Theorem on Spaces of Continuous Functions

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Definition 1 (Dense set)**.** *Let (X, d) be a metric space. A subset $A \subseteq X$ is said to be dense in X if for every $x \in X$ and every $\epsilon > 0$, there exists an element $a \in A$ such that $d(x, a) < \epsilon$. Equivalently, A is dense in X if for every $x \in X$, there exists a sequence $\{a_n\}$ of elements in A such that $a_n \to x$ as $n \to \infty$.*

Definition 2 (Countable set)**.** *A set A is said to be countable if there exists a bijection between A and the set of natural numbers $\mathbb{N}$.*

Definition 3 (Separable set)**.** *A is separable if there exists a countable set $B \subseteq A$ such that B is dense in A.*

Theorem 1. *The space of all continuous functions on $[a, b]$ is separable.*

Proof. Consider the space of continuous functions (X, d) on $[a, b]$, where $X = C[a, b]$ and $d(f(x), g(x)) = \sup_{x \in [a,b]} |f(x) - g(x)|$. We aim to prove that $C[a, b]$ is separable.

Step 1: Existence of a Dense Subset

By the Stone-Weierstrass theorem, for any continuous function f on $[a, b]$, there exists a sequence of polynomials $\{P_n\}$ in $C[a, b]$ such that $\lim_{n\to\infty} P_n = f(x)$ uniformly on $[a, b]$. This implies that given $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that $|P_n - f(x)| < \epsilon$ for all $n > N(\epsilon)$. Therefore, the set of polynomials is dense in $C[a, b]$.

Step 2: Countability of the Dense Subset

Next, we show that there exists a countable subset of the set of polynomials that is dense in $C[a, b]$. Consider polynomials of the form $P_n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where $a_i \in \mathbb{Q}$. The set of coefficients $(a_0, a_1, a_2, \ldots, a_n) \in \mathbb{Q}^{n+1}$ is countable because $\mathbb{Q}$ is countable in $\mathbb{R}$, and the Cartesian product of countable sets is countable. Thus, the set of polynomials with rational coefficients is countable.

Since the set of polynomials with rational coefficients is dense in the set of all polynomials (which is dense in $C[a, b]$), it follows that the set of polynomials with rational coefficients is dense in $C[a, b]$.

Conclusion Given that we have found a countable dense subset (the set of polynomials with rational coefficients) in $C[a, b]$, we conclude that $C[a, b]$ is separable. □