Methods of Mathematical Physics

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Conventions

- In view of the lack of quality of the author, this review handout is for reference only, please contact the author in time if there are any errors(QQ mailbox:1446507095@qq.com) .
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1 Basic Knowledge Points

Exam Question Type:

I.Writing Definite Solution Problems (9 points total: 4 points + 5 points) Exercise 1 (Questions 4 and 5)

II.Fill in the blanks (10 points: 2 points each) Basic knowledge, expansion, intrinsic value problems, traveling wave method, etc.

III.single-choice questions (8 points, 2 points each) basic knowledge, classification of equations, power functions.

IV.Simplification (to standard form) (10 points) Example 1.4 (Example 4), traveling wave method.

V.Legendre polynomials (10 points total) Find P1(x), P2(x), and P3(x), expanding, as in Example 6.3 (Example 3)

VI. Find the intrinsic values and intrinsic functions of the following problems (11 points total) Exercise 2 (14(1), 14(3))

VII. Solve the following fixed-solution problems (42 points in total, 14 points each) Example 2.1 (Example 1), Example 2.3 (Example), Example 2.5 (Example 2), Exercise 2 (Question 4), "Trial and Error Method"

2 Reference Solution To The Exercises After Class

Definition 2.1. Euler equation

An equation of the form:

$$x^n y^{(n)} + P_1 x^{n-1} y^{(n-1)} + \cdots + P_{n-1} x y^{'} + P_n y = f(x) \tag{1}$$

(where $P_1, P_2, \cdots, P_n$is a constant), is known as **Euler's equation**.
Here, we might as well make $x = e^t$,then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \cdot \frac{dy}{dt} \tag{2}$$

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} = \frac{d(\frac{1}{x}\frac{dy}{dt})}{dx} = \frac{-\frac{1}{x^2} \cdot \frac{dy}{dt} dx + \frac{1}{x} \cdot d(\frac{dy}{dt})}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x} \cdot \frac{d(\frac{dy}{dt})}{dt} \cdot \frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} \\ + \frac{1}{x^2}\frac{d^2y}{dt^2} = \frac{1}{x^2}(\frac{d^2y}{dt^2} - \frac{dy}{dt})\end{aligned} \tag{3}$$

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}) \tag{4}$$

If the notation D is used at this point to denote the operation of derivation from t, i.e., $\frac{d}{dt}$ denotes.
So we can get

$$\begin{aligned} xy' &= Dy, \\ x^2y'' &= D(D-1)y, \\ x^3y''' &= D(D-1)(D-2)y \end{aligned} \tag{5}$$

Generally,

$$x^k y^{(k)} = D(D-1)(D-2)\cdots(D-k+1)y \tag{6}$$

3 Reference Sample Volume

The following will be a reference answer to the sample volume of mathematical equations in 2023.

3.1 Definitive Soluton Questions

Proposition 3.1. Write a solution to the problem

Question 1: Homogeneous thin rod of length d, side insulation; One end of the rod ($x = 0$) has a constant temperature of zero and the other end ($x = d$) there is a constant heat flow q flowing out of the rod; The initial temperature distribution of the rod is $9x(d-x)$ and is set within the rod.There is no heat source. Try to write out the corresponding solution problem.

Proof.

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} \quad (0 < x < d, t > 0)\,, \\ u\Big|_{t=0} = 9x(d-x) \\ u\Big|_{x=0} = 0\,, \quad k\dfrac{\partial u}{\partial x}\Big|_{x=d} = -q. \end{cases} \tag{7}$$

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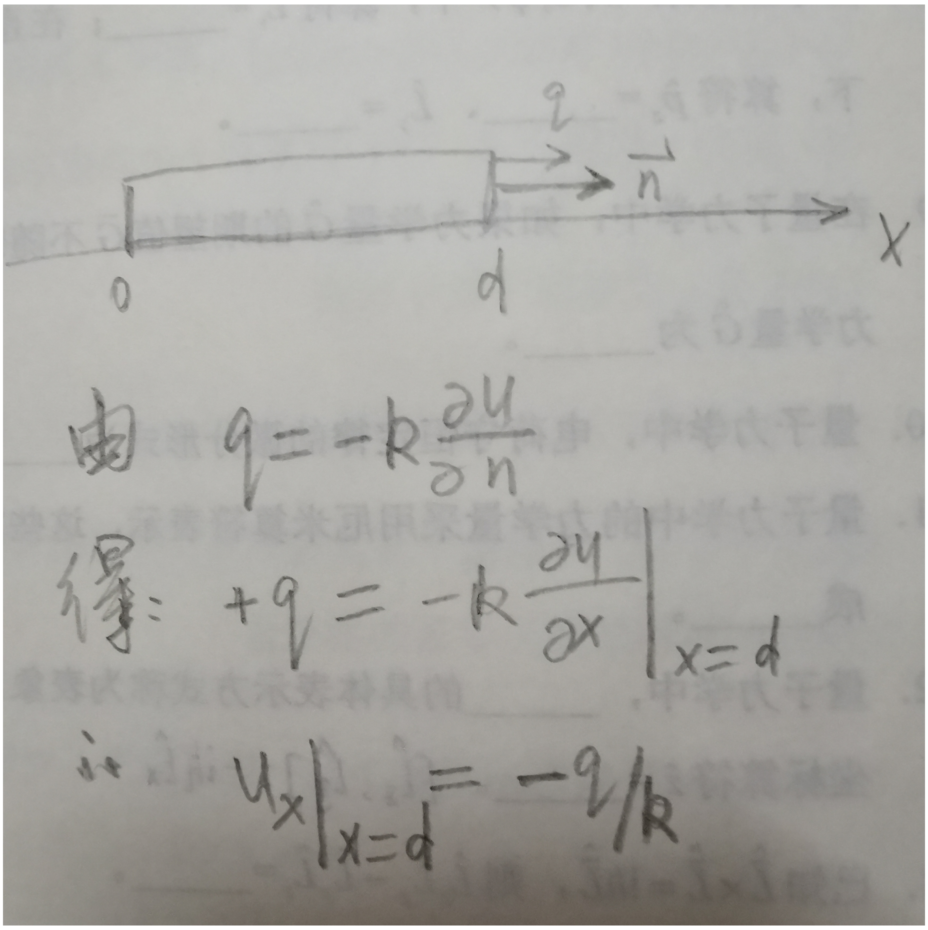

Figure 1: Example figure

Remark.
- In the same direction as the x-positive direction, take the “+” sign,finally "-".
- In the opposite direction to the x-positive direction, the “-” sign is taken,finally "+".

3.2 Fill-in-the-blank Questions

Proposition 3.2

Question 2: The **existence, uniqueness, and stability** of the solution problem are collectively referred to as the suitability of the solution problem.

Proposition 3.3

Question 3: The physical conditions that are met by the physical quantities that characterize a process at the boundaries of the system are called **boundary conditions**.

Proposition 3.4. The Hermitian polynomial

Question 4: The function $f(x) = 7 + 3x$ expands into a series by the Hermitian polynomial, which can be expressed as?(The differential representation of the Hermitian polynomial $H_n(x)$ is known to be $H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2}$.)

Theorem 3.5. The Hermitian polynomial

$$H_n(x) = \sum_{m=0}^{[\frac{n}{2}]} (-1)^m \frac{n!}{m!\,(n-2m)!}(2x)^{n-2m}. \tag{8}$$

The above equation is the nth order Hermitian polynomial.The first six concrete expressions for $H_n(x)$ are

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= 2x, \\ H_2(x) &= 4x^2 - 2, \\ H_3(x) &= 8x^3 - 12x, \\ H_4(x) &= 16x^4 - 48x^2 + 12, \\ H_5(x) &= 32x^5 - 160x^3 + 120x. \end{aligned} \tag{9}$$

Proof. Obviously, by comparing the first six formulas(equations 9), we can get

$$f(x) = 7 + 3x = 7H_1(x) + \frac{3}{2}H_2(x)$$

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Definition 3.6. D'Alembert's formula

Question 5:D'Alembert's formula for the free vibration of an infinitely long string is

$$u(x,t) = \frac{1}{2}\Big[\varphi(x-at) + \varphi(x+at)\Big] + \frac{1}{2a}\Big]_{x-at}^{x+at} \psi(x)\mathrm{d}x\,(a>0)\,, \tag{10}$$

where the law of vibration described by the function $\varphi(x+at)$ is called the **left** propagating wave.

Remark. Left plus right minus.

Proposition 3.7. Hybrid problem

Question 6:A problem that consists of **boundary conditions and initial conditions** is called a hybrid problem.

3.3 Multiple Choice Questions

Question 7:C
Question 8:C A

Definition 3.8. Second-order partial differential equations

According to the textbook, the general formula of the second-order linear bivariate partial differential equation is as follows

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \tag{11}$$

where A, B, C, D, E, F,and G are all functions of x and y, but not u. Equation (11) is non-homogeneous,if $G = 0$, it becomes homogeneous.

Now discussing the classification of equations, we will show that the equation is divided according to the value of $\Delta = B - AC$ into three categories, and can be transformed into the following standard forms:

- $\Delta > 0$,Hyperbolic equations.
- $\Delta = 0$,Parabolic equations.
- $\Delta < 0$,Ellipse equations.

Question 9:C
Question 10:A

Proof.
①$\lambda < 0$

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} \tag{12}$$

$$\Rightarrow A = B = 0 \tag{13}$$

$$X(x) \equiv 0$$

.

②$\lambda = 0$,$X(x) = Ax + B$

$$\Rightarrow A = B = 0 \tag{14}$$

③$\lambda > 0$

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \tag{15}$$

$$\Rightarrow A = 0, 3\sqrt{\lambda} = \left(\frac{2n+1}{2}\right)\pi, n = 0, 1, 2, \cdots \tag{16}$$

$$\lambda = \left[\frac{(2n+1)}{6}\pi\right]^2 \tag{17}$$

$$\therefore X(x) = B\sin\left[\frac{(2n+1)\pi}{6}x\right] \tag{18}$$

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3.4 Legend's Polynomial

Question 11: The differential expression of the Legendre polynomial $P_n(x)$ is known to be

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n, \tag{19}$$

1. Try $P_1(x), P_2(x)$.
2. Try to expand the function $f(x) = 7 + 2x + 8x^2(-1 < x < 1)$ into a Fourier-Legendre series.

Theorem 3.9. The Legende polynomial

$$P_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]}(-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!}\,x^{n-2m} \tag{20}$$

where $[\frac{n}{2}]$ is the largest integer not greater than $\frac{n}{2}$. This polynomial is called the nth Legende polynomial. In particular, when $n = 0, 1, 2, 3, 4, 5$, gets

$$\begin{aligned} &P_0(x) = 1, && P_1(x) = x, \\ &P_2(x) = \frac{1}{2}(3x^2-1), && P_3(x) = \frac{1}{2}(5x^3-3x), \\ &P_4(x) = \frac{1}{8}(35x^4-30x^2+3), && P_5(x) = \frac{1}{8}(63x^5-70x^3+15x). \end{aligned} \tag{21}$$

Proof. 2.Obviously, it can be obtained $f(x) = 7 + 2x + 8x^2 = \frac{16}{3}P_2(x) + 2P_1(x) + \frac{29}{3}P_0(x)$

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3.5 Simplification of Questions

Question 12: Try to convert Equation $u_{tt} - 9u_{xx} + 2x + 5 = 0$ into a standard form.

Proof. $\because \Delta > 0$,Hyperbolic equations.
Its characteristic equation is

$$\left(\frac{dx}{dt}\right)^2 - 9 = 0 \tag{22}$$

Here we set:$\xi = x - 3t, \eta = x + 3t$,then $x = \frac{\xi+\eta}{2}$

$$\begin{aligned} &u_x = u_\xi + u_\eta, \\ &u_t = -3u_\xi + 3u_\eta, \\ &u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\ &u_{tt} = 9u_{\xi\xi} - 18u_{\xi\eta} + 9u_{\eta\eta}. \end{aligned} \tag{23}$$

$$\Rightarrow -36u_{\xi\eta} + \xi + \eta + 5 = 0 \tag{24}$$

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3.6 Intrinsic Values and Intrinsic Functions

Question 13:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = X(7) = 0. \end{cases} \tag{25}$$

Proof. Similar to **Question 10**, only the answer is given here.

$$\lambda = \left[\frac{(2n+1)}{14}\pi\right]^2, X(x) = A\cos\left[\frac{(2n+1)\pi}{14}x\right], n = 0,1,2,\cdots \tag{26}$$

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3.7 Solve The Following Solution Problems

Question 14:

$$\begin{cases} u_{tt} = a^2 u_{xx}\ (0 < x < 3,\ t > 0), \\ u(0,\ t) = u(3,\ t) = 0, \\ u(x,\ 0) = 6\sin(3\pi x),\ u_t(x,\ 0) = 5\,(3-x)\,. \end{cases} \tag{27}$$

Proof.

$$u(x,t) = X(x)\cdot T(t)$$

$$\begin{cases} T''(t) + \lambda a^2 T(t) = 0 \\ X''(x) + \lambda X(x) = 0 \end{cases}$$

$\lambda > 0$

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \tag{28}$$

$$\Rightarrow A = 0, X(3) = B\sin 3\sqrt{\lambda} = 0 \tag{29}$$

$$\lambda = \left(\frac{n\pi}{3}\right)^2, n = 1,2,3,\cdots$$

$$\therefore X_n(x) = B_n\sin\left(\frac{n\pi x}{3}\right), n = 1,2,\cdots$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi at}{3}\right) + b_n\sin\left(\frac{n\pi at}{3}\right)\right)\sin\left(\frac{n\pi x}{3}\right) \tag{30}$$

$$\begin{aligned} a_n &= \begin{cases} 6, n = 9 \\ 0, n \neq 9. \end{cases} \\ b_n\frac{n\pi a}{l} &= \frac{2}{l}\int_0^l 5(3-x)\sin\left(\frac{n\pi x}{l}\right)dx \quad (l = 3) \\ (note &: \int udv = uv - \int vdu) \\ \Rightarrow b_n &= \frac{90}{n^2\pi^2 a} = \frac{10}{9\pi^2 a} \end{aligned} \tag{31}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty}\left(6\cos(3\pi at) + \frac{10}{9\pi^2 a}\sin(3\pi at)\right)\sin(3\pi x) \tag{32}$$

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Question 15:

$$\begin{cases} u_t = a^2 u_{xx}\ (0 < x < \pi,\ t > 0), \\ u_x(0,\ t) = u_x(\pi,\ t) = 0, \\ u(x,\ 0) = 12 + 4x. \end{cases} \tag{33}$$

Proof.

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}a_n e^{-(\frac{n\pi a}{l})^2 t}\cos\left(\frac{n\pi x}{l}\right) \tag{34}$$

According to the question, here we get $l = \pi$

$$\Rightarrow u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}a_n e^{-(na)^2 t}\cos(nx) \tag{35}$$

Then

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}(12+4x)\cos(nx) = \frac{8}{n^2\pi}(\cos(n\pi) - 1) \\ a_0 &= \frac{2}{\pi}\int_0^{\pi}(12+4x)dx = 24 + 4\pi \end{aligned} \tag{36}$$

$$\therefore u(x,t) = 12 + 2\pi + \sum_{n=1}^{\infty}\left[\frac{8}{n^2\pi}(\cos(n\pi) - 1)\right]e^{-(na)^2 t}\cos(nx) \tag{37}$$

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Question 16:

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 \; (0 < r < R), \\ u\Big|_{r=R} = 21\sin(3\theta) + 17\cos(5\theta). \end{cases} \tag{38}$$

Proof.

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n \tag{39}$$

$$\begin{aligned} a_n &= \frac{1}{\pi r_0^2} \int_0^{2\pi} (21\sin(3\theta) + 17\cos(5\theta)) \cdot \cos(n\theta) d\theta = \begin{cases} \dfrac{17}{R^5} & n = 5, \\ 0 & n \neq 5. \end{cases} \\ b_n &= \frac{1}{\pi r_0^2} \int_0^{2\pi} (21\sin(3\theta) + 17\cos(5\theta)) \cdot \sin(n\theta) d\theta = \begin{cases} \dfrac{21}{R^3} & n = 3, \\ 0 & n \neq 3. \end{cases} \\ &\Rightarrow u(r,\theta) = \frac{21}{R^3} r^3 \sin(3\theta) + \frac{17}{R^5} r^5 \cos(5\theta) \end{aligned} \tag{40}$$

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References

[1] Litskevich, M., Hossain, M.S., Zhang, SB. et al. Boundary modes of a charge density wave state in a topological material. Nat. Phys. (2024). https://doi.org/10.1038/s41567-024-02469-1

Dear readers, if you find any problems and questions, please feel free to contact me, I would appreciate it.
Have fun! 😎