Soft Margin SVM derivation :

Minimize ,

$$\frac{1}{2}\|w\|^2 + C\sum_i \xi_i$$

under the constraints

$$\xi_i \geq 0$$

$$\& \quad y_i(w x_i + b) \geq 1 - \xi_i$$

Proof :

$$L(w, b, \xi_i) = \frac{1}{2}\|w\|^2 + C\sum_i \xi_i - \sum_i \alpha_i \left(y_i(w x_i + b) - 1 + \xi_i\right) - \sum_i \beta_i \xi_i \quad \text{--- (3)}$$

such that

$\alpha_i \geq 0$,

$\beta_i \geq 0$

$$\frac{\partial L}{\partial w} = 0$$

$$\Rightarrow w + 0 - \sum_i \alpha_i y_i x_i = 0$$

$$\Rightarrow \boxed{w = \sum_i \alpha_i y_i x_i} \quad \text{--- (1)}$$

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$$\frac{\partial L}{\partial b} = 0$$

$$\Rightarrow 0 + 0 - \sum_i \alpha_i y_i = 0$$

$$\Rightarrow \boxed{\sum_i \alpha_i y_i = 0} \quad - \ (2)$$

For a particular ξ_i,

$$\frac{\partial L}{\partial \xi_i} = 0$$

$$\Rightarrow 0 + C - \alpha_i - \beta_i = 0$$

$$\Rightarrow \boxed{\alpha_i = C - \beta_i}$$ } For a particular ξ_i, If $\beta_i > 0$ Then $\alpha_i < C$

$\therefore$ $\boxed{0 \le \alpha_i <= C}$

Box constraint

Substituting (1) & (2), in (3) we get :

$$\frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j y_i y_j x_i x_j + C \sum_i \xi_i$$

$$- \sum_i \sum_j \alpha_i \alpha_j y_i y_j x_i x_j$$

$$-b \sum_i \alpha_i y_i = 0 \quad \text{from (2)}$$

$$+ \sum_i \alpha_i$$

$$- \sum_i \alpha_i \xi_i - \sum_i \beta_i \xi_i$$

$$= 0$$

$$\Rightarrow \theta(\alpha, \beta) = \sum \alpha_i - \frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j y_i y_j x_i x_j$$

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From KKT,

$$\boxed{\varepsilon_i \beta_i = 0}$$

WKT, $\alpha_i = c - \beta_i$

$\Downarrow$

if x_i is a support vector

then $\varepsilon_i = 0$

$\Rightarrow \beta_i \neq 0$ and wKT $\beta_i \geq 0$

$\Rightarrow \boxed{\alpha_i < C}$ } for support vectors

if $\varepsilon_i \neq 0$, then $\beta_i = 0$

$\Rightarrow \boxed{\alpha_i = c}$ } for noisy data where $\varepsilon_i \neq 0$