everyday mathematical fact or facts (see below), there was some hope that undecidable sentences might be circumscribed to a very pathological realm within arithmetic theory or its extensions.

Alas, this proved not to be true.

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A road map for this chapter

We present here a kind of primer on Gödel incompleteness, both in pure mathematics and in some domains of applied mathematics. It will be seen that undecidability and incompleteness exist everywhere in "reasonable" theories, so to say, and that they may affect innocent-looking, quite ordinary mathematical questions.

Moreover, it is possible that some big open questions might turn out to be undecidable — might turn out to be Gödelian specimens, we may say again — in strong axiomatic frameworks. Among those are Goldbach's conjecture and the P vs. NP question (which may turn out to be independent even from a very rich axiom system such as ZFC (see Ben-David and Halevy, s/d) and da Costa and Doria, 2016).

Since we require an axiomatic background in order to have incompleteness, we will discuss techniques here for the axiomatization of scientific theories that require mathematics as its main language. The axiomatization techniques we describe are based on a suggestion by P. Suppes, and provide a quite straightforward procedure to axiomatize theories in physics and also in mathematical economics, mathematical ecology, and so on. Out of that, we will construct examples of relevant undecidable sentences in those theories.

Actually this chapter is a survey of a few results that have been explored by N. da Costa and the author for over 30 years.

Gödel's 1931 paper on the incompleteness of arithmetics and related systems

Gödel's great 1931 paper, "On formally undecidable sentences of *Principia Mathematica* and related systems, I" (see Gödel, 1931) has two main results. Suppose that the axiomatic theory we consider contains "enough arithmetic"[1]:

[1] Lots of handwaving here! But anyway, we require arithmetic in classical predicate logic, plus the trichotomy axiom.

(1) If the axiomatic theory we consider is consistent, then there is in it a sentence which can neither be proved nor be disproved (always within the axiomatic framework we have chosen). However, that sentence can be easily seen to be true in the standard model for arithmetic.
(2) If the same theory is consistent, then we cannot (within it) prove a sentence that asserts the theory's consistency.

Gödel's undecidable sentence is weird and doesn't seem to have an everyday mathematical meaning. It is constructed as follows:

- He first shows that the sentence "ξ cannot be proved" can be formalized within his axiomatic framework.
- Then he diagonalizes it. The sentence he obtains is interpreted as "I cannot be proved." The sentence is true, but — as said — doesn't have an obvious everyday mathematical meaning.

The second incompleteness theorem can be seen as a kind of corollary to the first one. Briefly, for a theory based on classical first-order logic, it is consistent if and only if it doesn't prove a contradiction. It proves a contradiction if and only if it proves all its sentences. However, an incomplete theory doesn't prove at least two of its sentences, say ξ and $\neg\xi$. (For a more detailed discussion see Chaitin *et al.* (2011).)

A summary of the chapter

We give in the present chapter an intuitive presentation of the ideas involved, list some of the questions that were shown to be undecidable with the authors' techniques and apply them in detail to systems in economics and ecology. Those applications show that Gödel incompleteness may be one of the chief hindrances (besides nonlinearity) in the prediction of the future behavior of those systems in our current formal representations for social phenomena.

So, our knowledge about society may also have computational, predictive limits imposed by the Gödel phenomenon.

We discuss two techniques to construct examples of the Gödel phenomenon in scientific disciplines that have mathematics as their main language. We call the first technique — or the first collection of tricks — *Playing Games with the Theta Function.* The second technique is discussed afterwards and originally arose out of computer science; it may be called *The Monster in the Belly of Complexity Theory.*

Part I. Playing games with the theta function

We have here a long, detailed, conceptual discussion on undecidability and incompleteness. Then we get to hard facts. Section 11 presents the concepts and ideas involved in our incompleteness proofs. That presentation sketches previous contributions to those questions and relates them to our work. Section 16 describes in nontechnical detail our contribution to several open questions in mathematical physics and related areas. Section 17 applies our results to economics and to ecological models. In Section 15, we review the formal background for our results, that is, the theory of Suppes predicates and state without proof our main undecidability and incompleteness theorems, while section 20 gives details about the ecological models we use in Section 17.

Well, and the θ function? Wait, please.

The present chapter may be looked upon as lending strong support to Suppes' contention that there is no essential difference between "a piece of pure mathematics and a piece of theoretical science." We show that Gödel-like phenomena occur everywhere and in rather intuitive contexts within the language of classical analysis. They therefore necessarily occur within any theory where the underlying language is that of classical analysis. As Suppes (1988) remarks:

> It is difficult to predict the future of axiomatic methods in the empirical sciences. There are signs, at least, that the large gap that presently separates the methods used in physics from those in mathematics is beginning to close. In any case, axiomatic methods are now widely used in foundational investigation of particular sciences, as well as in the pursuit of certain general questions of methodology, especially those concerning probability and induction. The use of such methods permits us to bring to the philosophy of science the standards of rigor and clarity that are very much an accepted part of the closely related discipline of logic.[2]

[2]I would also like to point out that a related, albeit more refined approach to the axiomatization of the empirical sciences can be found in the book of Balzer *et al.* (1987).

The main point in our exposition is: physics, both classical and quantum, is here seen as an outcome, or as an extension of classical mechanics.[3] The Lagrangian and Hamiltonian formalisms, for systems of particles and then for fields, are seen as a basic, underlying construct that specializes to the several theories considered. A course in theoretical physics usually starts from an exposition of the Lagrangian and Hamiltonian (the so-called analytico-canonical) formalisms, how they lead to a general formal treatment of field theories, and then one applies those formalisms to electromagnetic theory, to Schrödinger's quantum mechanics — which is obtained out of geometrical optics and the eikonal equation, which in turn arise from Hamilton–Jacobi theory — and gravitation and gauge fields, which grow out of the techniques used in the formalism of electromagnetic theory. Here we use a variant of this approach.

We stress that this chapter is intended as an overview of the results obtained by N. da Costa and the author in the search of undecidability and incompleteness — the so-called Gödel phenomenon — in physics and in other mathematized domains. We present here the main "abstract" details (the construction of the many θ functions, which code the halting function in computer science and beyond) and then their use in the construction of several examples of, let us say, Gödelian behavior in physics and beyond.

A note on sources for this chapter

Our sources are the texts listed in the bibliography, and we will liberally use several papers of ours, and quote from them. To quote a few of them, we take results from a widely circulated preprint (Doria, 2014) as well as three published texts (Chaitin *et al.*, 2011; da Costa and Doria, 1994b, 2007).

[3]This is the actual way most courses in theoretical physics are taught.

2. Axiom systems: mathematics

We will restrict here our attention to two axiom systems for mathematics, Peano Arithmetic (PA) and Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC). PA will be mainly used when we are dealing with the concept of computation and its consequences; and we will require ZFC in order to axiomatize the kind of mathematics that is used by professional mathematicians. However our arguments are valid for recursive extensions of these theories, that is, extensions whose theorems are also recursively enumerable.

A first look at Gödel incompleteness

Axiomatic systems like Zermelo–Fraenkel set theory, or Peano arithmetic, can be formulated as computer programs that list (i.e., recursively enumerate) all theorems of the theory. These theories are machines that produce theorems — the sentences which are valid in the theory. So, suppose that S is one such axiomatic theory. Suppose that S is able to talk about computer programs, that is, we can talk about partial recursive functions in the language of S. We are interested in the recursive functions that are total, that is, which are defined for all natural numbers $0, 1, 2, \ldots$.

Then we try listing (i.e., we try to recursively enumerate) all S-total recursive functions, that is, those recursive functions that S can recognize as total, or better, which S can prove to be total. This is the starting point of our argument, which stems from Kleene (1936):

- We need two preliminary suppositions: first, axiomatic system S is supposed to be consistent (i.e., it doesn't prove a contradiction such as, e.g., $0 \neq 0$). Also, S must be sound, that is, S doesn't prove sentences that are false in the standard interpretation for arithmetic.
- Start the program that lists the theorems of S.
- Pick up those theorems that say: "function f is total and computable."
- Out of that, we can build another list, $f_0, f_1, f_2, \ldots$, of S-total computable functions (functions that are proved as such in S),

together with their values:

$$\begin{gathered} f_0(0), f_0(1), f_0(2), f_0(3), \ldots \\ f_1(0), f_1(1), f_1(2), f_1(3), \ldots \\ f_2(0), f_2(1), f_2(2), f_2(3), \ldots \\ f_3(0), f_3(1), f_3(2), f_3(3), \ldots \\ \vdots \end{gathered}$$

- Now define a function F:

$$\begin{aligned} F(0) &= f_0(0) + 1 \\ F(1) &= f_1(1) + 1 \\ F(2) &= f_2(2) + 1 \\ &\vdots \end{aligned}$$

- F is different from f_0 at value 0, from f_1 at 1, from f_2 at 2, and so on.

We can now conclude our reasoning. The $f_0, f_1, f_2, \ldots$ functions are said to be *provably total* in our theory S, as they are proved to be total functions and appear as such in the listing of the theory's theorems. However F cannot be provably total in S, since it differs at least once from each function we have listed. Yet F is obviously computable and total in the standard model for arithmetic, and given programs for the computation of $f_0, f_1, f_2, \ldots$ we can compute F too.

So the sentence "F is total" cannot be proved in our theory.

Also, if we suppose that the theory is *sound*, that is, if it doesn't prove false facts, then the sentence "F isn't total" cannot be proved too, as F is clearly total in the so-called standard model for arithmetic. Therefore, it is an undecidable sentence within our theory S.

Ladies and gentlemen, "F is total" and "F isn't total" are examples of the *Gödel incompleteness phenomenon* in S: they are sentences that can neither be proved nor disproved within S. And because of the soundness of our theory, "F is total" is, we may say, naïvely true in the standard interpretation for the arithmetics of S.[4]

We call "F is total" and "F isn't total" *undecidable sentences* in S. This example is quite simple, and has an obvious mathematical meaning: it talks about computer programs and their domains. So, Gödel incompleteness does matter, after all.

A first example

We can present here a first example of incompleteness that directly stems from the metamathematical properties of F. The BGS set $\mathcal{S} = \langle \mathsf{M}_m, |x|^{F(n)} + F(n) \rangle$, $n = 0, 1, 2, \ldots$ has the following two properties, among many others of interest (see the discussion in the *Belly* sections for details):

- It is a set of Turing machines M_m bound by a clock that stops it after $|x|^{F(n)} + F(n)$ computation steps, where x is the input to the machine and $|x|$ the binary length of x.
- $\mathcal{S}$ is a set of poly Turing machines in the standard model.
- The sentence "$\mathcal{S}$ is a set of Turing poly machines" is true of the standard model for the arithmetic portion of theory S.
- "$\mathcal{S}$ is a set of Turing poly machines" is undecidable in theory S.

3. The Gödel phenomenon in physics and in other mathematized sciences

Now, are there Gödel undecidable sentences in physics? In mathematical economics? Yes.

In order to look for undecidable sentences in physics, one must axiomatize the theories where our formal sentences are cradled. This is the gist of a remarkable question formulated at the very end of the 19th century, Hilbert's Sixth Problem. That seems to be a formidable obstacle; let's take a look at Hilbert's formulation of it.

[4]There are examples of theories like S where one cannot find a "natural" interpretation like the one in our example.

Hilbert's Sixth Problem

The Mathematical Treatment of the Axioms of Physics.

The investigations on the foundations of geometry suggest the problem: *to treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probability and mechanics.*

As to the axioms of the theory of probabilities, it seems to me to be desirable that their logical investigation be accompanied by a rigorous and satisfactory development of the method of mean values in mathematical physics, and in particular in the kinetic theory of gases.

Important investigations by physicists on the foundations of mechanics are at hand; I refer to the writings of Mach..., Hertz..., Boltzmann..., and Volkman... It is therefore very desirable that the discussion of the foundations of mechanics be taken up by mathematicians also. Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, those merely indicated, which lead from the atomistic view to the laws of continua. Conversely, one might try to derive the laws of motion of rigid bodies by a limiting process from a system of axioms depending upon the idea of continuously varying conditions on a material filling all space continuously, these conditions being defined by parameters. For the question as to the equivalence of different systems of axioms is always of great theoretical interest.

If geometry is to serve as a model for the treatment of physical axioms, we shall try first by a small number of axioms to include as large a class as possible of physical phenomena, and then by adjoining new axioms to arrive gradually at the more special theories. At the same time, Lie's principle of subdivision can perhaps be derived from the profound theory of infinite transformation groups. The mathematician will have also to take account not only of those theories coming close to reality, but also, as in geometry, of all logically possible theories. We must be always alert to obtain a complete survey of all conclusions derivable from the system of axioms assumed.

> Further, the mathematician has the duty to test exactly in each instance whether the new axioms are compatible with the previous ones. The physicist, as his theories develop, often finds himself forced by the results of his experiments to make new hypotheses, while he depends, with respect to the compatibility of the new hypotheses with the old axioms, solely upon these experiments or upon a certain physical intuition, a practice that in the rigorously logical building up of a theory is not admissible. The desired proof of the compatibility of all assumptions seems to me also of importance, because the effort to obtain such a proof always forces us most effectively to an exact formulation of the axioms.

There are two main questions in Hilbert's Sixth Problem:

- To give an axiom system for the whole of physics;
- To show that the axiom systems we formulate are consistent.

We already know that, if PA is included in our axiomatics, then by Gödel's second incompleteness theorem we cannot prove the consistence of our axiomatized theories for physics.

But how about incompleteness? There is a folklore conjecture that tries to connect quantum physics and incompleteness; one wonders whether there is a relation between Gödel incompleteness and Heisenberg's uncertainty principle. If one had a strict equivalence here, a consequence of that conjecture would be the nonexistence of metamathematical phenomena such as undecidability and incompleteness at the classical level, in physics.

Yet we show here that such a conjecture is false. Classical physics — which is usually taken to be the realm of determinism — is as marred by the Gödel phenomenon as its quantum counterpart (da Costa and Doria, 1991a,b).

4. Physics as an archetype for the mathematized sciences

Physics stands out as a kind of road map, or better, as a kind of archetype, for the development of a mathematical backbone in sciences that go from mathematical economics to mathematical ecology and to the theory of social systems. Actually, most of physics can be axiomatized in a straightforward way, as there is a standard and unified formalism that stands behind every physical theory since the 18th century.

I mean the so-called analytico-canonical formalism. Its starting point is:

A theory in physics is a representation of the analytico-canonical formalism. Its dynamical laws are derived from the variational principles:

$$\delta \int_a^b L dt = 0,$$

or

$$\delta \int_{Domain} \mathcal{L} d\sigma = 0,$$

where L is a lagrangian and $\mathcal{L}$ is a lagrangian density.

This is Hamilton's Principle; the integral acted upon by the variational operator δ is the Action Integral. Hamilton's Principle has a simple interpretation; it is a kind of least effort principle. Physical systems move through the easiest path or paths. And effort is measured by L or $\mathcal{L}$, which is, let us say, a kind of "free energy" available for a system to use in its motion.

5. Axiomatics for physics: preliminary steps

What can we know about the world through a formal language? Which are the limitations imposed on our empirical, everyday knowledge when we try to describe the things around us with the help of a formalized language?

We show that strong enough formal languages exhibit the Gödel phenomenon, that is to say, they have undecidable sentences.[5] But,

[5]We stress that consistency of the underlying axiomatic apparatus must be assumed throughout this chapter.

again, can we find undecidable sentences that make meaningful assertions about the world within reasonable axiomatics for the empirical sciences?

"Meaningful" undecidable sentences

Let us now quote a specific query, the decision problem for chaotic dynamical systems (Hirsch, 1985):

> An interesting example of chaos — in several senses — is provided by the celebrated *Lorenz System*. [...] This is an extreme simplification of a system arising in hydrodynamics. By computer simulation Lorenz found that trajectories seem to wander back and forth between two particular stationary states, in a random, unpredictable way. Trajectories which start out very close together eventually diverge, with no relationship between long run behaviors.
>
> But this type of chaotic behavior has not been *proved*. As far as I am aware, practically nothing has been proved about this particular system. Guckenheimer and Williams proved that there do indeed exist many systems which exhibit this kind of dynamics, in a rigorous sense; but it has not been proved that Lorenz's system is one of them. It is of no particular importance to answer this question; but the lack of an answer is a sharp challenge to dynamicists, and considering the attention paid to this system, it is something of a scandal.
>
> The Lorenz system is an example of (unverified) *chaotic dynamics*; most trajectories do not tend to stationary or periodic orbits, and this feature is persistent under small perturbations. Such systems abound in models of hydrodynamics, mechanics and many biological systems. On the other hand experience (and some theorems) show that many interesting systems can be expected to be nonchaotic: most chemical reactions go to completion; most ecological systems do not oscillate unpredictably; the solar system behaves fairly regularly. In purely mathematical systems we expect heat equations to have convergent solutions, and similarly for a single hyperbolic conservation law, a single reaction–diffusion equation, or a gradient vectorfield.

> A major challenge to mathematicians is to determine which dynamical systems are chaotic or not. Ideally one should be able to tell from the form of the differential equations. The Lorenz system illustrates how difficult this can be.

In 1990 da Costa and Doria showed that there is no general algorithm for the solution of Hirsch's decision problem, *no matter which definition for chaos is adopted* (da Costa and Doria, 1991a,b). That result led to several undecidability and incompleteness results in dynamical systems theory; all stem from Gödel's original incompleteness theorem for arithmetic through another of the Hilbert problems, the 10th Problem. Actually such examples of incompleteness and undecidability all stem from a very general Rice-like theorem proved by da Costa and Doria (1991a). However, in order to have incompleteness in physics, we must have an axiomatic framework. How do we proceed?

6. Axiomatics for physics: guidelines

Let us now look closely at a few examples. From da Costa and Doria (2007), we have the following data.

Axiomatics for classical mechanics: preliminary data

The first efforts toward an unification of mechanics are to be found in Lagrange's *Traité de Mécanique Analytique* (1811) and in Hamilton's results.

- Hertz is the author of the first unified, mathematically well-developed presentation of classical mechanics in the late 1800s, in a nearly contemporary mathematical language. His last book, *The Principles of Mechanics*, published in 1894, advances many ideas that will later resurface not just in 20th century analytical mechanics, but also in general relativity (see Hertz, 1956).
- Half a century later, in 1949, we have two major developments in the field: first C. Lanczos publishes *The Variational Principles of Mechanics*, a brilliant mathematical essay (see Lanczos, 1977) that for the first time, presents classical mechanics from the unified viewpoint of differential geometry and Riemannian geometry.

Concepts like kinetic energy or Coriolis force are made into geometrical constructs (respectively, Riemannian metric and affine connection); several formal parallels between mechanical formalism and that of general relativity are established.
However, the style of Lanczos' essay is still that of late 19th and early 20th century mathematics, and is very much influenced by the traditional, tensor-oriented, local coordinate domain oriented, presentations of general relativity.

- Then: new and (loosely speaking) higher-order mathematical constructs appear when Steenrod's results on fiber bundles and Ehresmann's concepts of connection and connection forms on principal fiber bundles are gradually applied to mechanics; those concepts go back to the late 1930s and early 1940s, and make their way into the mathematical formulations of mechanics in the late 1950s.
- Folklore has that the use of symplectic geometry in mechanics first arose in 1960 when a top-ranking unnamed mathematician[6] circulated a letter among colleagues, which formulated Hamiltonian mechanics as a theory of flows over symplectic manifolds, that is, a Hamiltonian flow is a flow that keeps invariant the symplectic form on a given symplectic manifold. The symplectic manifold was the old phase space; invariance of the symplectic form directly led to Hamilton's equations, to Liouville's theorem on the incompressibility of the phase fluid, and to the well-known Poincaré integrals — and here the advantage of a compact formalism was made clear, as the old, computational, very cumbersome proof for the Poincaré invariants was substituted for an elegant two-line, strictly geometrical proof.

  High points in this direction are Sternberg's lectures (see Sternberg, 1964), MacLane's monograph (see MacLane, 1968) and then the great Abraham–Marsden (1978) treatise, *Foundations of Mechanics*.
- Again one had at that moment a physical theory fully placed within the domain of a rigorous (albeit intuitive) mathematical

[6]Said to be Richard Palais.

framework, as in the case of electromagnetism, gauge field theory, and general relativity. So, the path was open for an axiomatic treatment.

For electromagnetism

The first conceptually unified view of electromagnetic theory is given in Maxwell's treatise, dated 1873 (for a facsimile of the 1891 edition see Maxwell (1954)).

- Maxwell's treatment was given a more compact notation by J. Willard Gibbs with the help of vector notation.
- A sort of renewed presentation of Maxwell's main conceptual lines appears in the treatise by Sir James Jeans (1925).
- Then there is Stratton's (1941) textbook with its well-known list of difficult problems.
- And then Jackson's (1962) book, still the main textbook in the 1970s and 1980s.

When one looks at the way electromagnetic theory is presented in these books one sees that:

- The mathematical framework is calculus — the so-called advanced calculus, plus some knowledge of ordinary and partial differential equations — and linear algebra.
- Presentation of the theory's kernel becomes more and more compact; its climax is the use of covariant notation for the Maxwell equations.
- However, covariant notation only appears as a development out of the set of Maxwell equations in the traditional Gibbsian "gradient-divergence-rotational" vector notation.
- Finally, the Maxwell equations are shown to be derived from a variational principle, out of a Lagrangian density.

So, the main trend observed in the presentation of electromagnetic theory is: the field equations for electromagnetic theory are in each case presented as a small set of coordinate–independent equations with a very synthetic notation system. When we need to do actual computations, we fall back into the framework of classical, 19th-century analysis, since for particular cases (actual, real-world, situations), the field equations open up, in general, to complicated, quite cumbersome differential equations to be solved by mostly traditional techniques.

A reliable reference for the early history of electromagnetism (even if theoretically very heterodoxical) is O'Rahilly's (1965) text.

General relativity and gravitation

The field equations for gravitation we use today, that is, the Einstein field equations, are already born in a compact, coordinate-independent form (1915/1916) (see Einstein, s/d). The Einstein gravitational field equations can also be derived from a variational principle where the Lagrangian density is $\sqrt{-g}\,R$, where R is the pseudo-Riemannian curvature scalar with respect to g, the four-dimensional metric tensor with signature $+2$.

- We find in Einstein's original presentation an explicit striving for a different kind of unification, that of a *conceptual* unification of all domains of physics. An unified formalism at that moment meant that one derived all different fields from a single, unified, fundamental field. That basic field then "naturally" splits up into the several component fields, very much like, or in the search of an analogy to, the situation uncovered by Maxwell in electromagnetism, where the electric and the magnetic fields are different concrete aspects of the same underlying unified electromagnetic field.
- This trend starts with Weyl's (1968) theory in 1918 just after Einstein's introduction in 1915 of his gravitation theory, and culminates in Einstein's beautiful, elegant, but physically unsound unified theory of the nonsymmetric field (1946; see Einstein, 1967).
- On the other, hand Weyl's ideas lead to developments that appear in the treatise by Corson (1953), and which arrive at the gauge field equations, or Yang–Mills equations (1954), which were for the first time examined in depth by Utiyama (1956).

- An apparently different approach appears in the Kaluza–Klein unified theories. Originally unpromising and clumsy-looking, the blueprint for these theories goes back to Kaluza (1921) and then to Klein (1926); see Tonnelat (1965).

  In its original form, the Kaluza–Klein theory is basically the same as Einstein's gravitation theory over a five-dimensional manifold, with several artificial-looking constraints placed on the fifth dimension; that extra dimension is associated with the electromagnetic field.
- The unpleasantness of having to deal with extraneous conditions that do not arise out of the theory itself was elegantly avoided when A. Trautmann in the late 1960s and then later Cho (1975), showed that the usual family of Kaluza–Klein-like theories arises out of a simile of Einstein's theory over a principal fiber bundle on space time with a semi-simple Lie group G as the fiber.

  Einstein's Lagrangian density over the principal fiber bundle endowed with its natural metric tensor splits up as Einstein's usual gravitational Lagrangian density with the so-called cosmological term plus an interacting gauge field Lagrangian density; depending on the group G one gets electromagnetic theory, isospin theory, and so on. The cosmological constant arises in the Cho–Trautmann model out of the Lie group's structure constants, and thus gives a possible geometrical meaning to its interpretation as dark energy.
- Here, conceptual unification and formal unification go hand in hand, but, in order to do so, we must add some higher-order objects (principal fiber bundles and the associated spaces, plus connections and connection forms) to get our more compact, unified treatment of gravitation together with gauge fields, which subsume the electromagnetic field.

We are but a step away from a rigorous axiomatic treatment.

From classical to quantum mechanics

Quantum mechanics has always been snugly cradled in the classical theory, at least when considered by theoretical and mathematical physicists, far from the cloudy popular misconceptions that have surrounded the domain since its inception in the late 1920s. The Bohr–Sommerfeld quantization conditions in the first, "old," quantum theory, arise from the well-known Delaunay conditions in celestial mechanics; so much for the old quantum theory. The new, or Schrödinger–Heisenberg–Dirac quantum mechanics is nearly empirical in its inception (see van der Waerden, 1968), but when Schrödinger and Dirac appear on stage (see Dirac, 1967), we clearly see that the theory's conceptual roots and formalism arise out of classical mechanics. Schrödinger's wave equation is a kind of reinterpretation of the eikonal equation in geometrical optics, which, in turn, is a consequence of the Hamilton–Jacobi equation; the Dirac commutators and Heisenberg's motion equations are new avatars of well-known equations in the classical theory that involve Poisson brackets. We can also look at the motion equations:

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \{H, G\}$$

as the definition of a partial connection given by the Hamiltonian H on a manifold.

A surprising technical development stems from the efforts by Wightman to place quantum mechanics and the second-quantization theories on a firm mathematical ground. The starting point here was von Neumann's view in the early 1930s that quantum mechanics was a linear dynamical theory of operators on some Hilbert space. The Murray and von Neumann theory, of what we now know as von Neumann algebras (1936), later expanded to the theory of C^* algebras, allowed a group of researchers to frame several quantum-theoretic constructions in a purely algebraic way. Its realization in actual situations is given by a quantum state that induces a particular representation for the system (representation is here taken as the meaning used in group theory). This is the so-called Gelfand–Naimark–Segal construction (see Emch, 1972).

The C^* algebra approach covers many aspects of quantum field theory and is again framed within a rigorous, albeit intuitive mathematical background. It also exhibits some metamathematical phenomena, since the existence of some very general representations for C^* algebras are dependent on the full axiom of choice.

To sum it up: physics has strived for conceptual unification during the 20th century. This unification was attained in the domains we just described through a least-effort principle (Hamilton's Principle) applied to some kind of basic field, the Lagrangian or Lagrangian density, from which all known fields should be derived.

Most of physics is already placed on a firm mathematical ground, so that a strict axiomatic treatment of the main physical theories is possible. Still, there are mathematically uncertain procedures that are part of the everyday activity of the theoretical physicist, like Feynmann integration — but in this particular example, we can take Feynmann's technique as an algorithm for the generation of a series of Feynmann diagrams, that is, as a strictly symbolic computational procedure. Other theoretical physics constructs that do not have a clear mathematical formulation (e.g., Boltzmann's H–theorem) can perhaps be approached in a similar way, as when we obtain formal series expansions out of the entropy integral, while one waits for a sound mathematical formulation of it.

7. Suppes predicates

Suppes predicates give us a simple way of axiomatizing empirical theories within set theory; one simply defines a set-theoretic predicate that formally characterizes the empirical theory:

> In the first place, it may be well to say something more about the slogan "To axiomatize a theory is to define a set–theoretical predicate." It may not be entirely clear what is meant by the phrase "set–theoretical predicate." Such a predicate is simply a predicate that can be defined within set theory in a completely formal way. For a set theory based only on the primitive predicate of membership, "$\in$" in the usual notation, this means that ultimately any set-theoretical predicate can be defined solely in terms of membership.
>
> ...

> It is one of the theses of this book that there is no theoretical way of drawing a sharp distinction between a piece of pure mathematics and a piece of theoretical science. The set-theoretical definition of the theory of mechanics, the theory of thermodynamics, and a theory of learning, to give three rather disparate examples, are on all fours with the definitions of the purely mathematical theories of groups, rings, fields, etc.

(See Suppes, 1988)

Construction of Suppes predicates

References are da Costa and Chuaqui (1988) and Suppes (1967, 1988). In the present version, a Suppes predicate is a conjunction of two pieces. The first one gives the mathematical setting for the objects we are going to deal with, constructed through set-theoretic operations out of known objects in the set-theoretic universe, while the second component adds the dynamical equations that rule the process (they act as a kind of postulate for our domain). For examples see da Costa and Doria (1992a, b) and da Costa *et al.* (1990).

8. Axiomatics for physics: the main ideas

The usual formal[7] treatment for physics goes as follows: one writes down a Lagrangian or a Lagrangian density for the phenomena we are interested in, and then use the variational principle as a kind of algorithmic procedure to derive the Euler–Lagrange equations, which give us the dynamics of the system. The variational principle also allows us to obtain a conservation-law, symmetry dependent interpretation of interaction as in the case of the introduction of gauge fields out of symmetry conditions imposed on some field (see Corson, 1953; Utiyama, 1956).

[7]We will proceed in an informal way, and leave to the archetypical interested reader the toil and trouble of translating everything that we have done into a fully formal, rigorous treatment of our presentation.

We take a slightly different approach here. We describe the arena where physics happens — phase space, spacetime, fibered spaces — and add the dynamics through a Dirac-like equation.

Our results are not intended as a complete, all-encompassing, axiomatics for the whole of physics: there are many interesting areas in physics with uncertain mathematical procedures at the moment, such as statistical mechanics or quantum field theory, and the present framework may be adequate for them. But we may confidently say that our axiomatization covers the whole of classical mechanics, classical field theory, and first-quantized quantum mechanics.

We follow the usual mathematical notation here. We use Suppes predicates as our main tool. As said, a Suppes predicate is essentially a set-theoretical conjunction with two parts:

- First, conjunct describes the mathematical objects we use in our theory (spacetime, vectors, tensors, and so on).
- Then, the second conjunct gives the dynamics for the theory. It may be given as a variational principle, or (as we may present it here) as an explicit set of differential equations.

More precisely, the species of structures of essentially all main classical physical theories can be formulated as particular dynamical systems derived out of the triple $P = \langle X, G, \mu \rangle$, where X is a topological space, G is a topological group, and μ is a measure on a set of finite rank over $X \cup G$ and it is easy to put it in the form of a species of structures.

Thus we can say that the mathematical structures of physics arise out of the geometry of a topological space X. More precisely, physical objects are (roughly) the elements of X that

- exhibit invariance properties with respect to the action of G. (Actually the main species of structures in "classical" theories can be obtained out of two objects, a differentiable finite-dimensional real Hausdorff manifold M and a finite-dimensional Lie group G.)
- are "generic" with respect to the measure μ for X.
  This means, we deal with objects of probability 1. So, we only deal with "typical" objects, not the "exceptional" ones.

That condition isn't always used, we must note, but anyway measure μ allows us to identify the exceptional situations in any construction.

Let's now give all due details:

Definition 8.1. The species of structures of a *classical physical theory* is given by the 9-tuple

$$\Sigma = \langle M, G, P, \mathcal{F}, \mathcal{A}, \mathcal{I}, \mathcal{G}, B, \nabla\varphi = \iota\rangle,$$

which is thus described:

(1) *The Ground Structures.* $\langle M, G\rangle$, where M is a finite-dimensional real differentiable manifold and G is a finite-dimensional Lie group.
(2) *The Intermediate Sets.* A fixed principal fiber bundle $P(M, G)$ over M with G as its fiber plus several associated tensor and exterior bundles.
(3) *The Derived Field Spaces.* Potential space $\mathcal{A}$, field space $\mathcal{F}$ and the current or source space $\mathcal{I}$. $\mathcal{A}$, $\mathcal{F}$, and $\mathcal{I}$ are spaces (in general, manifolds) of cross-sections of the bundles that appear as intermediate sets in our construction.
(4) *Axiomatic Restrictions on the Fields.* The dynamical rule $\nabla\varphi = \iota$ and the relation $\varphi = d(\alpha)\alpha$ between a field $\varphi \in \mathcal{F}$ and its potential $\alpha \in \mathcal{A}$, together with the corresponding boundary conditions B. Here $d(\alpha)$ denotes a covariant exterior derivative with respect to the connection form α, and ∇ a covariant Dirac-like operator. As an alternative, we may give the dynamics through a variational principle. The advantage (in that case) is that motion (or whatever is described by our theory's equations) is seen to arise out of a kind of "least effort principle."[8]
(5) *The Symmetry Group.* $\mathcal{G} \subseteq \mathrm{Diff}(M) \otimes \mathcal{G}'$, where $\mathrm{Diff}(M)$ is the group of diffeomorphisms of M and $\mathcal{G}'$ the group of gauge transformations of the principal bundle P.

[8]Not always so, as a variational principle only gives us an *extremal* behavior, which can be a maximum or a minimum.

(6) *The Space of Physically Distinguishable Fields.* If $\mathcal{K}$ is one of the $\mathcal{F}$, $\mathcal{A}$, or $\mathcal{I}$ field manifolds, then the space of physically distinct fields is $\mathcal{K}/\mathcal{G}$.

(In more sophisticated analyses, we can replace our concept of theory for a more refined one. Actually, in the theory of science, we proceed as in the practice of science itself by the means of better and better approximations. However, for the goals of the present work, our concept of empirical theory is enough.)

9. Axiomatics for physics: examples

Again, we base our exposition in da Costa and Doria (2007). What we understand as the classical portion of physics up to the level of first-quantized theories easily fits into the previous scheme. We discuss in detail several examples: Maxwellian theory, Hamiltonian mechanics, general relativity, and classical gauge field theory.

Maxwell's electromagnetic theory

Let $M = \mathsf{R}^4$ with some differential structure, exotic or standard. Anyway, physics may be seen as a *local* phenomenon, and so the global properties of the underlying spacetime may be unimportant.[9] Thus consider the standard case. Let us endow M with the Cartesian coordination induced from its product structure, and let $\eta = \mathrm{diag}(-1, +1, +1, +1)$ be the symmetric constant metric Minkowskian tensor on M.

Then M is Minkowski spacetime, the physical arena where we do special relativity theory. As it is well known, out of the linear transformations that keep invariant tensor η, we obtain the well-known relativistic contraction and dilation phenomena.

We use standard physics notation. If $F_{\mu\nu}(x)$ are the components of the electromagnetic field, that is, a differentiable covariant 2-tensor

[9]Not always, as the exotic underlying structure of spacetime may be seen as generated by some energy–momentum tensor.

field on M, $\mu, \nu = 0, 1, 2, 3$, then Maxwell's equations are:

$$\partial_\mu F^{\mu\nu} = j^\nu,$$
$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0.$$

The contravariant vectorfield whose components are given by the set of four smooth functions $j^\mu(x)$ on M is the current that serves as source for Maxwell's field $F_{\mu\nu}$. (We allow piecewise differentiable functions to account for shock-wave-like solutions.)

It is known that Maxwell's equations are equivalent to the Dirac-like set

$$\nabla\varphi = \iota,$$

where

$$\varphi = (1/2)F_{\mu\nu}\gamma^{\mu\nu},$$

and

$$\iota = j_\mu\gamma^\mu,$$

$$\nabla = \gamma^\rho\partial_\rho,$$

(where the $\{\gamma^\mu : \mu = 0, 1, 2, 3\}$ are the Dirac gamma matrices with respect to η, that is, they satisfy the anticommutation rules $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$). Those equation systems are to be understood together with boundary conditions that specify a particular field tensor $F_{\mu\nu}$ "out of" the source j^ν (see Doria, 1977). Here $\gamma^{\mu\nu} = (1/2)[\gamma^\mu, \gamma^\nu]$, where brackets denote the commutator.

The symmetry group of the Maxwell field equations is the Lorentz–Poincaré group that acts on Minkowski space M and, in an induced way on objects defined over M. However since we are interested in *complex* solutions for the Maxwell system, we must find a reasonable way of introducing complex objects in our formulation. One may formalize the Maxwellian system as a gauge field. We sketch the usual formulation: again we start from $M = \langle \mathsf{R}^4, \eta\rangle$, and construct the trivial circle bundle $P = M \times S^1$ over M, since Maxwell's field is the gauge field of the circle group S^1 (usually written in that respect as $U(1)$). We form the set $\mathcal{E}$ of bundles associated with P whose fibers are finite-dimensional vectorspaces. The set of physical fields in our theory is obtained out of some of the bundles in $\mathcal{E}$: the set of electromagnetic field tensors is a set of cross-sections of the bundle $F = \Lambda^2 \otimes s^1(M)$ of all s^1-valued 2-forms on M, where s^1 is the group's Lie algebra. To be more precise, the set of all electromagnetic fields is $\mathcal{F} \subset C^k(F)$, if we are dealing with C^k cross-sections (actually a submanifold in the usual C^k topology due to the closure condition $dF = 0$).

Finally we have two group actions on $\mathcal{F}$: the first one is the Lorentz–Poincaré action L which is part of the action of diffeomorphisms of M; then we have the (here trivial) action of the group $\mathcal{G}'$ of gauge transformations of P when acting on the field manifold $\mathcal{F}$. As it is well known, its action is *not* trivial in the non-Abelian case. Anyway, it always has a nontrivial action on the space $\mathcal{A}$ of all gauge potentials for the fields in $\mathcal{F}$. Therefore, we take as our symmetry group $\mathcal{G}$ the product $L \otimes \mathcal{G}'$ of the (allowed) symmetries of M and the symmetries of the principal bundle P.

We must also add the spaces $\mathcal{A}$ of potentials and of currents, $\mathcal{I}$, as structures derived from M and S^1. Both spaces have the same underlying topological structure; they differ in the way the group $\mathcal{G}'$ of gauge transformations acts upon them. We obtain $I = \Lambda^1 \otimes s^1(M)$ and $\mathcal{A} = \mathcal{I} = C^k(I)$. Notice that $\mathcal{I}/\mathcal{G}' = \mathcal{I}$ while $\mathcal{A}/\mathcal{G}' \neq \mathcal{A}$.

Therefore we can say that the 9-tuple

$$\langle M, S^1, P, \mathcal{F}, \mathcal{A}, \mathcal{G}, \mathcal{I}, B, \nabla\varphi = \iota \rangle$$

where M is Minkowski space, and B is a set of boundary conditions for our field equations $\nabla\varphi = \iota$, represents the species of mathematical structures of a Maxwellian electromagnetic field, where P, $\mathcal{F}$, and $\mathcal{G}$ are derived from M and S^1. The Dirac-like equation

$$\nabla\varphi = \iota$$

should be seen as an axiomatic restriction on our objects; the boundary conditions B are (i) a set of derived species of structures from M and S^1, since, as we are dealing with Cauchy conditions, we must specify a local or global spacelike hypersurface C in M to which (ii) we add sentences of the form $\forall x \in C\, f(x) = f_0(x)$, where f_0 is a set of (fixed) functions and the f are adequate restrictions of the field functions and equations to C.

Consistency of the added axioms

Loosely speaking, it suffices to get a specific example of an electromagnetic field and see that it satisfies the preceding formal constructions. That applies to the next examples too.

Hamiltonian mechanics

Hamiltonian mechanics is here seen as the dynamics of the "Hamiltonian fluid" (see Abraham and Marsden, 1978; Lanczos, 1977). Our ground structure for mechanics starts out of basic sets, which are a $2n$-dimensional real smooth manifold, and the real symplectic group $\mathrm{Sp}(2n, \mathsf{R})$. Phase spaces in Hamiltonian mechanics are symplectic manifolds: even-dimensional manifolds like M endowed with a symplectic form, that is, a nondegenerate closed 2-form Ω on M. The imposition of that form can be seen as the choice of a reduction of the linear bundle $L(M)$ to a fixed principal bundle $P(M, \mathrm{Sp}(2n, \mathsf{R}))$; however, given one such reduction, it does not automatically follow that the induced 2-form on M is a closed form.

All other objects are constructed in about the same way as in the preceding example. However, we must show that we still have here a Dirac-like equation as the dynamical axiom for the species of structures of mechanics. Hamilton's equations are

$$i_X \Omega = -dh,$$

where i_X denotes the interior product with respect to the vector-field X over M, and h is the Hamiltonian function. That equation is (locally, at least) equivalent to

$$L_X \Omega = 0,$$

or

$$d(i_X \Omega) = 0,$$

where L_X is the Lie derivative with respect to X. The condition $d\varphi = 0$, with $\varphi = i_X\Omega$, is the degenerate Dirac-like equation for Hamiltonian mechanics. We don't get a full Dirac-like operator $\nabla \neq d$ because M, seen as a symplectic manifold, doesn't have a canonical metrical structure, so that we cannot define (through the Hodge dual) a canonical divergence δ dual to d. The group that acts on M with its symplectic form is the group of canonical transformations; it is a subgroup of the group of diffeomorphisms of M so that symplectic forms are mapped onto symplectic forms under a canonical transformation. We can take as "potential space" the space of all Hamiltonians on M (which is a rather simple function space), and as "field space" the space of all "Hamiltonian fields" of the form $i_X\Omega$.

Of course, we can directly use a variational principle here for the dynamics, and start the axiomatics out of the Lagrangian formulation.[10]

Interpretations are immediate: h is the system's Hamiltonian, which (given some simple conditions) can be seen as the system's total energy. Invariance of the symplectic form by the Lie derivative with respect to a Hamiltonian flow is equivalent both to Poincaré's integral invariant theorem and to Liouville's theorem — just as a flavor of the way our treatment handles well-known concepts and results in mechanics.

General relativity

General relativity is a theory of gravitation that interprets this basic force as originating in the pseudo-Riemannian structure of spacetime. That is to say, in general relativity, we start from a spacetime manifold (a four-dimensional, real, adequately smooth manifold),[11] which is endowed with an pseudo-Riemannian metric tensor. Gravitational effects originate in that tensor.

[10] There are several delicate points here, as the Lagrangian we start from should be a hyperregular Lagrangian; see Abraham and Marsden (1978).
[11] Exotic manifolds are allowed here.

Given any four-dimensional, noncompact, real, differentiable manifold M, we can endow it with an infinite set of different, nonequivalent pseudo-Riemannian metric tensors with a Lorentzian signature (that is, $- + ++$). That set is uncountable and has the power of the continuum. (By nonequivalent metric tensors, we mean the following: form the set of all such metric tensors and factor it by the group of diffeomorphisms of M; we get a set that has the cardinality of the continuum. Each element of the quotient set is a different gravitational field for M.)

Therefore, neither the underlying structure of M as a topological manifold, nor its differentiable structure determines a particular pseudo-Riemannian metric tensor, that is, a specific gravitational field. From the strictly geometrical viewpoint, when we choose a particular metric tensor g of Lorentzian signature, we determine a g-dependent reduction of the general linear tensor bundle over M to one of its pseudo-orthogonal bundles. The relation

$$g \mapsto$$

g-dependent reduction of the linear bundle to a pseudo-orthogonal bundle is one-to-one.

We now follow our recipe:

- We take as basic sets a four-dimensional real differentiable manifold of class C^k, $1 \leq k \leq +\infty$, and the Lorentz pseudo-orthogonal group $O(3,1)$.
- We form the principal linear bundle $L(M)$ over M; that structure is solely derived from M, as it arises from the covariance properties of the tangent bundle over M. From $L(M)$, we fix a reduction of the bundle group $L(M) \to P(M, O(3,1))$, where $P(M, O(3,1))$ is the principal fiber bundle over M with the $O(3,1)$ group as its fiber.

  Those will be our derived sets. We, therefore, inductively define a Lorentzian metric tensor g on M, and get the couple $\langle M, g\rangle$, which is spacetime.

  (Notice that the general relativity spacetime arises quite naturally out of the interplay between the theory's "general covariance" aspects, which appear in $L(M)$, and — as we will see in the next section — its "gauge-theoretic features", which are clear in $P(M, O(3,1))$.)

- Field spaces are:
  — The first is the set (actually a manifold, with a natural differentiable structure) of all pseudo-Riemannian metric tensors,
  $$\mathcal{M} \subset C^k(\odot^2 T_*(M)),$$
  where $C^k(\odot^2 T_*(M))$ is the bundle of all C^k symmetric covariant 2-tensors over M.
  — Also out of M and out of adequate associated bundles we get $\mathcal{A}$, the bundle of all Christoffel connections over M, and $\mathcal{F}$, the bundle of all Riemann–Christoffel curvature tensors over M.
- We need the space of source fields, $\mathcal{I}$, that includes energy–momentum tensors, and arise out of adequate associated tensor bundles over M.
- $\mathcal{G}$ is the group of C^k-diffeomorphisms of M.
- If $\mathcal{K}$ is any of the field spaces above, then $\mathcal{K}/\mathcal{G}$ is the space of physically distinct fields.
- Finally, the dynamics are given by Einstein's equations (there is also a Dirac-like formulation for those, first proposed by R. Penrose in 1960 as a neutrino-like equation; see Doria (1975)).

The quotient $\mathcal{K}/\mathcal{G}$ is the way we distinguish concrete, physically diverse, fields, as for covariant theories, one has that any two fields related by an element of $\mathcal{G}$ "are" the "same" field.

Classical gauge fields

The mathematics of classical gauge fields can be found in Utiyama (1956). We follow here the preceding examples, and, in particular, the treatment of general relativity:

- The basic sets are a spacetime $\langle M, g\rangle$, and a finite dimensional, semi-simple, compact Lie group G.
- The derived set is a fixed principal bundle $P(M, G)$ over M with G as the fiber.

- The group of gauge transformations $\mathcal{G}$ is the subgroup of all diffeomorphisms of $P(M, G)$ that reduce to a diffeomorphism on M and to the group action on the fiber.
- If $\ell(G)$ is the Lie algebra of G, we get:
  - — Connection-form space, or the space of potentials, noted $\mathcal{A}$, is the space of all C^k-cross-sections of the bundle of $\ell(G)$-valued 1-forms on M.
  - — Curvature space, or the space of fields $\mathcal{F}$, is the space of all C^k-cross-sections of $\ell(G)$-valued 2-forms on M, such that $F \in \mathcal{F}$ is the field with potential $A \in \mathcal{A}$.
  - — Source space $\mathcal{I}$ coincides with $\mathcal{A}$, but is acted upon in a different way by the group $\mathcal{G}$ of gauge transformations. (Currents in $\mathcal{I}$ are *tensorial* 1-forms, while gauge-potentials in $\mathcal{A}$ are transformed via an inhomogeneous transformation.)
- The space of physically different fields is $\mathcal{K}/\mathcal{G}$, where $\mathcal{K}$ is any of the above field spaces.
- Dynamics are given by the usual gauge-field equations, which are a nonlinear version of the electromagnetic field equations. There is also a Dirac-like equation for gauge fields (see Doria *et al.*, 1986). Or, again, we can start from a variational principle.

To sum it up with the help of the schema presented at the beginning of the section, we can say that the structure of a physical theory is an ordered pair $\langle \mathcal{F}, \mathcal{G} \rangle$, where $\mathcal{F}$ is an infinite-dimensional space of fields, and $\mathcal{G}$ is an infinite-dimensional group that acts upon field space. To get the Suppes predicate, we must add the information about the dynamical equations $D(\phi) = 0, \phi \in \mathcal{F}$, for the fields ϕ.

Notice that general relativity can be seen as a kind of degenerate gauge field theory, more precisely a gauge theory of the $O(3, 1)$ group.

Quantum theory of the electron

The Dirac electron theory (and the general theory of particles with any spin) can be easily formalized according to the preceding schemata. One uses as geometrical background the setting for special relativity; dynamics is given either by Dirac's equation or Weyl's equation, for the case of zero-mass particles. Higher spin fields are dealt with the help either of the Bargmann–Wigner equations or their algebraic counterpart (see Doria, 1977). The Schrödinger equation is obtained from the Dirac set out of a — loosely speaking — "standard" limiting procedure, which can be formally represented by the addition of new axioms to the corresponding Suppes predicate.

General field theory

Sometimes one may wish to discuss field theory in a very general, motion-equation independent, way. We then use as geometrical background the construction of Minkowski space and take as dynamical axioms the field-theoretic Euler–Lagrange equations, or, as we've said, we can take the variational principle as a formal algorithm to derive the dynamics of the system.

Summing it up

We will briefly mention a few results of our own (with da Costa) on the axiomatics of physics.

Proposition 9.1. *Classical mechanics, Schrödinger's quantum mechanics, electromagnetism, general relativity and gauge field theory can all be axiomatized within Zermelo–Fraenkel set theory.*

Therefore, all known results in theoretical physics become ZF theorems, as long as they can be given rigorous formulations, let us say, by the usual mathematical standards and without the help of "very large" objects (that last condition is given within quotation marks to mean that we exclude large cardinal extensions of ZF).

Proposition 9.2. *There are "physically meaningful" undecidable sentences in any consistent* ZF *axiomatization of physics.*

By "physically meaningful", we mean sentences that describe actual situations in physics, e.g., for a particular Lagrangian L, the formalized version of the sentence "L describes a harmonic oscillator" is formally undecidable within ZF, or, as in our first example, "X is a chaotic vectorfield." We can say that any two ZF axiomatizations of physics as sketched have the same undecidable sentences.

This means that, for a wide variety of constructions, choosing a particular axiomatics for physics is just a matter of taste. It won't affect the results we can derive (or that we can't derive) from our axiomatic framework. For more details, see da Costa and Doria (2007).

10. Beyond physics

We can extend the preceding techniques to several scientific domains. For example, the bulk of economics, as presented, say, in Samuelson's *Foundations of Economic Analysis* (Samuelson, 1967), or some specific results, such as the Nash equilibrium theorem (da Costa and Doria, 2005), easily fit within our construction — we can find in a straightforward way a Suppes predicate for results in mathematical economics (da Costa and Doria, 1991b). The same goes with mathematical biology (Lotka, 1956).

We have proceeded from start with a specific goal in mind: we wished to follow Hilbert's program in his 6th Problem, that is, we proposed an axiomatization of physics that allows us to explore many interesting mathematical consequences of those theories.

We now wish to obtain specific examples of Gödel sentences — undecidable sentences — within the axiomatic versions of those theories, and in a more general cadre, we wish to see the effects and consequences of metamathematical results and techniques when applied to those theories, or to their axiomatic versions.

11. The incompleteness of analysis

The first explicit constructions of actual unsolvable problems in analysis were only made in the 1960s by Scarpellini (in 1963) and, a few years later, by Adler and Richardson (see da Costa and Doria, 2007). Richardson's results are by far the most interesting, since they amount to the construction of a functor from the theory of formal systems into classical elementary analysis. His results were frequently quoted after they were published; however, their applications were until recently restricted to computer science, to set computational bounds for techniques in algebraic computation.

From here onward, we suppose that our formal constructions are made within a first-order axiomatic theory T, which can be thought to include Zermelo–Fraenkel (ZF) set theory plus at least some portion of the Axiom of Choice.

In our version, the Richardson functor starts out of a polynomial p in q indeterminates over the integers Z. Richardson tells us how to explicitly and algorithmically construct two expressions out of that p:

- First, an expression for a q-variable function $f(p)$, which includes polynomial terms, sine functions, and the number π. $f(p)$ satisfies the following conditions:
  1. The Diophantine equation $p = 0$ has no solutions over the integers if and only if, for all values of its variables, $f(p) > 1$.
  2. $p = 0$ has a solution over the integers if and only if $f(p)$ dips beyond 1 and has zero values.
- Second, an expression for a 1-variable function, which can be explicit constructed with elementary functions and which has a similar behavior according to the existence of roots in $p = 0$.

So, the idea in Richardson's transform is that there is a kind of strip of finite width, which is never crossed by the values of $f(p)$ if the Diophantine equation $p = 0$ has no solutions. If it does have solutions, that strip will be crossed at points depending on the Diophantine roots. If we now add an expression for the absolute value function $|\ldots|$ to our language, we can obtain a new function $c(f)$ with the following behavior:

- $c(f) = 0$ if and only if $p = 0$ has no solutions as a Diophantine equation.
- $c(f) > 0$ somewhere if and only if $p = 0$ does have integer solutions.

Richardson's interest seems to have been restricted to the construction of a few unsolvable problems in analysis. However, we realized that his two maps amounted to a true-blood functor from axiomatizable systems into classical analysis. In the case of our axiomatic system T, we can represent its proofs by a Turing machine M_T that halts whenever one of its theorems ξ (or a decidable sentence) is input; if we input an undecidable sentence ζ, $M_T(\zeta)$ will never halt over it.

If M_T halts over ξ, then we can explicitly obtain a Diophantine equation $p(m_\xi, x_1, \ldots) = 0$, which has solutions ($m_\xi$ is a Gödel number for ξ); if M_T doesn't halt over ζ, then $p(m_\zeta, \ldots) = 0$ has no integer solutions. Therefore, with the help of Richardson's maps, we can code within the language of classical analysis the whole deductive machinery of an axiomatizable formal system.

The whole thing turns out to be wide-ranging. Out of a suggestion by Suppes we proved a general undecidability and incompleteness theorem — a Rice-like theorem — within classical analysis (see Proposition 15.1). Let P be an arbitrary nontrivial[12] property in the language of analysis. The blueprint for our undecidable sentences out of those constructions is, informally:

Proposition 11.1. *There is a term-expression ζ in the language of analysis such that neither $T \vdash P\zeta$ nor $T \vdash \neg P\zeta$.*

There is a corresponding undecidability result, and the associated decision problems can be made as high as one wishes in the arithmetical hierarchy, and even beyond (da Costa and Doria, 1994a). See below.

12. Generalized incompleteness

This is a technical section. Notation is standard. We follow here da Costa and Doria (2007) and some previous papers where these questions have been introduced and discussed such as da Costa and Doria (1991a, 2005). We deal (among other objects) with algorithmic functions here. These are given by their programs coded in Gödel numbers e (see Rogers, 1967). We will sometimes use Turing machines (noted

[12] A property that isn't satisfied by either all or none of the objects in its domain.

by sans-serif letters with the Gödel number as index M_e) or partial recursive functions, noted $\{e\}$. Peano Arithmetic is noted PA. We require Russell's ι symbol: $\iota_x P(x)$ is, roughly, the x such that $P(x)$.

The *standard interpretation* for PA is: the variables $x, y, \ldots$ range over the natural numbers, and **0** and **1** are seen as, respectively, zero and one. PA is strong enough to formally include Turing machine theory (see da Costa and Doria, 2005). Rigorously, for PA, we have:

Definition 12.1. A Turing machine of Gödel number e operating on x with output y, $\{e\}(x) = y$ is *representable* in PA if there is a formula $F_e(x, y)$ in the language of our arithmetic theory so that:

(1) $\mathrm{PA} \vdash F_e(x, y) \wedge F_e(x, z) \rightarrow y = z$, and
(2) For natural numbers a, b, $\{e\}(a) = b$ if and only if $\mathrm{PA} \vdash F_e(a, b)$.

Proposition 12.2. *Every Turing machine is representable in Peano Arithmetic. Moreover there is an effective procedure that allows us to obtain F_e from the Gödel number e.*

A theory is *arithmetically sound* if it has a model with standard arithmetic for its arithmetical segment.

A simple example of generalized incompleteness

The argument below is valid for all theories that contain enough arithmetic, have a model where arithmetic is standard, and have a recursively enumerable set of theorems.

Suppose that our theory S has Russell's description symbol ι. Let P be a predicate symbol so that for closed terms ξ, ζ such that $S \vdash \xi \neq \zeta$, $S \vdash P(\xi)$ and $S \vdash \neg P(\zeta)$ (we call such P, nontrivial predicates). Then, for the term:

$$\eta = \iota_x[(x = \xi \wedge \alpha) \vee (x = \zeta \wedge \neg\alpha)],$$

where α is an undecidable sentence in S:

Proposition 12.3. $S \nvdash P(\eta)$ *and* $S \nvdash \neg P(\eta)$.

This settles our claim. From now on, we will consider theories S, T, like the one characterized above.

Our main tool here will be an explicit expression for the Halting Function, that is, the function that settles the halting problem (see Rogers, 1967). We have shown elsewhere that it can be constructed within the language of classical analysis.

Proposition 12.4. *If $\{e\}(a) = b$, for natural numbers a, b, then we can algorithmically construct a polynomial p_e over the natural numbers so that $\{e\}(a) = b \leftrightarrow \exists x_1, x_2, \ldots, x_k \in \omega\, p_e(a, b, x_1, x_2, \ldots, x_k) = 0$.*

Proposition 12.5. *$a \in R_e$, where R_e is a recursively enumerable set, if and only if there are e and p so that $\exists x_1, x_2, \ldots, x_k \in \omega\, (p_e(a, x_1, x_2, \ldots, x_k) = 0)$.*

The Halting Function

Remark 12.6. Let $\mathsf{M}_m(a) \downarrow$ mean: "Turing machine of Gödel number m stops over input a and gives some output." Similarly $\mathsf{M}_m(a) \uparrow$ means, "Turing machine of Gödel number m enters an infinite loop over input a." Then we can define the halting function θ:

- $\theta(m, a) = 1$ if and only if $\mathsf{M}_m(a) \downarrow$.
- $\theta(m, a) = 0$ if and only if $\mathsf{M}_m(a) \uparrow$.

$\theta(m, a)$ is the halting function for M_m over input a.

θ isn't algorithmic, of course (see Rogers, 1967), that is, there is no Turing machine that computes it.

Then, if σ is the sign function, $\sigma(\pm x) = \pm 1$ and $\sigma(0) = 0$:

Expressions for the Halting Function

Proposition 12.7 (The Halting Function). *The halting function $\theta(n, q)$ is explicitly given by:*

$$\theta(n, q) = \sigma(G_{n,q}),$$

$$G_{n,q} = \int_{-\infty}^{+\infty} C_{n,q}(x) e^{-x^2} dx,$$
$$C_{m,q}(x) = |F_{m,q}(x) - 1| - (F_{m,q}(x) - 1).$$
$$F_{n,q}(x) = \kappa_P p_{n,q}.$$

Here $p_{n,q}$ is the two-parameter universal Diophantine polynomial and κ_P an adequate Richardson transform.

Undecidability and incompleteness

Lemma 12.8. *There is a Diophantine set D so that*

$$m \in D \leftrightarrow \exists x_1, \ldots, x_n \in \omega\, p(m, x_1, \ldots, x_n) = 0,$$

p a Diophantine polynomial, and D is recursively enumerable but not recursive.

Corollary 12.9. *For an arbitrary $m \in \omega$ there is no general decision procedure to check whether $p(m, x_1, \ldots) = 0$ has a solution in the positive integers.*

Main undecidability and incompleteness result

Therefore, given such a p, and $F = \kappa_P(p)$, where κ_P is an adequate Richardson transform (see da Costa and Doria, 1991), we have the following corollary.

Corollary 12.10. *For an arbitrary $m \in \omega$, there is no general decision procedure to check whether, for F and G adequate real-defined and real-valued functions*:

1. *There are real numbers $x_1, \ldots, x_n$ such that $F(m, x_1, \ldots, x_n) = 0$.*
2. *There is a real number x so that $G(m, x) < 1$.*
3. *Whether we have $\forall x \in \mathsf{R}\ \theta(m, x) = 0$ or $\forall x \in \mathsf{R}\ \theta(m, x) = 1$ over the reals.*
4. *Whether for an arbitrary $f(m, x)$ we have $f(m, x) \equiv \theta(m, x)$.*

Let $\mathcal{B}$ be a sufficiently large algebra of functions and let $P(x)$ be a nontrivial predicate. If ξ is any word in that language, we write $\|\xi\|$ for its complexity, as measured by the number of letters from ZFC's alphabet in ξ. We define the *complexity of a proof* $C_{\mathrm{ZFC}}(\xi)$ of ξ in the language of ZFC to be the minimum length that a deduction of ξ from the ZFC axioms can have, as measured by the total number of letters in the expressions that belong to the proof.

Proposition 12.11. *If* ZFC *is arithmetically sound, then:*

(1) *There is an $h \in \mathcal{B}$ so that neither* $\mathrm{ZFC} \nvdash \neg P(h)$ *nor* $\mathrm{ZFC} \nvdash P(h)$*, but $\mathbf{N} \models P(h)$, where $\mathbf{N}$ makes* ZFC *arithmetically sound.*
(2) *There is a denumerable set of functions $h_m(x) \in \mathcal{B}$, $m \in \omega$, such that there is no general decision procedure to ascertain, for an arbitrary m, whether $P(h_m)$ or $\neg P(h_m)$ is provable in* ZFC*.*
(3) *Given the set $K = \{m : \mathrm{ZFC} \vdash \phi(\widehat{m})\}$, and given an arbitrary total recursive function $g : \omega \to \omega$, there is an infinite number of values for m so that $C_{\mathrm{ZFC}}(P(\widehat{m})) > g(\|P(\widehat{m})\|)$.*

Proof. Let θ be as above. Let f_0, g_0 satisfy our conditions on P, that is, $\mathrm{ZFC} \vdash P(f_0)$ and $\mathrm{ZFC} \vdash \neg P(g_0)$. Then define:

$$h(m, x) = \theta(m, x) f_0 + (1 - \theta(m, x)) g_0.$$

This settles (2). Now let us specify θ so that the corresponding Diophantine equation $p = 0$ is never solvable in the standard model for arithmetic, while that fact cannot be proved in ZFC. We then form, for such an indicator function,

$$h = \theta f_0 + (1 - \theta) g_0.$$

This settles (1). Finally, for (3), we notice that as K is recursively enumerable but not recursive, it satisfies the conditions in the Gödel–Ehrenfeucht–Mycielski theorem about the length of proofs. $\square$

13. Higher degrees

Here we will directly quote from (da Costa and Doria, 2007), and give full details. Our main result in this section is:

Proposition 13.1. *If T is arithmetically sound, then we can explicitly and algorithmically construct in the language $\mathcal{L}_T$ of T an expression for the characteristic function of a subset of ω of degree $\mathbf{0}''$.*

Remark 13.2. We can obtain an expression in a recursive way, but such an expression isn't computable — every effort to compute it falls into an infinite loop.

That expression depends on recursive functions defined on ω and on elementary real-defined and real-valued functions plus the absolute value function, a quotient and an integration, or perhaps an infinite sum, as in the case of the β and θ functions associated to the halting problem.

Proof. We could simply use Theorem 9-II in Rogers (1967, p. 132). However, for the sake of clarity, we give a detailed albeit informal proof. Actually, the degree of the set described by the characteristic function whose expression we are going to obtain will depend on the fixed oracle set A; so, our construction is a more general one. $\square$

Let us now review a few concepts. Let $A \subset \omega$ be a fixed infinite subset of the integers.

Definition 13.3. The *jump of* A is noted A'; $A' = \{x : \phi_x^A(x) \downarrow\}$, where ϕ_x^A is the A-partial recursive algorithm of index x.

In order to make things self-contained, we review here some ideas about A-partial recursive functions.

From Turing machines to oracle Turing machines

(1) An oracle Turing machine ϕ_x^A with oracle A can be visualized as a two-tape machine where tape 1 is the usual computation tape, while tape 2 contains a listing of A. When the machine enters the oracle state s_0, it searches tape 2 for an answer to a question of the form "does $w \in A$?" Only finitely many such questions are asked during a converging computation; we can separate the positive and negative answers into two disjoint finite sets $D_u(A)$ and $D_v^*(A)$ with (respectively) the positive and negative answers for those questions; notice that $D_u \subset A$, while $D_v^* \subset \omega - A$. We

can view those sets as ordered k- and k^*-ples; u and v are recursive codings for them (see Rogers, 1967). The $D_u(A)$ and $D^*_v(A)$ sets can be coded as follows: only finitely many elements of A are queried during an actual converging computation with input y; if k' is the highest integer queried during one such computation, and if $d_A \subset c_A$ is an initial segment of the characteristic function c_A, we take as a standby for D and D^* the initial segment d_A where the length $l(d_A) = k' + 1$.

We can effectively list all oracle machines with respect to a fixed A, so that, given a particular machine, we can compute its index (or Gödel number) x, and given x we can recover the corresponding machine.

(2) Given an A-partial recursive function ϕ^A_x, we form the oracle Turing machine that computes it. We then do the computation $\phi^A_x(y) = z$ that outputs z. The initial segment $d_{y,A}$ is obtained during the computation.

(3) The oracle machine is equivalent to an ordinary two-tape Turing machine that takes as input $\langle y, d_{y,A}\rangle$; y is written on tape 1 while $d_{y,A}$ is written on tape 2. When this new machine enters state s_0 it proceeds as the oracle machine. (For an ordinary computation, no converging computation enters s_0, and $d_{y,A}$ is empty.)

(4) The two-tape Turing machine can be made equivalent to a one-tape machine, where some adequate coding places on the single tape all the information about $\langle y, d_{y,A}\rangle$. When this third machine enters s_0 it scans $d_{y,A}$.

(5) We can finally use the standard map τ that codes n-ples one-to-one onto ω and add to the preceding machine a Turing machine that decodes the single natural number $\tau(\langle y, d_{y,A}\rangle)$ into its components before proceeding to the computation.

Let w be the index for that last machine; we note the machine ϕ_w.

If x is the index for ϕ^A_x, we write $w = \rho(x)$, where ρ is the effective one-to-one procedure above described that maps indices for oracle machines into indices for Turing machines. Therefore,

$$\phi^A_x(y) = \phi_{\rho(x)}(\langle y, d_{y,A}\rangle).$$

Now let us now write down an universal polynomial $p(n, q, x_1, \ldots, x_n)$. We can define the jump of A as follows:

$$A' = \{\rho(z) : \exists x_1, \ldots, x_n \in \omega\, p(\rho(z), \langle z, d_{z,A}\rangle, x_1, \ldots, x_n) = 0\}.$$

With the help of the Richardson map described above, we can now form a function modeled after the θ function that settles the Halting Problem; it is the desired characteristic function:

$$c_{\emptyset'}(x) = \theta(\rho(x), \langle x, d_{x,\emptyset'}\rangle).$$

(Actually we have proved more; we have obtained

$$c_{A'}(x) = \theta(\rho(x), \langle x, d_{x,A}\rangle),$$

with reference to an arbitrary $A \subset \omega$.)

Finally, we write $\theta^{(2)}(x) = c_{\emptyset''}(x)$. □

We recall the following definition (see Rogers, 1967).

Definition 13.4. The complete Turing degrees $\mathbf{0}, \mathbf{0}', \mathbf{0}'', \ldots, \mathbf{0}^{(p)}$, $\ldots$, $p < \omega$, are Turing equivalence classes generated by the sets $\emptyset, \emptyset', \emptyset'', \ldots, \emptyset^{(p)}, \ldots$.

Now let $\mathbf{0}^{(n)}$ be the nth complete Turing degree in the arithmetical hierarchy. Let $\tau(n, q) = m$ be the pairing function in recursive function theory (see Rogers, 1967). For $\theta(m) = \theta(\tau(n, q))$, we have the following corollary.

Corollary 13.5 (Complete Degrees). *If T is arithmetically sound, for all $p \in \omega$ the expressions $\theta^p(m)$ explicitly constructed below represent characteristic functions in the complete degrees $\mathbf{0}^{(p)}$.*

Proof. From Proposition 13.1,

$$\begin{cases} \theta^{(0)} = c_{\emptyset}(m) = 0, \\ \theta^{(1)}(m) = c_{\emptyset'}(m) = \theta(m), \\ \theta^{(n)}(m) = c_{\emptyset^{(n)}}(m), \end{cases}$$

for c_A as in Proposition 13.1. □

Incompleteness theorems

We suppose, as already stated, that PA $\subset T$ means that there is an interpretation of PA in T.

The next results will be needed when we consider our main examples. We recall that "$\dot{-}$" (the truncated sum) is a primitive recursive operation on ω:

- For $a > b$, $a \dot{-} b = a - b$.
- For $a < b$, $a \dot{-} b = 0$.

In the next result, Z is the set of integers. Let $\mathbf{N}$ be a model, $\mathbf{N} \models T$, and $\mathbf{N}$ makes T arithmetically sound.

Proposition 13.6. *If T is arithmetically sound, then we can algorithmically construct a polynomial expression*

$$q(x_1, \ldots, x_n)$$

over Z such that

$$\mathbf{N} \models \forall x_1, \ldots, x_n \in \omega\, q(x_1, \ldots, x_n) > 0\},$$

but

$$T \nvdash \forall x_1, \ldots, x_n \in \omega\, q(x_1, \ldots, x_n) > 0$$

and

$$T \nvdash \exists x_1, \ldots, x_n \in \omega\, q(x_1, \ldots, x_n) = 0.$$

Proof. Let $\xi \in \mathcal{L}_T$ be an undecidable sentence obtained for T with the help of Gödel's diagonalization; let n_ξ be its Gödel number and let m_T be the Gödel coding of proof techniques in T (of the Turing machine that enumerates all the theorems of T). For an universal polynomial $p(m, q, x_1, \ldots, x_n)$ we have:

$$q(x_1, \ldots, x_n) = (p(m_T, n_\xi, x_1, \ldots, x_n))^2. \qquad \square$$

Corollary 13.7. *If* PA *is consistent then we can find within it a polynomial p as in Proposition* 13.6.

Now a weaker version of Proposition 13.6 is as follows.

Proposition 13.8. *If T is arithmetically sound, there is a polynomial expression over* Z $p(x_1, \ldots, x_n)$ *such that* $\mathbf{N} \models \forall x_1, \ldots, x_n \in \omega\, p(x_1, \ldots, x_n) > 0$, *while*

$$T \not\vdash \forall x_1, \ldots, x_n \in \omega\, p(x_1, \ldots, x_n) > 0$$

and

$$T \not\vdash \exists x_1, \ldots, x_n \in \omega\, p(x_1, \ldots, x_n) = 0.$$

Proof. If $p(m, x_1, \ldots, x_n)$, $m = \tau\langle q, r\rangle$, is an universal polynomial with τ being Cantor's pairing function (see Rogers, 1967), then $\{m : \exists x_1 \ldots \in \omega\, p(m, x_1, \ldots) = 0\}$ is recursively enumerable but not recursive. Therefore there must be an m_0 such that $\forall x_1 \ldots \in \omega\, (p(m_0, x_1, \ldots))^2 > 0$. □

Proposition 13.9. *If* PA *is consistent and* $\mathbf{N} \models$ PA *is standard, and if P is nontrivial then there is a term-expression $\zeta \in \mathcal{L}_{\mathrm{PA}}$ such that* $\mathbf{N} \models P(\zeta)$ *while* PA $\not\vdash P(\zeta)$ *and* PA $\not\vdash \neg P(\zeta)$.

Proof. Put $\zeta = \xi + r(x_1, \ldots, x_n)\nu$, for $r = 1 \dot{-} (q + 1)$, q as in Proposition 13.6 (or as p in Proposition 13.8). □

Remark 13.10. Therefore, every nontrivial arithmetical P in theories from formalized arithmetic upward turns out to be undecidable. We can generalize that result to encompass other theories T that include arithmetic; see below.

14. θ functions and the arithmetical hierarchy

Definition 14.1. The sentences $\xi, \zeta \in \mathcal{L}_T$ are *demonstrably equivalent* if and only if $T \vdash \xi \leftrightarrow \zeta$.

Definition 14.2. The sentence $\xi \in \mathcal{L}_T$ is *arithmetically expressible* if and only if there is an arithmetic sentence ζ such that $T \vdash \xi \leftrightarrow \zeta$.

Then, for $\mathbf{N} \models T$, a model makes it arithmetically sound.

Proposition 14.3. *If T is arithmetically sound, then for every $m \in \omega$ there is a sentence $\xi \in T$ such that $\mathbf{N} \models \xi$ while for no $k \leq n$ there is a Σ_k sentence in* PA *demonstrably equivalent to ξ.*

Proof. The usual proof for PA is given in Rogers (1967, p. 321). However we give here a slightly modified argument that imitates Proposition 13.8. First notice that

$$\emptyset^{(m+1)} = \{x : \phi_x^{\emptyset^{(m)}}(x)\}$$

is recursively enumerable but not recursive in $\emptyset^{(m)}$. Therefore, $\overline{\emptyset^{(m+1)}}$ isn't recursively enumerable in $\emptyset^{(m)}$, but contains a proper $\emptyset^{(m)}$-recursively enumerable set. Let us take a closer look at those sets.

We first need a lemma: form the theory $T^{(m+1)}$ whose axioms are those for T plus a denumerably infinite set of statements of the form "$n_0 \in \emptyset^{(n)}$," "$n_1 \in \emptyset^{(m)}$,"..., that describe $\emptyset^{(m)}$. Of course, this theory doesn't have a recursively enumerable set of theorems. Then,

Lemma 14.4. *If $T^{(n+1)}$ is arithmetically sound, then $\phi_x^{\emptyset^{(m)}}(x) \downarrow$ if and only if*

$$T^{(m+1)} \vdash \exists x_1, \ldots, x_n \in \omega\, p(\rho(z), \langle z, d_{y,\emptyset^{(m)}} \rangle, x_1, \ldots, x_n) = 0.$$

Proof. Similar to the proof in the nonrelativized case; see Machtey and Young (1979, p. 126 ff). □

Therefore, we have that the oracle machines $\phi_x^{\emptyset^{(m)}}(x) \downarrow$ if and only if

$$T^{(m+1)} \vdash \exists x_1, \ldots, x_n \in \omega\, p(\rho(z), \langle z, d_{y,\emptyset^{(m)}} \rangle, x_1, \ldots, x_n) = 0.$$

However, since $\overline{\emptyset^{(m+1)}}$ isn't recursively enumerable in $\emptyset^{(m)}$, then there will be an index $m_0(\emptyset^{(m)}) = \langle \rho(z), \langle z, d_{y,\emptyset^{(m)}} \rangle \rangle$ such that

$$\mathbf{N} \models \forall x_1, \ldots, x_n [p(m_0, x_1, \ldots, x_n)]^2 > 0,$$

while it cannot be proved neither disproved within $T^{(m+1)}$. It is therefore demonstrably equivalent to a Π_{m+1} assertion.

Now let $q(m_0(\emptyset^{(m)}), x_1, \ldots) = p(m_0(\emptyset^{(m)}), x_1, \ldots))^2$ be as in Proposition 14.3. Then, we have the following corollary.

Corollary 14.5. *If T is arithmetically sound, then, for*

$$\beta^{(m+1)} = \sigma(G(m_0(\emptyset^{(n)}))),$$

$$G(m_0(\emptyset^{(n)})) = \int_{-\infty}^{+\infty} \frac{C(m_0(\emptyset^{(n)}), x)\mathrm{e}^{-x^2}}{1 + C(m_0(\emptyset^{(n)}), x)} \mathrm{d}x,$$

$$C(m_0(\emptyset^{(n)}), x) = \lambda q(m_0(\emptyset^{(n)}), x_1, \ldots, x_r),$$

$\mathbf{N} \models \beta^{(m+1)} = 0$ *but for all* $n \leq m+1$, $\neg\{T^{(n)} \vdash \beta^{(m+1)} = 0\}$ *and* $\neg\{T^{(n)} \vdash \neg(\beta^{(m+1)} = 0)\}$.

We have used here a variant of the construction of θ and β which first appeared in da Costa and Doria (1991a). Then, we have the following corollary.

Corollary 14.6. *If T is arithmetically sound and if $\mathcal{L}_T$ contains expressions for the $\theta^{(m)}$ functions as given in Proposition* B.3, *then for any nontrivial arithmetical predicate P there is a $\zeta \in \mathcal{L}_T$ such that the assertion $P(\zeta)$ is T-demonstrably equivalent to and T-arithmetically expressible as a Π_{m+1} assertion, but not equivalent to and expressible as any assertion with a lower rank in the arithmetic hierarchy.*

Proof. As in the proof of Proposition 13.9, we write:

$$\zeta = \xi + [1 \dot{-} (p(m_0(\emptyset^m), x_1, \ldots, x_n) + 1)]\nu,$$

where $p(\ldots)$ is as in Proposition 14.3. □

An extension of the preceding result is as follows.

Corollary 14.7. *If T is arithmetically sound then, for any nontrivial P there is a $\zeta \in \mathcal{L}_T$ such that $P(\zeta)$ is arithmetically expressible, $\mathbf{N} \models P(\zeta)$ but only demonstrably equivalent to a Π^0_{n+1} assertion and not to a lower one in the hierarchy.*

Proof. Put

$$\zeta = \xi + \beta^{(m+1)}\nu,$$

where one uses Corollary 14.5. □

Beyond arithmetic

Definition 14.8.

$$\emptyset^{(\omega)} = \{\langle x, y\rangle : x \in \emptyset^{(y)}\},$$

for $x, y \in \omega$.

Definition 14.9.

$$\theta^{(\omega)}(m) = c_{\emptyset^{(\omega)}}(m),$$

where $c_{\emptyset^{(\omega)}}(m)$ is obtained as in Proposition 13.1.

Definition 14.10.

$$\emptyset^{(\omega+1)} = (\emptyset^{(\omega)})'.$$

Corollary 14.11. $\mathbf{0}^{(\omega+1)}$ *is the degree of* $\emptyset^{(\omega+1)}$.

Corollary 14.12. $\theta^{(\omega+1)}(m)$ *is the characteristic function of a nonarithmetic subset of* ω *of degree* $\mathbf{0}^{(\omega+1)}$.

Corollary 14.13. *If* T *is arithmetically sound, then, for*

$$\beta^{(\omega+1)} = \sigma(G(m_0(\emptyset^{(\omega)})),$$

$$G(m_0(\emptyset^{(\omega)})) = \int_{-\infty}^{+\infty} \frac{C(m_0(\emptyset^{(\omega)}), x)e^{-x^2}}{1 + C(m_0(\emptyset^{(\omega)}), x)} dx,$$

$$C(m_0(\emptyset^{(\omega)}), x) = \lambda q(m_0(\emptyset^{(\omega)}), x_1, \ldots, x_r),$$

$\mathbf{N} \models \beta^{(\omega+1)} = 0$ *but* $T \nvdash \beta^{(\omega+1)} = 0$ *and* $T \nvdash \neg(\beta^{(\omega+1)} = 0)$.

Proposition 14.14. *If* T *is arithmetically sound then, given any nontrivial predicate* P,

(1) *there is a family of terms* $\zeta_m \in \mathcal{L}_T$ *such that there is no general algorithm to check, for every* $m \in \omega$, *whether or not* $P(\zeta_m)$;
(2) *there is a term* $\zeta \in \mathcal{L}_T$ *such that* $\mathbf{M} \models P(\zeta)$ *while* $T \nvdash P(\zeta)$ *and* $T \nvdash \neg P(\zeta)$;
(3) *neither the* ζ_m *nor* ζ *are arithmetically expressible.*

Proof. We take:

(1) $\zeta_m = x\theta^{(\omega+1)}(m) + (1 - \theta^{(\omega+1)}(m))y$.
(2) $\zeta = x + y\beta^{(\omega+1)}$.
(3) Neither $\theta^{(\omega+1)}(m)$ nor $\beta^{(\omega+1)}$ are arithmetically expressible. □

Remark 14.15. We have thus produced out of every nontrivial predicate in T intractable problems that cannot be reduced to arithmetic problems. Actually, there are infinitely many such problems for every ordinal α, as we ascend the set of infinite ordinals in T. Also, the general nonarithmetic undecidable statement $P(\zeta)$ has been obtained without the help of any kind of forcing construction.

For the way one proceeds with those extensions, we refer the reader to references on the hyperarithmetical hierarchy (see Ash and Knight, 2000; Rogers, 1967).

Corollary 14.16. *There is an explicit expression for a function β such that $T \vdash \beta = 0 \vee \beta = 1$, while neither $T \vdash \beta = 0$ nor $T \vdash \beta = 1$.*

(Proof follows from the fact that the θ function has a recursively enumerable set of nonzero values, with a nonrecursive complement, and from the existence of a Diophantine equation, which has no roots in the standard model **M** while that fact can neither be proved nor disproved by T.)

Notice that since equality is undecidable in the language of analysis, there is no general algorithmic procedure to check whether a given expression in that language equals, say, the θ^n or the β.

In order to sum it up, I'll restate now the chief undecidability and incompleteness theorem.

15. Statement of the main undecidability and incompleteness results

We suppose that our theories are formalized within a first-order classical language with equality and the description operator.

We follow the notation of da Costa and Doria (1991a); ω denotes the set of natural numbers, Z is the set of integers, and R are the real numbers. Let T be a first-order consistent axiomatic theory that contains formalized arithmetic N and such that T is strong enough to include the concept of set plus the whole of classical elementary analysis and dynamical systems theory. (We can simply take $T = \text{ZFC}$, where ZFC is Zermelo–Fraenkel set theory with the Axiom of Choice.) Moreover, T has a model $\mathbf{M}$ where the arithmetic portion of T is standard.

A general undecidability and incompleteness theorem

If L_T is the formal language of T, we suppose that we can form within T a recursive coding for L_T so that it becomes a set L_T of formal expressions in an adequate interpretation of T. Objects in T will be noted by lower case italic letters. Predicates in T will be noted $P, Q, \ldots$.

From time to time, we play with the distinction between an object and the expression in L_T that represents it. If x, y are objects in the theory, $\xi, \zeta \in L_T$ are term-expressions for those objects in the formal language of T. In general, there is no one-to-one correspondence between objects and expressions; thus, we may have different expressions for the same functions: "$\cos \frac{1}{2}\pi$" and "0" are both expressions for the constant function 0. We note by $\lceil x \rceil$ an expression for x in L_T. We say that a predicate P defined for a set X is *nontrivial* if neither $T \vdash \forall x \in X P(x)$ nor $T \vdash \forall x \in X \neg P(x)$. In what follows $\mathcal{B}$ is a set of (expressions for) functions that includes elementary real analysis.

Let P be any nontrivial predicate. Our main theorem is as follows.

Proposition 15.1.

(1) *There is an expression $\xi \in \mathcal{B}$ so that $T \nvdash \neg P(\xi)$ and $T \nvdash P(\xi)$, but $\mathbf{M} \models P(\xi)$.*
(2) *There is a denumerable set of expressions for functions $\xi_m(x) \in \mathcal{B}$, $m \in \omega$, such that there is no general decision procedure to ascertain, for an arbitrary m, whether $P(\xi_m)$ or $\neg P(\xi_m)$ is provable in T.*

(3) *Given the set* $K = \{m : T \vdash P(\widehat{m})\}$, *and given an arbitrary total recursive function* $g : \omega \to \omega$, *there is an infinite number of values for* m *so that* $C_T(P(\widehat{m})) > g(\|P(\widehat{m})\|)$.
(*Here the* $\widehat{m}$ *recursively code the set* ξ_m *of expressions in* L_T; C_T *is the shortest length of a proof for* $P\widehat{m}$ *in* T, *and* $\|P\widehat{m}\|$ *is the length of* $P\widehat{m}$ *in* L_T.)

(4) *There is a* $\zeta \in L_T$ *such that the assertion* $P(\zeta)$ *is arithmetically expressible,* $\mathbf{M} \models P(\zeta)$ *but only demonstrably equivalent to a* Π_{n+1} *assertion and not to a lower one in the arithmetic hierarchy.*

(5) *There are expressions* ζ_m *and* ζ *in* L_T *which are not arithmetically expressible, and*:

   (a) *there is a family of those expressions* $\zeta_m \in L_T$ *such that there is no general algorithm to check, for every* $m \in \omega$, *whether or not* $P(\zeta_m)$ *in* T;
   (b) *there is an expression* $\zeta \in L_T$ *as above such that* $\mathbf{M} \models P(\zeta)$ *while* $T \nvdash P(\zeta)$ *and* $T \nvdash \neg P(\zeta)$.

(Recall that ζ is arithmetically expressible within T if we can make it formally equivalent to an arithmetic expression with the tools available in T.)

16. Questions settled with those techniques

We immediately noticed that our tools led to the negative solution of several open problems in dynamical systems theory and related areas. They are discussed below.

The integrability problem in classical mechanics

That's an old question. We quote a mention of that problem (see Lichtenberg and Lieberman, 1983):

> Are there any general methods to test for the integrability of a given Hamiltonian? The answer, for the moment, is no. We can turn the question around, however, and ask if methods can be found to construct potentials that give rise to integrable Hamiltonians. The

answer here is that a method exists, at least for a restricted class of problems and so on.

We can divide the integrability question into three items:

- Given any Hamiltonian h, do we have an algorithm to decide whether the associated flow X_h can be integrated by quadratures?
- Given an arbitrary Hamiltonian h such that X_h can be integrated by quadratures, can we algorithmically find a canonical transformation that will do the trick?
- Can we algorithmically check whether an arbitrary set of functions is a set of first integrals for a Hamiltonian system?

No, in all three cases. There is no general algorithm to decide, for a given Hamiltonian, whether or not it is integrable. Also, there will be sentences such as $\xi =$ "h is integrable by quadratures," where, however, $T \nvdash \xi$ and $T \nvdash \neg\xi$ (see da Costa and Doria, 1991a).

The Hirsch problem: the decision problem for chaos

That problem was discussed above. Is there an algorithm to check for chaos given the expressions of a dynamical system? No. There is no such a general algorithm, and there will be systems that look chaotic on a computer screen (that is to say, they are chaotic in a standard model $\mathbf{M}$, see section 15) but such that proving or disproving their chaotic behavior is impossible in T (see da Costa and Doria, 1991a; Hirsch, 1985).

That result applies to *any* nontrivial characterization for chaos in dynamical systems.

Wolfram's conjecture and Penrose's thesis

Wolfram had long conjectured that simple phenomena in classical physics might lead to undecidable questions (see Wolfram, 1984):

> One may speculate that undecidability is common in all but the most trivial physical theories. Even simply-formulated problems in theoretical physics may be found to be provably insoluble.

On the other side, Penrose asserted as a kind of thesis that classical physics offers no examples of noncomputable phenomena. We proved Wolfram's conjecture and thus gave a counterexample to Penrose's thesis (see da Costa and Doria, 1991b; Penrose, 1989; Stewart, 1991).

Arnol'd's problems

Arnol'd formulated in the 1974 AMS Symposium on the Hilbert Problems (see Arnol'd, 1976) a question dealing with algorithmic decision procedures for polynomial dynamical systems over Z (see Arnol'd, 1976):

> *Is the stability problem for stationary points algorithmically decidable?* The well-known Lyapounov theorem solves the problem in the absence of eigenvalues with zero real parts. In more complicated cases, where the stability depends on higher order terms in the Taylor series, there exists no *algebraic* criterion.
>
> *Let a vector field be given by polynomials of a fixed degree, with rational coefficients. Does an algorithm exist, allowing to decide, whether the stationary point is stable?*
>
> A similar problem: Does there exist an algorithm to decide, whether a plane polynomial vector field has a limit cycle?

For those questions, there is no general algorithm available since T contains the sine function, the absolute value function and π; also the corresponding theory is incomplete (see da Costa and Doria, 1994a). In the polynomial case again, there is no algorithm to decide whether a fixed point at the origin is stable or not (see da Costa and Doria, 1993a,b).

Problems in mathematical economics

Lewis and Inagaki (1991b) pointed out that our results entail the incompleteness of the theory of Hamiltonian models in economics. They also entail the incompleteness of the theory of Arrow–Debreu equilibria and (what is at first sight surprising) the incompleteness of the theory of *finite* games with Nash equilibria (see da Costa and

Doria, 1994a; Tsuji *et al.*, 1998). Those two last questions are discussed below in Section 17.

"Smooth" problems equivalent to hard number-theoretic problems

Common wisdom among mathematicians has that number-theoretic problems are in general much more difficult than "smooth" problems. We showed that that is definitely not the case. We gave an explicit example of a dynamical system where the proof that there will be chaos is equivalent to the proof of Fermat's last theorem (or the proof of Riemann's hypothesis, or the decision of the P vs. NP question). We also proved that (given some conditions) those "nasty" problems are dense in the space of all dynamical systems (see da Costa *et al.*, 1993).

Simple problems worse than any number-theoretic problem

The language of analysis is much richer than the language of arithmetic, as we can express the halting function in analysis. Also we can explicitly construct "natural"-looking and quite simple problems with our techniques that lie beyond the arithmetical hierarchy (see da Costa and Doria, 1994a). For example, we can explicitly define a procedure to obtain an expression $\theta^{(\omega)}(n)$ for a characteristic function in $\mathbf{0}^{(\omega)}$. That function equals either 0 or 1, but the actual computation of $\theta^{(\omega)}$'s values reaches beyond the arithmetic hierarchy.

Generic, faceless objects

One of the features of the main set-theoretic forcing constructions is that we add "generic," faceless sets to our formal theories. However, there are no explicit expressions for those objects. With the help of our techniques, we exhibited an expression for a "faceless" Hamiltonian (see da Costa and Doria, 1994a): the only thing we can prove about it is that it definitely is a Hamiltonian, and nothing more.

Undecidable sentences as bifurcation points in a formal theory

Our techniques allow a simple coding of undecidable sentences in T into a bifurcating vectorfield (see da Costa and Doria, 1993a,b). One such example goes (roughly) as follows: given the extended theories $T^+ = T + P(\xi)$, $T^- = T + \neg P(\xi)$, we have that $T^+ \vdash P$ if and only if a certain vectorfield X undergoes a Hopf bifurcation, while $T^- \vdash \neg P$ if and only if X doesn't undergo a Hopf bifurcation; moreover, we can algorithmically obtain an expression for that X.

17. Undecidability and incompleteness in the social sciences

Our main examples in this section have to do with economics (competitive markets) and the theory of social structures as modeled by population dynamics equations. We are especially interested in the difficulties of forecasting in economics and in the applied social sciences.

Whenever we describe social phenomena by dynamical systems, uncertainties in forecasting are usually supposed to be due to the nonlinearities in the systems considered, that is to say, they are related to the sensitivity those systems exhibit when small changes are made in the initial conditions. Linear systems do not have that kind of behavior and so are supposed to be strictly deterministic.

Our results contradict that belief. We show that equilibrium prices in competitive markets are in general *noncomputable*, and so fall outside the scope of the techniques available in the usual formal modeling tools; competitive market equilibrium is, however, equivalent to determining a minmax solution for a noncooperative game, which is a *linear* problem. So, there are also obstacles to forecasting when one deals with linear systems. The remaining two examples in this section have to do with the ecology of populations ("do we have cycles or chaotic behavior?") and a model for class structures in a population ("will the middle class survive?").

Undecidability and incompleteness in the theory of finite games

We start from the usual mathematical definitions in game theory.

Definition 17.1. A *noncooperative game* is given by the von Neumann triplet $\Gamma = \langle N, S_i, u_i \rangle$, with $i = 1, 2, \ldots N$, where N is the number of players, S_i is the strategy set of player i and u_i is the real-valued utility function $u_i : S_i \to \mathsf{R}$, where each $s_i \in S_i$.

Definition 17.2. A strategy vector $s^* = \langle s_1^*, \ldots \rangle$, $s_k^* \in S_k$ is a *Nash equilibrium vector* for a finite noncooperative game Γ if for all strategies and all i,

$$u_i(s^*) = u_i(\langle s_1^*, \ldots, s_k^*, \ldots \rangle) \geq u_i(\langle s_1^*, \ldots, s_k, \ldots \rangle),$$

for $s_k \neq s_k^*$.

The main result goes as follows: we suppose that game theory has been formalized through a Suppes predicate within our theory T; therefore, when we talk about games in T, we discuss objects that can be formally proved to equal explicitly defined games in T.

Proposition 17.3.

(1) *Given any nontrivial property P of finite noncooperative games, there is an infinite denumerable family of finite games Γ_m such that for those m with $T \vdash$ "$P(\Gamma_m)$," for an arbitrary total recursive function $g : \omega \to \omega$, there is an infinite number of values for m such that the shortest length of a proof of $P\Gamma_m$ in T, $C_T(P\Gamma_m) > g(\|P\Gamma_m\|)$.*
(2) *Given any nontrivial property P of finite noncooperative games, there is one of those games Γ such that $T \vdash$ "$P(\Gamma)$" if and only if $T \vdash$ "Fermat's Conjecture."*
(3) *There is a noncooperative game Γ where each strategy set S_i is finite but such that we cannot compute its Nash equilibria.*
(4) *There is a noncooperative game Γ where each strategy set S_i is finite and such that the computation of its equilibria is T-arithmetically expressible as a Π_{m+1} problem, but not to any Σ_k problem, $k \leq m$.*

(5) *There is a noncooperative game* Γ *where each strategy set* S_i *is finite and such that the computation of its equilibria isn't arithmetically expressible.*

So nasty things may crop up in game theory, even if that theory turns out to be linear; suffices to embed it into the language of analysis.

To take a closer look: when it comes to the theory of finite games, the situation looks very neat at first. If we can describe the game by tables, it should be decidable; if not, it may be undecidable. However, what do we mean when we say that we are "describing a game by a table"? A table of payoffs and outcomes is easily handled when there are just a few participants in the game, but in an actual market situation when we have thousands of players, we may rather naturally relax the condition of an explicit presentation of the table of payoffs in favor of the following arrangement:

- There is a partial recursive function with a finite set of values that lists the participants in the game. (The game's players are *recursively presented.*)
- The payoff matrix is given by a partial recursive function.

In our case, the payoff matrix u is given by

$$u = u'\theta + u''(1 - \theta),$$

where θ is the function we have previously introduced (or one of its higher-degree variants θ^n), and u' and u'' are noncoincident payoff matrices with different Nash solutions.

The first condition in the preceding itemization means simply that we have a procedure that lists all players in the game (presumably out of a larger universe); the second condition means that we can summarize within the bounds of an algebraic expression all gains and losses in the game. Why don't we require that the players and the payoff matrix be given by *total* recursive functions? Because for most complicated situations, we can't algorithmically check that a given partial recursive function is a total function; therefore, if we add that requirement to a situation where thousands of variables are to be handled, we would add an idealized condition that nobody would be able to decide if needed.

Our results here satisfy the above set of conditions. The undecidable payoff functions can easily (even if clumsily) be translated as Turing machines that once started in their computation will run forever (or until they physical counterparts break down, or are stopped by an external agent). So, our payoff functions are represented by partial recursive functions as required.

Markets in equilibrium may have noncomputable prices

Those results have an immediate consequence for a question of both historical and practical importance: the controversy on economic planning between L. von Mises and O. Lange (see, on that controversy, Seligman (1971, I, p. 115ff)).

The central problem of economic planning is an allocation problem. Very frequently, allocation is to be done on the basis of maximizing (or minimizing) simple functions over finite sets. We proved that trouble is to be expected even when the problem of planning is reduced to the problem of determining equilibria in *finite* noncooperative Nash games, which is formally equivalent to the determination of equilibrium prices in a competitive market.

So, the main argument by Lange in favor of a planned economy (by the way, an argument also shared by von Mises) clearly breaks down. Lange thought that given the (possibly many) equations defining an economy, a huge and immensely powerful computer would *always* be able to figure out the equilibrium prices, therefore allowing (at least theoretically) the existence of an efficient global policy maker. However, our results (as well as the weaker previous results by Lewis (1991a); Lewis and Inagaki (1991b)) disprove Lange's conjecture.

Therefore, those that argue that "the market knows better" may find a strong theoretical support in our conclusions (or in Lewis' already quoted results), since the equilibrium point is reached (at least according to theory) while we cannot, in general, compute it beforehand.

The axiomatic background is made explicit for clarity. The incompleteness phenomenon means that within a (consistent) prescribed axiomatic framework, certain facts cannot be proved. Assuredly, if we add stronger axioms to our system, a few of those unprovable facts may be proved. Yet the stronger axioms may also be debatable on philosophical grounds, so that the proof of a desired fact from the enriched system eventually turns out to be technically correct but philosophically (and perhaps empirically) doubtful. For details see da Costa and Doria (1994a) and Tsuji *et al.* (1998).

Oscillating populations or a chaotic demography?

The Lotka–Volterra equations (LV) describe two interacting populations, namely a "preyed" population x and a "predator" population y. If left by themselves, the x blow up exponentially, while the y decay exponentially. When x and y interact, they start to move in nonlinear cycles. The LV model describes in a reasonable way a number of ecological cycles (see Goel *et al.*, 1971; Nicolis and Prigogine, 1977).

Now consider the following situation: let $\langle x, y\rangle$ and $\langle x', y'\rangle$ be two uncoupled LV systems. We show in Section 20 that they can be described by a Hamiltonian system Ξ, and that there is a perturbed system

$$\Xi' = \Xi + \epsilon\Upsilon$$

where all populations are (in general) coupled and Ξ' has a Smale horseshoe. So Ξ' is chaotic in its behavior.

We slightly modify Ξ' and get

$$\Xi^* = \Xi + \epsilon\beta\Upsilon,$$

where β is given in Corollary 14.16. Then the assertions "Ξ^* describes two sets of oscillating, nonchaotic, uncoupled populations" and "Ξ^* describes four coupled, chaotic populations" are both undecidable in our formal theory T.

Will the middle class survive?

Section 20, equation (6), describes a beautiful model (elaborated by Prigogine) for the interaction of three coupled populations, x, y, z, where z is a small-sized "upper" or "warrior" class, y is a "lower" or "working" class, and x is a "middle class." The model predicts two stable equilibria: in the first, the "middle class" vanishes; in the second, the "warrior" class vanishes while "workers" and the "middle class" thrive. The "middle class" will grow to a steady state value if

$$\kappa(N_0 - y_0 - z_0) - \delta - \rho z_0 > 0,$$

where ρ is a measure of the aggressiveness of the "upper class" against the "middle class," and N_0 measures the total wealth of the ecological niche; y_0, z_0 are constant values obtained out of the system's parameters, like δ (see Section 20).

We now write $\rho^* = \beta\rho' + (1-\beta)\rho''$, where $\rho' \gg \rho''$, and ρ' doesn't satisfy the preceding condition, while ρ'' satisfies it. Then, out of

$$\kappa(N_0 - y_0 - z_0) - \delta - \rho^* z_0 > 0,$$

the sentences "The middle class will thrive" and "The middle class will fade away" are both undecidable in T.

More details in Section 20.

18. Forcing and our techniques

Let us now restate a few ideas from one of our papers (see da Costa and Doria, 1994b). Our techniques certainly look very different from the well-known forcing constructions that have led to so many undecidable statements in mathematics. Thus, which is the relation between our techniques for the construction of undecidable statements and the Cohen–Solovay kind of forcing? In order to answer this question, we must conceive a theory as a Turing machine that accepts strings of symbols — well-formed formulas — and stops whenever those strings of symbols are theorems of the theory. If not, it never halts and enters an infinite loop.

Now consider Zermelo–Fraenkel axiomatic set theory, ZF. If M_{ZF} is the corresponding proof machine for ZF, and if CH is the Continuum Hypothesis, we know that $M_{\mathrm{ZF}}(\mathrm{CH})$ never halts. Accordingly, there is a Diophantine polynomial $p_{\mathrm{ZF}}(\mathrm{CH}, x_1, \ldots)$ that has no roots over Z, but since CH is independent of the axioms of ZF, there can be no proof (within ZF) of the statement "$p_{\mathrm{ZF}}(\mathrm{CH}, x_1, \ldots) = 0$ has no roots over Z." (If there were one such proof, we would then be able to decide CH in ZF.) With the help of our techniques, we can obtain a two-step function $\theta_{\mathrm{ZF}}(m)$ such that, if m_{CH} is a Gödel number for CH, then both $\mathrm{ZF} \nvdash \theta_{\mathrm{ZF}}(m_{\mathrm{CH}}) = 0$ and $\mathrm{ZF} \nvdash \theta_{\mathrm{ZF}}(m_{\mathrm{CH}}) = 1$. Therefore, every undecidable statement constructed with the help of forcing within ZF (or even within weaker theories, provided that they include elementary arithmetic) gives rise to undecidable statements according to the present tools.

Moreover, the converse isn't true, that is, there are some (actually, infinitely many) undecidable statements that can be constructed according to the present techniques, but such that no forcing statement will be mapped on them if we follow the preceding procedure. Finite objects are (set-theoretic) forcing-absolute, but we have seen that we can construct undecidable statements about finite objects in ZF say, again through the θ function. If m_{Fin} is the Gödel-coding for one of those statements, then "$\theta_{\mathrm{ZF}}(m_{\mathrm{Fin}}) = 0$" cannot be proved in ZF. So, there is a (metamathematical) algorithmic procedure that goes from every undecidable statement in ZF onto undecidable statements about the θ_{ZF} function; and yet forcing statements are only a portion of that map, since there is much more in it (see da Costa and Doria, 1992a).

19. Evaluation of the results

There are two possible opposing views concerning Gödel-like undecidable statements in mathematics.

Following the first view, Gödel-like undecidable sentences are usually seen as warning posts that indicate blocked routes in axiomatic systems; according to that viewpoint, they mean that one can't go farther along some direction. They were already known to appear in lost backalleys; yet, as told here, the authors had long striven to show that the incompleteness phenomenon is part of the actual practice in any axiomatized science, and their endeavor proved a fruitful one when they showed that simple questions in dynamical systems theory ("Given a dynamical system, can we check whether it is chaotic? Can we prove that it is chaotic?") led to undecidability and incompleteness (see da Costa and Doria, 1991a). We call that the "negative" viewpoint, since it is usually supposed to imply that incompleteness means that there is an essential deficit in our knowledge when it is obtained through some formal system.

Assuredly incompleteness means that we can't compute some result. But should we take that fact as some kind of absolute obstacle to our knowledge of the world through formal languages?

The second point of view is the "optimistic" one; it is the position adopted by the authors. Undecidable sentences are seen as degrees of freedom, as bifurcation points in our theories. They reveal some kind of inner freedom in the possibilities we have when trying to describe the world within a formal system. They show the existence of *open* possibilities, choices available in the formalism; they cannot be looked upon as limitations to our knowledge. That point of view is reinforced when one considers that there is an actual functor that goes from the theory of formal systems into the theory of bifurcating dynamical systems, as described for instance in da Costa and Doria (1993b). Very much as if the whole of mathematics were to be redrawn onto a small spot over its own belly.

20. More on population dynamics

We sketch here the main technical details we require from the theory of population dynamics.

The Lotka–Volterra equations in Hamiltonian form

The Lotka–Volterra (LV) equations describe two competing populations; its variables denote the number of individuals in each population. Here x represents the *prey* population while y is the *predator* population. For $x > 0$, $y > 0$, they interact through:

$$\begin{aligned} \dot{x} &= +\alpha x - \beta xy, \\ \dot{y} &= -\gamma y + \delta xy, \end{aligned} \tag{1}$$

where α, β, γ, δ are positive real constants. There is an immediate first integral,

$$\exp H = y^{\alpha} x^{\gamma} \exp[-(\beta y + \delta x)] = k, \tag{2}$$

where k is a positive constant. Solutions for the LV equations are shown to be nonlinear cycles.

If we put $u = \log x$, $v = \log y$, then there is a time-independent Hamiltonian

$$H = \alpha v + \gamma u - (\beta e^{v} + \delta e^{u}) = \log k, \tag{3}$$

so that equation (1) becomes:

$$\begin{aligned} \dot{u} &= +\partial H/\partial v, \\ \dot{v} &= -\partial H/\partial u. \end{aligned} \tag{4}$$

For the references, see Goel *et al.* (1971) and Nicolis and Prigogine (1977).

Competing populations with chaotic dynamics

We use a famous result by Ruelle and Takens.

Proposition 20.1. *If $\omega = \langle \omega_1, \omega_2, \omega_3, \omega_4 \rangle$ is a constant vectorfield on the torus T^4, where $\omega_1 \leq \omega_2 \leq \omega_3 \leq \omega_4$, then given every C^3-neighborhood of ω there is an open set of vectorfields with strange attractors.*

Proof. See Ruelle and Takens (1971). □

Now consider two sets of four LV predator–prey populations that obey two uncoupled LV systems (equation (1)),

$$\Xi = \langle \langle x, y \rangle, \langle x', y' \rangle \rangle.$$

Due to equations (1)–(4) they can be cast in Hamiltonian form and (through the Liouville–Arnol'd theorem), they can be canonically mapped onto a constant vectorfield ω on T^4. Then, as a corollary of the Ruelle–Takens result, when we go back to the LV systems, we have the following corollary.

Corollary 20.2. *Close to Ξ there is an open set of perturbed LV equations with four population components and with a chaotic behavior due to strange attractors.*

The attractors are Smale horseshoes; a similar result within Hamiltonian mechanics is explicitly dealt with in Holmes and Marsden (1982).

Remark 20.3. If Ξ also denotes the LV-system for those four populations, then we note its chaotic perturbation:

$$\Xi' = \Xi + \epsilon\Upsilon, \tag{5}$$

where ϵ is a small positive constant and Υ is the perturbation.

Class structures

We consider here three interacting populations where the number of individuals in each is denoted by the variables $x, y, z > 0$ (see Nicolis and Prigogine, 1977, p. 460). The dynamics of the model is given by:

$$\begin{aligned}
\dot{x} &= \kappa x(N_0 - x - y - z) - \delta x - \rho xy,\\
\dot{y} &= \kappa y(N_0 - x - y - z) - \delta y - f(x,y),\\
\dot{z} &= f(x,y) - \delta z.
\end{aligned} \tag{6}$$

Here $f(x,y) = \alpha_1 z(y - \alpha_2 z)$ is a Verhulst term. Greek letters represent positive real constants; N_0 is a maximum level of resources for the competing populations. $y+z$ is to be understood as a population that split between a "working class" y and a (small-sized) "warrior class" z; x is a "middle class", which is preyed upon by the "warriors" since it competes within the same ecological niche with the coalescing "warriors" and "workers."

Stability analysis shows that the equilibrium state $x_0, y_0, z_0 \neq 0$ is unstable. One considers the equilibria $x_0, y_0 \neq 0; z_0 = 0$ (no warrior class) or $x_0 = 0; y_0, z_0 \neq 0$ (no middle class). Condition for the survival of the middle class is

$$\kappa(N_0 - y_0 - z_0) - \delta - \rho z_0 > 0. \tag{7}$$

Since z_0 is small, the x will vanish if either the workers y_0 are a large-sized population or if the warriors are very aggressive ($\rho \gg 0$) against the middle class x.

Recall that the Lotka–Volterra (LV) system,

$$\begin{aligned} \dot{x} &= -x + \alpha xy \\ \dot{y} &= +y - \beta xy, \end{aligned} \tag{8}$$

which describes two interacting populations, the predators (x) and preys (y), can be made equivalent through a simple variable change,

$$\begin{aligned} u &= \log x, \\ v &= \log y, \end{aligned} \tag{9}$$

which is derived from a Hamiltonian function H (not made explicit here; see Nicolis and Prigogine, 1977), so that we have:

$$\begin{aligned} \dot{u} &= -\partial H/\partial v, \\ \dot{v} &= +\partial H/\partial u. \end{aligned} \tag{10}$$

Moreover, since H is a first integral in that system, and since the LV system has only closed orbits, we may transform it into a harmonic oscillator system through a canonical transformation:

$$\begin{aligned} \dot{\xi} &= -\eta, \\ \dot{\eta} &= +\xi. \end{aligned} \tag{11}$$

Those results will be our starting point.

Two perennially competing populations or two populations that will fatally disappear?

We consider here a model derived from a Lotka–Volterra model which undergoes a Hopf bifurcation:

$$\dot{x} = -y + \alpha xy - x(\mu - x^2 - y^2), \tag{12}$$

$$\dot{y} = +x - \beta xy - y(\mu - x^2 - y^2). \tag{13}$$

It is now a simple exercise to obtain an undecidable dynamical system out of the preceding equations.

Part II. The monster in the belly of complexity theory

Appendix A. Preliminary remarks

We may use a more transparent (but colorless) title: incompleteness out of fast-growing computable functions. We explore here the fast-growing nature of several counterexample functions that appear in complexity theory, as they lead to examples of undecidability and incompleteness in reasonable axiomatic theories.[13] Whenever required, we use (again!) a formal framework S that has the following properties:

We suppose that theory S satisfies the following:

- Its underlying language is the first-order classical predicate calculus.
- It has a recursively enumerable set of theorems.
- It includes PA (Peano Arithmetic) in the following sense: it includes the language and proves all theorems of PA.
- It has a model with standard arithmetic.

(Think of S as PA or ZFC with models that have standard arithmetic.) Suppose that we want to build a function that tops all total recursive functions. Let's do it formally. We will work within S:

Remark A.1. For each n, $\mathsf{F}(n) = \max_{k \leq n}(\{e\}(k)) + 1$, that is, it is the sup of those $\{e\}(k)$ such that:

(1) $k \leq n$.
(2) $\lceil \mathrm{Pr}_S(\lceil \forall x\, \exists z\, T(e, x, z) \rceil) \rceil \leq n$.

$\mathrm{Pr}_S(\lceil \xi \rceil)$ means there is a proof of ξ in S, where $\lceil \xi \rceil$ means: the Gödel number of ξ. So $\lceil \mathrm{Pr}_S(\lceil \xi \rceil) \rceil$ means: "the Gödel number of sentence 'there is a proof of ξ in S.' " Condition 2 above translates as: there is a proof of [$\{e\}$ is total] in S whose Gödel number is $\leq n$.

[13]We follow here a suggestion by G. Kreisel in a private message to N. C. A. da Costa and F. A. Doria.

Proposition A.2. *We can explicitly compute a Gödel number* e_{F} *so that* $\{e_{\mathsf{F}}\} = \mathsf{F}$.

Proposition A.3. *If* S *is consistent then* $S \nvdash \forall m \exists n\, [\{e_{\mathsf{F}}\}(m) = n]$.

We do *not* get here a Busy Beaver like function; we get a partial recursive function (the Busy Beaver is noncomputable), which can neither be proved nor disproved total in S — it is total in the standard model for arithmetic, provided that S has a model with standard arithmetic.

Notice that this function is another version for Kleene's function F, which appears in the beginning of this chapter.

Sources

Again we will base our exposition in three papers of ours (see da Costa and Doria, 2015, 2016; Doria, 2016), which will be freely quoted in what follows.

Appendix B. Technicalities

We deal here with two possible formalizations for both $P = NP$ and $P < NP$. We have called the unusual formalizations the "exotic formalization." They are naïvely equivalent, but when we move to a formal framework like that of S, we have difficulties.

Let $t_m(x)$ be the primitive recursive function that gives the operation time of $\{m\}$ over an input x of length $|x|$. If $\{m\}$ stops over an input x, then

$$t_m(x) = |x| + [\text{number of cycles of the machine until it stops}].$$

t_m is primitive recursive and can in fact be defined out of Kleene's T predicate.

Definition B.1 (Standard formalization for $P = NP$). $[P = NP] \leftrightarrow_{\text{Def}} \exists m, a \in \omega\, \forall x \in \omega\, [(t_m(x) \leq |x|^a + a) \wedge R(x, m)]$.

$R(x, y)$ is a polynomial predicate; it formalizes a kind of "verifying machine" that checks whether or not x is satisfied by the output of $\{m\}$.

Definition B.2. $[P < NP] \leftrightarrow_{\text{Def}} \neg[P = NP]$.

Now suppose that $\{e_{\mathsf{f}}\} = \mathsf{f}$ is total recursive and strictly increasing:

Remark B.3. The naïve version for the exotic formalization is

$$[P = NP]^{\mathsf{f}} \leftrightarrow \exists m \in \omega, a \,\forall x \in \omega \,[(t_m(x) \leq |x|^{\mathsf{f}(a)} + \mathsf{f}(a)) \wedge \, R(x, m)].$$

However, there is no reason why we should ask that f be total; on the contrary, there will be interesting situations where such a function may be partial and yet provide a reasonable exotic formalization for $P < NP$.

Let f be in general a (possibly partial) recursive function that is strictly increasing over its domain, and let e_{f} be the Gödel number of an algorithm that computes f. Let $p(\langle e_{\mathsf{f}}, b, c\rangle, x_1, x_2, \ldots, x_k)$ be an universal Diophantine polynomial with parameters e_{f}, b, c; that polynomial has integer roots if and only if $\{e_f\}(b) = c$. We may suppose that polynomial to be ≥ 0. We omit the "$\in \omega$" in the quantifiers, since they all refer to natural numbers.

Definition B.4. $M_{\mathsf{f}}(x, y) \leftrightarrow_{\text{Def}} \exists\, x_1, \ldots, x_k \,[p(\langle e_{\mathsf{f}}, x, y\rangle, x_1, \ldots, x_k) = 0]$.

Actually $M_{\mathsf{f}}(x, y)$ stands for $M_{e_{\mathsf{f}}}(x, y)$, or better, $M(e_{\mathsf{f}}, x, y)$, as dependence is on the Gödel number e_{f}.

Definition B.5. $\neg Q(m, a, x) \leftrightarrow_{\text{Def}} [(t_m(x) \leq |x|^a + a) \rightarrow \neg R(x, m)]$.

Proposition B.6 (Standard formalization, again.).

$$[P < NP] \leftrightarrow \forall m, a \,\exists x \,\neg Q(m, a, x).$$

Definition B.7. $\neg Q_{\mathsf{f}}(m, a, x) \leftrightarrow_{\text{Def}} \exists a' \,[M_{\mathsf{f}}(a, a') \wedge \neg Q(m, a', x)]$.

Remark B.8. We will sometimes write $\neg Q(m, \mathsf{f}(a), x)$ for $\neg Q_{\mathsf{f}}(m, a, x)$, whenever f is provably recursive and total.

Definition B.9 (Exotic formalization).

$$[P < NP]^{\mathsf{f}} \leftrightarrow_{\mathrm{Def}} \forall m, a\, \exists x\, \neg Q_{\mathsf{f}}(m, a, x).$$

Notice that again this is a Π_2 arithmetic sentence:

$$\forall m, a\, \exists x, a', x_1, \ldots, x_k\, \{[p(\langle e_{\mathsf{f}}, a, a'\rangle, \ldots, x_1, \ldots, x_k) = 0] \\ \wedge \neg Q(m, a', x)\}.$$

(Recall that Q is primitive recursive.)

Definition B.10. $[P = NP]^{\mathsf{f}} \leftrightarrow_{\mathrm{Def}} \neg[P < NP]^{\mathsf{f}}$.

We will sometimes write $\neg Q(m, \mathsf{g}(a), x)$ for $\neg Q_{\mathsf{g}}(m, a, x)$, whenever g is S-provably total.

Appendix C. Hard stuff

For the definition of SAT (and a detailed presentation of the satisfiability problem), see Machtey and Young (1979); for the BGS recursive set of poly Turing machines, see Baker *et al.* (1975). In a nutshell, SAT is the set of all Boolean expressions in conjunctive normal form (cnf) that are satisfiable, and BGS is a recursive set of poly Turing machines that contains emulations of every conceivable poly Turing machines.

The full counterexample function, intuitive ideas

The full counterexample function f is defined as follows: let ω code an enumeration of the Turing machines. Similarly code by a standard code SAT onto ω:

- If $n \in \omega$ isn't a poly machine, $f(n) = 0$.
- If $n \in \omega$ codes a poly machine:
  - — $f(n) =$ first instance x of SAT so that the machine fails to output a satisfying line for x, plus 1, that is, $f(n) = x + 1$.

— Otherwise $f(n)$ is undefined, that is, if $P = NP$ holds for $n, f(n) =$ undefined.

As defined, f is noncomputable. It will also turn out to be at least as fast growing as the Busy Beaver function, since in its peaks it tops all intuitively total recursive functions.

The idea in the proof of that fact goes as follows:

- Use the s–m–n theorem to obtain Gödel numbers for an infinite family of "quasi-trivial machines" — soon to be defined. The table for those Turing machines involves very large numbers, and the goal is to get a compact code for that value in each quasi-trivial machine so that their Gödel numbers are in a sequence $\mathsf{c}(0), \mathsf{c}(1), \mathsf{c}(2), \ldots$, where c is primitive recursive.
- Then add the required clocks as in the BGS sequence of poly machines, and get the Gödel numbers for the pairs machine + clock. We can embed the sequence we obtain into the sequence of all Turing machines.
- Notice that the subsets of poly machines we are dealing with are (intuitive) recursive subsets of the set of all Turing machines. More precisely, if we formalize everything in some theory S, then the formalized version of the sentence "the set of Gödel numbers for these quasi–trivial Turing machines is a recursive subset of the set of Gödel numbers for Turing machines" holds for the standard model for arithmetic in S, and vice versa.
  However, S may not be able to prove or disprove that assertion, that is to say, such assertions will sometimes be formally independent of S.
- We define the counterexample functions over the desired set(s) of poly machines, and compare them to fast-growing total recursive functions over similar restrictions.

Definition C.1. For $f, g : \omega \to \omega$,

$$f \textbf{ dominates } g \leftrightarrow_{\text{Def}} \exists y \, \forall x \, (x > y \to f(x) \geq g(x)).$$

We write $f \succ g$ for f dominates g.

Quasi-trivial machines: intuitions

The counterexample function is highly oscillating. Since it collects the counterexamples for a problem in the class NP, out of a natural listing of the poly machines, there will be cases where the counterexample takes a long time to reveal itself, while the very next poly machine outputs a counterexample in the first trials. So the idea here is to construct a family of poly machines, which produce counterexamples that can be seen as part of a fast-growing functions; a function which grows as fast as one wishes.

So, we construct a family of poly machines that fits our purpose:

- Consider an exponential algorithm for the problem we are dealing with — it will settle all instances of the question, albeit in exponential time. Call it E.
- Build the following algorithm A:
  - $\mathsf{A} = \mathsf{E}$ for any input up to instance k.
  - $\mathsf{A} = 0$ for all inputs $> k$.
- A is a poly machine, and the value of the counterexample function at (the Gödel number of) A is $k + 1$.
- Now make k as large as one wishes.

Notice that k requires about $\log k$ bits to be described. As such, a value must be coded in our machine's Gödel number, which will be very large for large k. However, we need a kind of controlled growth in Gödel numbers for the A-machines family. So, instead of explicitly encoding k as a numeral, we use it as the value of a function; we put $k = \mathsf{g}(i)$, for a fast-growing function g. For the s–m–n theorem comes in handy and ensures us that the modified A's Gödel numbers will always grow as a primitive recursive function, but not beyond it.

Quasi-trivial machines: formal treatment

Recall that the operation time of a Turing machine is given as follows: given that x is the Turing machine's input (in binary form) and $|x|$ is its length, if M stops over an input x, then we have the following definition.

Definition C.2. The operation time over x,

$$t_{\mathsf{M}} = |x| + \text{number of cycles of the machine until it stops.}$$

Example C.3.

- **First trivial machine.** Note it O. O inputs x and stops.

  $$t_{\mathsf{O}} = |x| + \text{moves to halting state} + \text{stops.}$$

  So, operation time of O has a linear bound.
- **Second trivial machine.** Call it O'. It inputs x, always outputs 0 (zero) and stops.
  Again operation time of O' has a linear bound.
- **Quasi-trivial machines.** A *quasi-trivial machine* Q operates as follows: for $x \leq x_0$, x_0 a constant value, $\mathsf{Q} = \mathsf{R}$, R an arbitrary total machine. For $x > x_0$, $\mathsf{Q} = \mathsf{O}$ or O'.
  This machine has also a linear bound.

Remark C.4. Now let H be any fast-growing, superexponential total machine. Let H' be a total Turing machine. Form the following family $\mathsf{Q}_{\dots}$ of quasi-trivial Turing machines with subroutines H and H':

(1) If $x \leq \mathsf{H}(n)$, $\mathsf{Q}^{\mathsf{H},\mathsf{H}',n}(x) = \mathsf{H}'(x)$;
(2) If $x > \mathsf{H}(n)$, $\mathsf{Q}^{\mathsf{H},\mathsf{H}',n}(x) = 0$.

Proposition C.5. *There is a family* $\mathsf{R}_{\mathsf{g}(n,|\mathsf{H}|,|\mathsf{H}'|)}(x) = \mathsf{Q}^{\mathsf{H},\mathsf{H}',n}(x)$, *where* g *is primitive recursive, and* $|\mathsf{H}|, |\mathsf{H}'|$ *denotes the Gödel number of* H *and of* H'.

Proof. By the composition theorem and the *s–m–n* theorem.

Now let T be the usual, exponential algorithm that computes the truth values of the elements of SAT.

Remark C.6. *Very important!* Recall that we are interested in quasi-trivial machines where $\mathsf{H}' = \mathsf{T}$, that is, it is like, say, the standard truth-table exponential algorithm for SAT.

We first give a result for the counterexample function when defined over all Turing machines (with the extra condition that the counterexample function $= 0$ if M_m isn't a poly machine). We have the following proposition.

Proposition C.7. *If $N(n) = \mathsf{g}(n)$ is the Gödel number of a quasi-trivial machine as in Remark C.4, then $f(N(n)) = f(\mathsf{g}(n)) = \mathsf{H}$ $(n) + 1$.*

Proof. Use the machines in Proposition C.5 and Remark C.6. □

Appendix D. The counterexample function *f*

Our goal here is to prove the following result:

Proposition D.1. *For no total recursive function h does $\mathsf{h} \succ f$.*

Sketch of proof. The idea behind the argument goes as follows: suppose that there is a total recursive function, say, h, that tops — dominates — the counterexample function f. Then we try to identify some fast-growing segment in the counterexample function f, it overtakes h infinitely many times.

How are we to proceed? We use the quasi-trivial machines. Pick up some function g that dominates h. We will try to "clone" that function g in a subset of the values of f — or better, we'll show that g exists as if already cloned in f. We use the quasi-trivial machines for that purpose.

How do we proceed? We construct a denumerable family of quasi-trivial machines Q_n of Gödel numbers $\mathsf{c}(n)$ so that the values of f at those machines are given by

$$f(\mathsf{c}(n)) = \mathsf{g}(n).$$

$\mathsf{c}(n)$ is a primitive recursive function. Now we have $\mathsf{g}(\mathsf{c}(n)) > \mathsf{h}(\mathsf{c}(n))$ as we have, by construction, $\mathsf{g}(n) > \mathsf{h}(n)$. □

Appendix E. BGS-like sets

We require here the BGS (see Baker *et al.*, 1975) set of poly machines:

$$\langle \mathsf{M}_m, |x|^a + a \rangle,$$

where we couple a Turing machine M_m to a clock regulated by the polynomial $|x|^a + a$, that is, it stops M_m after $|x|^a + a$ steps in the operation over x, where x is the machine's binary input and $|x|$ its bit-length.

The BGS set is constructed out of a recursive enumeration of all Turing machines coupled to a clock as described (that is, another Turing machine that shuts its partner down once it has operated for a prescribed number of steps).

The BGS set has the following properties:

- Every machine in the BGS set is a poly Turing machine. (Obviously, by construction.)
- Every poly Turing machine has a BGS machine that emulates it.

In fact, given one particular poly machine there will be *infinitely many* BGS machines which emulate it. And:

Proposition E.1. *The* BGS *set is recursive.*

(This contrasts with the fact that the set of all poly machines isn't even recursively enumerable.)

A more general machine-clock couple will also be used here:

$$\langle \mathsf{M}_m, |x|^{(a)} + \mathsf{f}(a)\rangle \mapsto \mathsf{M}_{\mathsf{c}(m,|\mathsf{f}|,a)},$$

Its Gödel number is given by $\mathsf{c}(m, |\mathsf{f}|, a)$, with c primitive recursive by the s–m–n theorem, with f at least intuitively recursive.

Remark E.2. Notice that we can have c such that, for parameters a, b, if $a < b$, then $\mathsf{c}(\ldots a \ldots) < \mathsf{c}(\ldots b \ldots)$.

It is a generalization of the BGS set. It also satisfies:

- Every machine in the above described generalization of the BGS set is a poly Turing machine. (Obviously, by construction.)
- Every poly Turing machine has a generalized BGS machine that emulates it.

The generalized BGS set is also recursive.

Appendix F. An example

As an example recall that $P < NP$ is given by a Π_2 arithmetic sentence, that is, a sentence of the form "for every x there is an y so that $P(x, y)$," where $P(x, y)$ is primitive recursive. Given our theory S with enough arithmetic in it, S proves a Π_2 sentence ξ if and only if the associated Skolem function f_ξ is proved to be total recursive by S. For $P < NP$, the Skolem function is what we have been calling the counterexample function.

Remark F.1. However, there are infinitely many counterexample functions we may consider. Why is it so? For many adequate, reasonable theories S, we can build a recursive (computable) *scale of functions*[14] $\mathsf{f}_0, \mathsf{f}_1, \ldots, \mathsf{f}_k, \ldots$ with an infinite set of S-provably total recursive functions so that f_0 is dominated by f_1 which is then dominated by f_2, ..., and so on.

Given each function f_k, we can form a BGS-like set BGS^k, where clocks in the time-polynomial Turing machines are bounded by a polynomial:

$$|x|^{\mathsf{f}_k(n)} + \mathsf{f}_k(n),$$

where $|x|$ denotes the length of the binary input x to the machine. We can then consider the recursive set:

$$\bigcup_k \mathrm{BGS}^k$$

of all such sets. Each BGS^k contains representatives of all poly machines (time polynomial Turing machines). Now, what happens if there is no such an g, but there are functions g_k which dominate each particular f_k, while the sequence $\mathsf{g}_0, \mathsf{g}_1, \ldots$ is unbounded in S, that is, grows as the sequence $\mathsf{F}_0, \mathsf{F}_1, \ldots$ in S?

[14]Such a "scale of functions" exists and can be explicitly constructed.

Exotic $\mathrm{BGS}^{\mathsf{F}}$ machines

Now let F be a fast growing, intuitively total algorithmic function. We consider the exotic $\mathrm{BGS}^{\mathsf{F}}$ machines already described, that is, poly machines coded by the pairs $\langle m, a\rangle$, which code Turing machines M_m with bounds $|x|^{\mathsf{F}(a)} + \mathsf{F}(a)$. Since the bounding clock is also a Turing machine, now coupled to M_m, there is a primitive recursive map c so that:

$$\langle \mathsf{M}_m, |x|^{\mathsf{F}(a)} + \mathsf{F}(a)\rangle \mapsto \mathsf{M}_{\mathsf{c}(m,|\mathsf{F}|,a)},$$

where $\mathsf{M}_{\mathsf{c}(m,|\mathsf{F}|,a)}$ is a poly machine within the sequence of all Turing machines. We similarly obtain a g as above, and follows.

Proposition F.2. *Given the counterexample function f_k defined over the* BGS^k*-machines, for no* ZFC*-provable total recursive* h *does* $\mathsf{h} \succ \mathsf{f}_k$.

Proof. As in Proposition D.1, use Gödel number coding primitive recursive function c to give the Gödel numbers of the quasi-trivial machines we use in the proof. □

Remark F.3. Notice that we have a — may we call it reasonable? — formalization for our main question:

$$[P < NP]^k \leftrightarrow [P < NP]^{\mathsf{f}^k}.$$

Also, $S \vdash [P < NP]^k \leftrightarrow [\ \mathsf{f}_c^k \text{ is total}]$. Our analysis will give estimates for the growth rate of each counterexample function f_c^k.

We can state, for total f_c^k, the following proposition.

Proposition F.4. *For each j, there is a k, $k > j+1$, so that S proves the sentence "f_k doesn't dominate the* BGS^k *counterexample function f_c^k."*

A caveat: we cannot conclude that "for all j, we have that..." since that would imply that S proves "for all j, f_j is total" as a scholium, which cannot be done (as that is equivalent to "F_S is total," which again cannot be proved in S).

What can be concluded: let S' be the theory $S + \mathsf{F}_S$ is total. Then, we have the following proposition.

Proposition F.5. *If S is consistent and if f_c^k is total in a model with standard arithmetic for each k, then S' proves: there is no proof of the totality of f_c^k, any k, in S.*

Proof. See the discussion above. □

Remark F.6. Notice that:

- $S' \vdash \forall k\,([P < NP]^k \leftrightarrow [\mathsf{f}_c^k \text{ is total}])$, while S cannot prove it.
- $S' \vdash \forall k\,([P < NP]^k \leftrightarrow [P < NP])$ while again S cannot prove it.
- S' is $S + [S \text{ is } \Sigma_1 \text{ sound}]$.

Remark F.7. It means that we can conclude:

S' proves that, for every k, S cannot prove $[P < NP]^k$.

Now: does the $[P < NP]^k$ adequately translate our main question?

Remark F.8. Notice that theory S + "F_S is total" is the same as theory S + "S is Σ_1-sound."

Now forget about the technicalities and ponder for a while those results: waving hands, they mean that:

- S is totally blind with respect to some property X within itself.
- While S', which includes S, is positive about the following: S' asserts that there is no X within S.

Yet, if in fact no X exists within S, that remains an open question.

Appendix G. Discussion and more intuitions

f and the infinitely many f_c are very peculiar objects. They are fractal-like in the following sense: the essential data about NP-complete questions is reproduced mirror-like in each of the f (or over each BGS^k). The different BGS^k are distributed over the set of all Turing machines by the primitive recursive function $\mathsf{c}(m, k, a)$.

Also we cannot argue within S that for all k, f_k dominates ..., as that would imply the totality of the recursive function F_S.

It is interesting to keep in mind a picture of these objects. First notice that the BGS and BGS^k machines are interspersed among the Turing machines. The quasi-trivial Turing machines have their Gödel numbers given by the primitive recursive function $\mathsf{c}(k, n)$ — we forget about the other parameters — where:

- k refers to f^k and to BGS^k as already explained;
- n is the argument in $\mathsf{f}^k(n)$.

So fast-growing function f^k is sort of cloned among the values of the BGS^k counterexample function while slightly slowed down by c. (Recall that c is primitive recursive, and cannot compete in growth power with the f^k.)

Function f^k compresses what might be a very large number into a small code given by the Gödel number of g^k and by n (recall that the length of $\mathsf{f}^k(n)$ is the order of $\log \mathsf{f}^k(n)$). The effect is that all functions $\mathsf{f}^j, j < k$ embedded into the k-counterexample function via our quasi-trivial machines keep their fast-growing properties and allow us to prove that the counterexample function is fast-growing in its peaks for BGS^k.

For $j > k$, the growth power of f^k doesn't compensate the length of the parameters in the bounding polynomial that regulates the coupled clock in the BGS^k machines.

Finally while $j < k$, the compressed Gödel numbers of the quasi-trivial machines — they depend on the exponent and constant of the polynomial $x^{\mathsf{f}^k(n)} + \mathsf{f}^k(n)$ which regulates the clock — grow much slower that the growth rate of the counterexample function over these quasi-trivial machines (depending on f^j) and so their fast growing properties come out clearly.

A final remark: we stress here that something may look correct if we look at it, say, with naïve eyes. Yet it may be the case that the landscape perceived from within a formal framework like our S or S'

is wildly counterintuitive, and contradicts the naïve intuition. This seems to be what happens here.

Appendix H. An application: Maymin's theorem

A brief scenario

We start here from a recent intriguing result by Maymin (see Doria and Cosenza, 2016; Maymin, 2011), which relates efficient markets to the P vs. NP question.[15] Roughly, a Maymin market is a market coded by a Boolean expression. We are going to make some move in the market. Our move now is determined by a series of k previous moves.

Definition H.1.

- A k-run policy σ_k, k a positive integer, is a series of plays (b for buy and s for sell) of length k. There are clearly 2^k possible k-run policies.
- A map v from all possible k-run policies into $\{0, 1\}$ is a valuation; we have a "gain" if and only if $v(\sigma_k) = 1$; a "loss" otherwise.
- A policy is *successful* if it provides some gain (adequately defined); in that case we put $v(\sigma_k) = 1$. Otherwise $v(\sigma_k) = 0$.

There is a natural map between these objects and k-variable Boolean expressions (see below), if we take that $v(\sigma_k) = 1$ means that σ_k is satisfiable, and 0 otherwise. We say that a *market configuration* (k-steps market configuration, or simply k-market configuration) is coded by a Boolean expression in disjunctive normal form (dnf). That map between k-market configurations and k-variable Boolean expressions in dnf can be made one-to-one. The financial game for our simplified market is simple: we wish to discover the fastest way to algorithmically obtain a successful k-market configuration, given a particular market (i.e., a given k-variable Boolean expression). Finally the k-market configurations are Maymin-efficient (see below) if v can be implemented by a poly algorithm.

[15] Based on "A beautiful theorem," already cited.

We restrict our analysis to the so-called *weakly efficient* markets. Since one adds the condition that there is a time-polynomial algorithmic procedure to spread the data about the market, we name those markets *Maymin-efficient markets*, where (we stress) $v(\sigma_k)$ is computed by a time-polynomial Turing machine (or poly-machine).

So the existence of general poly procedures characterizes the market as *Maymin efficient*. Now, we have the following Maymin's theorem.

Proposition H.2. *Markets are* (*Maymin*) *efficient if and only if* $P = NP$.

Now we put: markets are *almost Maymin-efficient* if and only if there is an O'Donnell algorithm to determine its successful policies (see da Costa and Doria, 2016). Then, we have the following proposition.

Proposition H.3. *If* $P < NP$ *isn't proved by primitive recursive arithmetic then there are almost Maymin-efficient markets.*

Appendix I. Details

The main motive is very simple: we are going to code Maymin-efficient markets as Boolean expressions. We use a result by E. Post. The 2^k binary sequences naturally code integers from 0 to $2^k - 1$; more precisely, from

$$000\ldots 00, k \text{ digits},$$

to:

$$111\ldots 11, k \text{ digits}.$$

Fix one such coding; a k-digit binary sequence is seen as a sequence of truth values for a Boolean expression Ξ_k.

Proposition I.1. *Let* ξ_k *be a binary sequence of length* 2^k. *Then there is a Boolean expression* Ξ_k *on* k *Boolean variables so that* ξ_k *is its truth table.*

(We take 1 as "true" and 0 as "false."). The idea of the proof goes as follows:

$$\neg p_1 \wedge p_2 \wedge p_3 \wedge \neg p_4 \wedge \neg p_5$$

is satisfied by the binary 5-digit line:

$$01100$$

(When there is a $\neg$ in the conjunction put 0 in the line of truth-values; if not put 1.)

Trivially every k-variable Boolean expression gives rise to a 2^k-length truth table which we can code as a binary sequence of, again, size 2^k bits. The converse result is given by Post's theorem.

Sketch of proof. Consider the k-variable Boolean expression:

$$\zeta = \alpha_1 p_1 \wedge \alpha_2 p_2 \wedge \ldots \wedge \alpha_k p_k,$$

where the α_i are either nothing or $\neg$. Pick up the line of truth values $\zeta' = \alpha_1 \alpha_2 \ldots \alpha_k$, where "nothing" stands for 1 and $\neg$ for 0. ζ' satisfies ζ, while no other line of truth values does. Our Boolean expression ζ is satisfied by ζ' and by no other k-digit line of truth values.

The disjunction $\zeta \vee \xi$ where ξ is a k-variable Boolean expression as ζ, is satisfied by (correspondingly) two lines of truth values, and no more. And so on.

The rigorous proof of Post's theorem is by finite induction. $\square$

Definition I.2. The Boolean expression in dnf ζ is identified to a Maymin k-market configuration.

Proposition I.3. *There are Maymin-efficient markets if and only if $P = NP$.*

Proof. Such is the condition for the existence of a poly algorithmic map v. $\square$

Appendix J. The O'Donnell algorithm

We are now going to describe O'Donnell's algorithm (see da Costa and Doria, 2016); the O'Donnell algorithm is a quasi-polynomial algorithm for SAT.[16] We require the so-called BGS set of poly machines and f_c, which is the (now recursive) counterexample function to $[P = NP]$ (see Baker *et al.* (1975) and da Costa and Doria (2016) for details.)

Recall that a BGS machine is a Turing machine $\mathsf{M}_n(x)$ coupled to a clock that stops the machine when it has operated for $|x|^p + p$ steps, where x is the binary input to the machine and $|x|$ is its length in bits; p is an integer ≥ 1. Of course, the coupled system is a Turing machine. All machines in the BGS set are poly machines, and given any poly machine, there will be a corresponding machine in BGS with the same output as the original poly machine.

Again, f_c is the recursive counterexample function to $P = NP$. To get it:

- Enumerate all BGS machines in the natural order (one can do it, as the BGS set is recursive).
- For BGS machine P_n, $\mathsf{f}_c(n)$ equals the first instance of SAT, which is input to P_n and fails to output a satisfying line for that instance of SAT.

O'Donnell's algorithm is very simple: we list in the natural ordering all BGS machines. Given a particular instance $x \in \text{SAT}$, we input it to $\mathsf{P}_1, \mathsf{P}_2, \ldots$ up to the moment when the output is a satisfying line of truth values. When we compute the time bound to that procedure, we see that it is near polynomial, that is, the whole operation is bounded by a very slow-growing exponential.

Now some requirements:

- We use the enumeration of finite binary sequences

  $$0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots.$$

  If FB denotes the set of all such finite binary sequences, form the standard coding $FB \mapsto \omega$ which is monotonic on the length of the binary sequences.

[16] Actually we deal with a slightly larger class of Boolean expressions.

- We use a binary coding for the Turing machines, which is also monotonic on the length of their tables, linearly arranged, that is, a 3-line table s_1, s_2, s_3, becomes the line $s_1 s_2 s_3$.
  We call such monotonic codings *standard codings*.
- We consider the set of all Boolean expressions in cnf,[17] including those that are unsatisfiable, or totally false. We give it the usual coding, which is one-to-one and onto ω.
- Consider the poly Turing machine $\mathsf{V}(x, s)$, where $\mathsf{V}(x, s) = 1$ if and only if the binary line of truth values s satisfies the Boolean cnf expression x, and $\mathsf{V}(x, s) = 0$ if and only if s doesn't satisfy x.
- Consider the enumeration of the BGS (see Baker *et al.*, 1975) machines, $\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \ldots$.[18]

We start from x, a Boolean expression in cnf binarily coded:

- Consider x, the binary code for a Boolean expression in cnf form.
- Input x to $\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \ldots$ up to the first P_j so that $\mathsf{P}_j(x) = s_j$ and s_j satisfies x (that is, for the verifying machine $\mathsf{V}(x, s_j) = 1$).
- Notice that there is a bound $\leq j = \mathsf{f}_c^{-1}(x)$.
  This requires some elaboration. Eventually a poly machine (in the BGS sequence) will produce a satisfying line for x as its output given x as input. The upper bound for the machine with that ability is given by the first BGS index so that the code for x is smaller than the value at that index of the counterexample function.
  That means: we arrive at a machine M_m, which outputs a correct satisfying line up to x as an input, and then begins to output wrong solutions.
- Alternatively check for $\mathsf{V}(x, 0)$, $\mathsf{V}(x, 1)$, $\ldots$ up to — if it ever happens — some s so that $\mathsf{V}(x, s) = 1$; or,

[17] Conjunctive normal form.

[18] Have in mind that the BGS machine set is a set of time-polynomial Turing machines, which includes algorithms that mimic all time-polynomial Turing machines. See above and check (see Baker *et al.*, 1975).

- Now, if f_c is fast-growing, then as the operation time of P_j is bounded by $|x|^k + k$, we have that $k \leq j$, and therefore it grows as $O(\mathsf{f}_c^{-1}(x))$. This will turn out to be a very slowly growing function. Again this requires some elaboration. The BGS machines are coded by a pair $\langle m, k \rangle$, where m is a Turing machine Gödel index, and k is as above. So we will have that the index j by which we code the BGS machine among all Turing machines is greater than k, provided we use a monotonic coding.
  More precisely, it will have to be tested up to j, that is the operation time will be bounded by $\mathsf{f}_c^{-1}(x)(|x|^{\mathsf{f}_c^{-1}(x)} + \mathsf{f}_c^{-1}(x))$.
  Again notice that the BGS index $j \geq k$, where k is the degree of the polynomial clock that bounds the poly machine.

Appendix K. Almost Maymin-efficient markets

More on almost Maymin-efficient markets.

For a theory S with enough arithmetic — we leave it vague — and with a recursively enumerable set of theorems, for any provably total recursive function h there is a recursive, total, function g so that g dominates h.

Suppose now that we conjecture: the formal sentence $P < NP$ isn't proved by primitive recursive arithmetic. Then the counterexample function f_c will be at least of the order of growth of Ackermann's function.

Given that condition, we can state:

Proposition K.1. *If $P < NP$ isn't proved by primitive recursive arithmetic then there are almost Maymin-efficient markets.*

Notice that we require very little in our discussion — main tool is Post's theorem.

Appendix L. A wild idea: inaccessible cardinals?

We conjecture: the existence of some of those fast-growing functions is dependent on inaccessible cardinals. Recall that a strongly inaccessible cardinal λ satisfies:

(1) $\lambda > \omega$.
(2) If α is a cardinal and $\alpha < \lambda$, then $2^\alpha < \lambda$.
(3) For every family $\beta_i, i \in \iota, \iota < \lambda$, and for each $i, \beta_i < \lambda$, then $\sup_i(\beta_i) < \lambda$.

Let Consis(T) mean the (usual) sentence that asserts that theory T is consistent. Let Card(λ) mean that λ is a cardinal, and let $\mathrm{SInac}_T(\lambda)$ mean that λ is strongly inaccessible for theory T. Finally let F be the fast-growing, partial recursive function that appears in the exotic formulation:

Proposition L.1. *There is a* λ_F *so that*:

$$(\mathrm{ZFC} + [\mathsf{F} \textit{ is total}]) \vdash \mathrm{Card}(\lambda) \wedge \mathrm{SInac}_{\mathrm{ZFC}}\lambda_\mathsf{F}).$$

Tentative sketch of proof, to be further developed.

(1) As we suppose that Consis(ZFC + [F is total]) holds, then it has a model $\mathbf{M}$.
(2) Now, ZFC + [F is total] $\vdash$ Consis (ZFC).
(3) It is a theorem of ZFC that:

$$\mathrm{Consis}(\mathrm{ZFC}) \leftrightarrow \exists x[x \models \mathrm{ZFC}].$$

(We can also take this as a definition for Consis(T).
(4) Given that:

$$\mathrm{ZFC} + [\mathsf{F} \text{ is total}] \vdash \mathrm{Consis}\,(\mathrm{ZFC}),$$

there is a set $x \in \mathbf{M}$ that is a model for ZFC.
(5) Write $\mathbf{V}_\lambda = \mathbf{M} - x$.
(6) Since $\mathbf{V}_\lambda$ is nonempty and as the axiom of choice holds, there are ordinals in it.
(7) Therefore, there is at least a cardinal in $\mathbf{V}_\lambda$.
(8) Pick up the smallest of such cardinals; note it λ:
    (a) One easily sees that for each cardinal $\alpha \in V$, λ is different from 2^α.
    (b) Also for each sequence β_i, etc., λ is different from $\sup_i \beta_i$.

    (Both conditions hold because if not, λ would be in V.)

(9) Finally for all cardinals $\alpha \in V$, $\lambda > \alpha$. For if not, there would be a $\beta \in V$, and $\lambda < \beta$, and λ would be in V.
This also means that $\mathbf{V}$ is in fact a set, V. $\square$

This argument doesn't show that existence of this particular inaccessible cardinal λ proves $P < NP$; it only shows (or purports to show) that our extended theory ZFC + [F is total] implies the existence of an inaccessible cardinal. One must now show that $\mathsf{f}^*(\aleph_0)$ can be interpreted as an inaccessible cardinal, and that it then implies $P < NP$. We would then have $\lambda \leq \mathsf{f}^*(\aleph_0)$.

References

Abraham, R. and Marsden, J. (1978). *Foundations of Mechanics*, Second Edition. Addison-Wesley.

Adler, A. (1969). Some recursively unsolvable problems in analysis. *Proceedings of the American Mathematical Society*, **22**, 523–526.

Arnol'd, V. I. (1976). Problems of present day mathematics. XVII (Dynamical systems and differential equations). *Proceedings of Symposia in Pure Mathematics*, **28**, 59.

Ash, C. J. and Knight, J. (2000). *Computable Structures and the Hyperarithmetical Hierarchy*. Elsevier.

Baker, T., Gill, J. and Solovay, R. (1975). Relativizations of the $P = ?NP$ question. *SIAM Journal on Computing*, **4**, 431–442.

Balzer, W., Moulines, C. U. and Sneed, J. D. (1987). *An Architectonic for Science: the Structuralist Program*. Synthèse Library, D. Reidel.

Black, M. (1933). *The Nature of Mathematics*, London.

Ben-David, S. and Halevy, S. (s/d). On the independence of P vs. NP. Preprint Technion.

Chaitin, G. J., da Costa, N. C. A. and Doria, F. A. (2011). *Gödel's Way*. CRC Press.

Cho, Y. M. (1975). Higher-dimensional unifications of gravitation and gauge theories. *Journal of Mathematical Physics*, **16**, 2029–2037.

Corson, E. M., (1953). *Introduction to Tensors, Spinors and Relativistic Wave-Equations*. Blackie & Sons.

da Costa, N. C. A. (1980). *Ensaio sobre os Fundamentos da Lógica*. São Paulo: Hucitec-Edusp.

da Costa, N. C. A. and Chuaqui, R. (1988). On Suppes' set-theoretic predicates. *Erkenntnis*, **29**, 95–112.

da Costa, N. C. A. and Doria, F. A. (1991a). Undecidability and incompleteness in classical mechanics. *International Journal of Theoretical Physics*, **30**, 1041–1073.

da Costa, N. C. A. and Doria, F. A. (1991b). Classical physics and Penrose's thesis. *Foundation of Physics Letters*, **4**, 363–373.

da Costa, N. C. A. and Doria, F. A. (1992a). Gödel incompleteness in analysis, with an application to the forecasting problem in the social sciences. *Philosophica*, **50**, 1–24.

da Costa, N. C. A. and Doria, F. A. (1992b). Suppes predicates for classical physics, in J. Echeverría *et al.* (eds.), *The Space of Mathematics*. Berlin: Walter de Gruyter.

da Costa, N. C. A. and Doria, F. A. (1994a). Suppes predicates and the construction of unsolvable problems in the axiomatized sciences. in *Patrick Suppes, Mathematician, Philosopher*. Humphreys, P. (ed.), Kluwer Academic Publishers.

da Costa, N. C. A. and Doria, F. A. (2003/2006). Consequences of an exotic formulation for $P = NP$. *Applied Mathematics and Computation*, **145**, 655–665. Also Addendum. *Applied Mathematics and Computation*, **172**, 1364–1367.

da Costa, N. C. A. and Doria, F. A. (1993a). On Arnol'd's Hilbert symposium problems, in Gottlob, G., Leitsch, A., Mundici, D. (eds.), *Proceedings of the 1993 Kurt Gödel Colloquium: Computational Logic and Proof Theory*. Lecture Notes in Computer Science, vol. 713, p. 152. Springer.

da Costa, N. C. A. and Doria, F. A. (1993b). An undecidable Hopf bifurcation with an undecidable fixed point. Preprint CETMC-19.

da Costa, N. C. A. and Doria, F. A. (1994b). Gödel incompleteness in analysis, with an application to the forecasting problem in the social sciences. *Philosophia Naturalis*, **31**, 1–24.

da Costa, N. C. A. and Doria, F. A. (2003a). Metamathematics of physics. Preprint, IEA-USP.

da Costa, N. C. A. and Doria, F. A. (2005). Computing the future, in K. Vela Velupillai, (ed.), *Computability, Complexity and Constructivity in Economic Analysis*. Blackwell.

da Costa, N. C. A. and Doria, F. A. (2007). Janus-faced physics: on Hilbert's 6th problem, in Calude, C. (ed.), *Randomness and Complexity: from Leibniz to Chaitin*. World Scientific.

da Costa, N. C. A. and Doria, F. A. (2015). Variations on a complex theme. Preprint.

da Costa, N. C. A. and Doria, F. A. (2016). On the O'Donnell algorithm for NP-complete problems. *Review of Behavioral Economics*, **3**, 221–242.

da Costa, N. C. A., Doria, F. A. and Bir, E. (2007). On the metamathematics of the P *vs.* NP question. *Applied Mathematics and Computation*, **189**, 1223–1240.

da Costa, N. C. A., Doria, F. A. and Furtado-do-Amaral, A. F. (1993). *International Journal of Theoretical Physics*, **32**, 2187.

da Costa, N. C. A., Doria, F. A. and de Barros, J. A. (1990). A suppes predicate for general relativity and set–theoretically generic spacetimes. *International Journal of Theoretical Physics*, **29**, 935.

Davis, M. (1982). Hilbert's tenth problem is unsolvable. in *Computability and Unsolvability*, Dover.

DeMillo, R. A. and Lipton, R. J. (1979). Some connections between computational complexity theory and mathematical logic. in *Proceedings of the 12th Annual ACM Symposium on the Theory of Computing*, pp. 153–159.

DeMillo, R. A. and Lipton, R. J. (1980). The consistency of $P = NP$ and related problems with fragments of number theory. in *Proceedings of 12th Annual ACM Symposium on the Theory of Computing*, pp. 45–57.

Dirac, P. A. M. (1967). *The Principles of Quantum Mechanics*. Oxford University Press.

Doria, F. A. (1975). A Dirac-like equation for the gravitational field. *Lett Nuovo Cimento*, **14**, 480–483.

Doria, F. A. (1977). A Lagrangian formulation for noninteracting high–spin fields. *Journal of Mathematical Physics*, **18**, 564–571.

Doria, F. A. (2000). Is there a simple, pedestrian, arithmetic sentence which is independent of ZFC? *Synthese*, **125**, # 1/2, 69–79.

Doria, F. A. (2007). Informal vs. formal mathematics. *Synthèse*, **154**, 401–415.

Doria, F. A. (2016). A monster lurks in the belly of complexity theory. Preprint.

Doria, F. A., Furtado do Amaral, A. F. and Abrahão, S. M. (1986). A Dirac–like equation for gauge fields. *Progress of Theoretical Physics*, **75**, 1440–1446.

Doria, F. A. (2014). *Physics: A Primer*, Klam/LabFuzzy.

Doria, F. A. and Cosenza, C. A. (2016). A beautiful theorem. in Akama, S. (ed.), *Towards Paraconsistent Engineering*. Springer.

Emch, G. (1972). *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*. Wiley.

Einstein, A. (1967). *The Meaning of Relativity*. Methuen.

Einstein, A. *et al.* (s/d). *The Principle of Relativity*. Dover.

Gödel, K. (1931). Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I. *Monatshefte für Mathematik und Physik*, **38**, 173–198.

Goel, N. S., Maitra, S. C. and Montroll, E. W. (1971). On the Volterra and other nonlinear models of interacting populations. *Reviews of Modern Physics*, **44**, 231–276.

Guckenheimer, J. and Holmes, P. (1983). *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. Springer.

Hertz, H. (1956). *The Principles of Mechanics*. Translated by Jones, D. E. and Walley, J. T. Dover Publications.

Hirsch, M. (1985). The chaos of dynamical systems. in Fischer, P. and Smith, W. R. (eds.), *Chaos, Fractals and Dynamics*. Marcel Dekker.

Holmes, P. J. and Marsden, J. (1982). Horseshoes in perturbations of Hamiltonian systems with two degrees of freedom. *Communications in Mathematical Physics*, **82**, 523–544.

Jackson, J. D. (1962). *Classical Electrodynamics*. Wiley.

Jeans, J. H. (1925). *Mathematical Theory of Electricity and Magnetism*. Cambridge University Press.

Kleene, S. C. (1936). General recursive functions of the natural numbers. *Mathematische Annalen*, **112**, 727–742.

Kreisel, G. (1976). A notion of mechanistic theory. in Suppes, P. (ed.), *Logic and Probability in Quantum Mechanics*, Reidel, D.

Lanczos, C. (1977). *The Variational Principles of Mechanics*. Univeristy of Toronto Press.

Lewis, A. A. (1991a). On turing degrees of walrasian models and a general impossibility result in the theory of decision making. Preprint, University of California at Irvine, School of Social Sciences.

Lewis, A. A. and Inagaki, Y. (1991b). On the effective content of theories. Preprint, University of California at Irvine, School of Social Sciences.

Lichtenberg, A. J. and Lieberman, M. A. (1983). *Regular and Stochastic Motion*. Springer.

Lotka, A. J. (1956). *Elements of Mathematical Biology*. Dover.

Maymin, P. (2011). Markets are efficient if and only if $P = NP$. *Algorithmic Finance*, **1**(1), 1–22.

Machtey, M. and Young, P. (1979). *An Introduction to the General Theory of Algorithms*. North-Holland.

MacLane, S. (1968). *Geometrical Mechanics*. Monograph, Department of Mathematics, University of Chicago.

Maxwell, J. C. (1954). *A Treatise on Electricity and Magnetism, I and II*. Dover.

Nicolis, G. and Prigogine, I. (1977). *Self-organization in Nonequilibrium Systems*. Wiley.

O'Rahilly, A. (1965). *Electromagnetic Theory, I and II*. Dover.

Penrose, R. (1989). *The Emperor's New Mind*. Oxford University Press.

Richardson, D. (1968). Some undecidable problems involving elementary functions of a real variable. *Journal of Symbolic Logic*, **33**, 514–520.

Rogers, Jr. (1967). *Theory of Recursive Functions and Effective Computability*. McGraw-Hill.

Ruelle, D. and Takens, F. (1971). On the nature of turbulence. *Communications in Mathematical Physics*, **20**, 167–192.

Samuelson, P. A. (1967). *Foundations of Economic Analysis*, Atheneum.

Scarpellini, B. (1963). Zwei unentscheidbare probleme der analysis. *Z. Math. Logic & Grudl. Math.*, **9**, 265–289.

Seligman, B. B. (1971). *Main Currents in Modern Economics, I–III*. Chicago: Quadrangle Books.

Stewart, I. (1991). Deciding the undecidable. *Nature*, **352**, 664–665.

Sternberg, S. (1964). Deciding the undecidable. *Lectures on Differential Geometry*. Benjamin.

Stratton, J. A. (1941). *Electromagnetic Theory*. McGraw-Hill.

Suppes, P. (1967). *Set-Theoretical Structures in Science*. Mimeo. Stanford University.

Suppes, P. (1988). *Scientific Structures and their Representation*. Preliminary Version, Stanford University.

Tonnelat, M. A. (1965). *Les Théories Unitaires de l'Électromagnétisme et de la Gravitation*. Gauthier-Villars.

Tsuji, M., da Costa, N. C. A. and Doria, F. A. (1998). The incompleteness of the theory of games. *Journal of Philosophical Logic*, **27**, 553–568.

Utiyama, R. (1956). Invariant theoretical interpretation of interaction. *Physics Review*, **101**, 1597–1607.

van der Waerden, B. L. (ed.) (1968). *Sources of Quantum Mechanics*. Dover.

Weyl, H. (1952). *Space Time Matter*. Dover.

Wolfram, S. (1984). Computation theory of cellular automata. *Communications in Mathematical Physics*, **96**, 15.