Markov's Insight: Unveiling Active Views with Hidden Markov Models for Enhanced Black-Litterman Portfolio Optimization

QIUTONG(POLLY) PU, University of California, Berkeley, USA

This study presents the integration of Hidden Markov Models with the Black-Litterman framework as a method for portfolio optimization amidst financial market volatility. It introduces an automated process for real-time market state identification and investment strategy adjustment, leveraging historical returns to improve portfolio management in volatile settings. Utilizing data from July 1993 to July 2023, our findings demonstrate that portfolios optimized using the Hidden Markov-based Black Litterman framework, particularly the Best Sharpe Ratio portfolio, outperform traditional methods. The Best Sharpe Ratio portfolio achieved a 101.7788-fold increase in cumulative return per dollar invested initially and exhibited a Sharpe ratio of 0.9002 after adjusting for volatility to 15%, indicating superior risk-adjusted performance. This research serves as a navigational tool for the complexities of current financial markets and establishes a new benchmark for empirical investment strategy development, offering valuable insights for enhancing Robo-Advisor platforms.

1 INTRODUCTION

In the complex world of finance, where economic indicators frequently present conflicting signals, the challenge of making well-informed decisions intensifies. The Federal Reserve's aggressive interest rate hikes initiated in 2022 underscore the complexity of today's financial landscape. This period marked the most rapid increase in rates since the late 1980s, with a rise of 2.36 percentage points in a remarkably short span [27].

This financial turbulence coincided with significant inflationary pressures. Over the 12 months ending in June 2022, food prices surged by 10.4%, marking the most substantial annual increase since February 1981. Specifically, grocery prices jumped 12.2%, and dining out became 7.7% more expensive, both recording highs not seen for over four decades. The energy sector was hit even harder, with an overall 41.6% increase. Notably, motor fuel prices soared by over 60%, and natural gas prices climbed by 38.4% [4].

Despite these decisive monetary actions by the Federal Reserve, the persistence of high inflation coupled with strong employment figures creates a seemingly paradoxical situation that highlights the unpredictability of financial markets. This dynamic environment complicates both personal and professional decision-making, reflecting the pervasive uncertainty that characterizes the modern economic landscape.

This paper introduces a groundbreaking research project that leverages a Hidden Markov Model-based Black Litterman approach to navigate the quagmire of market signals. By constructing an automated pipeline, this research seeks to discern distinct market regimes and tailor portfolio compositions accordingly, aligning with investors' risk preferences. This initiative does not merely fulfill an academic purpose; it transcends scholarly boundaries to serve an ethical role, particularly salient in the wake of unforeseen financial cataclysms—often termed "black swan" events. The devastating financial toll of such crises, exemplified by the global financial meltdown of 2008 and the economic ramifications of the COVID-19 pandemic [5], highlights the vital role of data-driven decision making.

Ultimately, this project aspires to bolster financial literacy, fostering a culture of resilience and savvy decision-making amid economic adversities. By peeling back the layers of market operations and elucidating the underpinnings of financial turmoil, it aims to empower society at large. In doing so, the research not only contributes to academic discourse but also holds the promise of making a profound impact on the practical world, guiding individuals towards a more informed, secure, and financially literate future.

Author's address: Qiutong(Polly) Pu, pollypqt1101@berkeley.edu, University of California, Berkeley, 110 Sproul Hall, Berkeley, California, USA, 94720.

2 PRIOR WORK

2.1 The Black-Litterman Model: A Paradigm Shift in Portfolio Management

The Black-Litterman (B-L) model, emerging from the seminal work of Fischer Black and Robert Litterman [3], represents a pivotal advancement in portfolio management by addressing the inherent limitations of traditional mean-variance optimization. It was specifically designed to mitigate the model's sensitivity to input changes that often resulted in extreme and impractical portfolio weights. By integrating market equilibrium with subjective investor views, the B-L model offers a more robust framework for asset allocation, ensuring more stable and intuitive portfolio allocations and overcoming the over-reliance on subjective estimates that plagued earlier methods. Their model provided a systematic way to blend market equilibrium views, derived from the Capital Asset Pricing Model (CAPM) [22], with the investor's unique perspectives, offering a more intuitive and diversified portfolio construction method. This groundbreaking approach not only tackled the issues inherent in Harry Markowitz's mean-variance framework [18] but also enhanced the practical applicability of portfolio optimization, marking a departure from the extreme portfolio suggestions often resulting from purely subjective estimations.

Subsequent contributions to the B-L model literature include Thomas Idzorek's practical guide to implementing the B-L model, emphasizing the integration of user-specified confidence levels [8]. Idzorek's work addressed a key limitation of the original model—the quantification of subjective certainty—by allowing for explicit specification of confidence in investor views. Further, He and Litterman [7] expanded on the intuitive appeal and accessibility of the B-L model, making it more understandable and user-friendly for portfolio managers. Erindi Allaj [1] tackled the estimation of the parameter tau (τ) within the B-L model, proposing a consistent and intuitive estimation procedure. This work addressed the ambiguity surrounding τ's calibration, which represents the uncertainty in equilibrium market returns, and presented a new framework that enhances the model's transparency and applicability in real-world portfolio management. Their contributions have significantly influenced the broader adoption and application of the B-L model in investment management.

Bessler et al. [26] provided a comprehensive analysis of the B-L model's performance in multi-asset portfolios, highlighting its superiority in delivering enhanced out-of-sample performance and its robustness across different market conditions and monetary cycles. Their study showcased the B-L model's ability to produce portfolios with lower risk and broader diversification, effectively capturing upside potential relative to downside risk. Sun et al. [29] introduced a novel approach by integrating time-varying covariance into the B-L model for pension fund asset allocation. Their innovative method, leveraging Vector Autoregressive Moving-Average and Generalized Autoregressive Conditional Heteroskedasticity models, allowed for a more accurate and dynamic representation of risk, reflecting the time-varying nature of asset returns. This approach provided significant improvements in asset allocation effectiveness, as evidenced by higher Sharpe ratios and better risk-adjusted returns. Min et al. [19] explored the integration of machine learning algorithms with the B-L model to generate quantitative investor opinions. By harnessing the predictive capabilities of algorithms such as Random Forest and XGBoost, the authors demonstrated the superior performance of the B-L model variants in terms of higher cumulative returns and robust diversification. Their study underscores the potential of sophisticated, data-driven approaches to asset allocation, paving the way for future research in portfolio optimization strategies.

2.2 Hidden Markov Models: Unveiling Hidden States in Financial Markets

The Hidden Markov Model (HMM) has emerged as a versatile and powerful statistical tool for modeling and analyzing sequential data, especially in contexts where the process involves hidden states that are not directly observable. The foundational work by Leonard E. Baum and Ted Petrie [2] laid the theoretical groundwork for HMM, where they introduced algorithms for evaluating the likelihood of a sequence of observations and for uncovering the most probable sequence of hidden states, which have profound implications across various fields, including finance.

Advancements in HMMs have significantly influenced financial time series analysis and portfolio management. Zhang et al. [28] introduced a high-order HMM for financial trend prediction, enhancing traditional models by considering both short and long-term temporal dependencies, thus improving predictive accuracy. De la Torre-Torres et al. [21] further expanded this field by integrating Markov-Switching models with the B-L framework to optimize portfolio selection, demonstrating the efficacy of such combinations in navigating the dynamic and complex financial markets. Additionally, Jadhav et al. [10] showcased the utility of HMMs in forecasting stock prices for FAANG companies, an acronym that refers to the stocks of five prominent American technology companies: Meta, Amazon, Apple, Netflix and Alphabet, underlining the models' ability to decode hidden market dynamics with remarkable precision. Collectively, these studies underscore the evolving complexity and adaptability of HMMs in financial forecasting and portfolio optimization, highlighting their potential to address the nuanced challenges of market prediction and asset allocation.

2.3 Bridging Traditional and Advanced Forecasting Methodologies

The exploration of stock market prediction methodologies has evolved remarkably over the years, transitioning from traditional statistical models such as Autoregressive Integrated Moving Average [11] to more sophisticated machine learning techniques like Support Vector Machine [9, 14] and deep learning approaches, including Recurrent Neural Network [12, 13]. The advent of advanced language models like BloombergGPT [23] further underscores the rapid advancements in automating financial data analysis and trend prediction. Despite these technological strides, the foundational theories of the random walk [16] and the efficient market hypothesis [15] maintain a skeptical stance towards the predictability of stock prices, arguing that market fluctuations are inherently independent and random.

However, the integration of HMM into the realm of financial forecasting presents a compelling counter-narrative to these established theories. HMMs, with their unique ability to model the hidden state dynamics underlying observable financial data, offer a robust framework for capturing the intricate patterns and regime shifts in market behavior that traditional models and even some advanced machine learning algorithms may overlook. This attribute of HMMs to discern the unobservable factors influencing stock prices positions them as an invaluable tool for generating active views in the B-L model, providing a more informed and dynamic basis for portfolio optimization.

3 DATA

In this study, the dataset comprising the 30 Industry Portfolios Monthly Returns, curated by Kenneth R. French, is utilized. This dataset, embodying monthly return data across thirty varied industry sectors, where each column corresponds to the monthly returns of an industry, has been meticulously aggregated to mirror the dynamics of the broader market and the trends specific to each industry. Its utility in this project is paramount, as it allows for the delineation of market condition transitions and their subsequent integration within the B-L framework to optimize portfolio management. The selection of this dataset is predicated on its extensive historical breadth, encompassing several decades, thereby providing a solid basis for the longitudinal analysis of market behavior. Moreover, the dataset's architecture aids in the scrutiny of industry-specific returns, yielding insights into the performance and risk profiles of different sectors. This aspect is especially relevant to our application of a HMM-based approach, wherein the discernment of latent market states is fundamentally dependent on a detailed comprehension of these returns.

To ensure the integrity and reliability of our analysis, the dataset has undergone a rigorous preprocessing phase. To address potential gaps in the dataset, we employed linear interpolation—a method that estimates missing values by drawing on the known values preceding and following the gap. Recognizing the importance of logarithmic returns for financial time series analysis, due to their properties of time-additivity and better handling of volatility, we applied a logarithmic transformation to the returns. This step enhances the model's ability to capture the multiplicative effects of returns over time, providing a more accurate reflection of the compounding nature of financial returns.

In the later stages of our B-L model construction, we integrate two additional datasets to refine our portfolio optimization process and enhance the model's predictive accuracy. The inclusion of the one-month Treasury bill rate, sourced from Ibbotson Associates via the Ken French Database (specifically, the Fama-French 3 factors file), serves a dual purpose. Firstly, it provides a risk-free rate benchmark, which is essential for calculating excess returns over the risk-free rate on monthly data. Secondly, it is a fundamental component in determining expected returns under the B-L framework. This rate is pivotal in adjusting our return expectations to account for the time value of money and risk aversion, thereby aligning our model more closely with the realities of financial markets.

Moreover, the inclusion of Standard & Poor's 500 (S&P 500) monthly returns is crucial for the calculation of market-implied risk aversion—a parameter that reflects the market's collective attitude towards risk. By analyzing the historical returns of the S&P 500, we gain insights into the market's performance over time, allowing us to estimate the level of risk aversion inherent in market pricing. Consequently, this enhances the assumptions of our model concerning the pricing of risk into assets, facilitating a more refined methodology for portfolio optimization. The S&P 500, representing a broad cross-section of the U.S. economy, is an ideal proxy for the market portfolio, making it a suitable choice for these calculations. However, many papers argue the dominance of single stocks may tilt the movement of the index to be biased. We still use it as proxy as it is one of the most widely followed and most quoted indexes [20].

3.1 Explanatory Data Analysis

The efficacy of our statistical models is contingent upon the attributes of return distributions; thus, an initial analysis of the empirical distribution pertaining to the returns of the 30 industries is imperative. This examination, rooted in the scrutiny of empirical returns and augmented by the analytical insights derived from Quantile-Quantile (Q-Q) plots for monthly industry returns, uncovers substantial departures from normal distribution. Illustrations within Figures 1 and 2 selectively highlight these anomalies for the first and last eight industries, selected for illustrative lucidity. The empirical dataset, extending from 1926 to 2023, unequivocally demonstrates that the returns are characterized by non-normal features, particularly evident through elevated kurtosis and nonzero skewness.

The empirical observation intuitively suggests the suitability of a Gaussian Mixture Model (GMM) for our analysis. A GMM, by design, incorporates multiple Gaussian distributions, each defined by unique parameters of mean and variance, offering a versatile framework to model complex, multi-modal distributions. Such adaptability makes GMMs particularly apt for capturing the complex distributional features of financial returns, including the observed fat tails and elevated kurtosis. Moreover, real-world financial markets are often composed of multiple underlying subpopulations, each representing distinct market behaviors or industry characteristics. The GMM approach enables the explicit modeling of these subpopulations through individual Gaussian components, enhancing the model's ability to represent the detailed distributional properties of industry returns.

An essential aspect of customizing the GMM for our financial dataset is the determination of the optimal number of Gaussian components. Selecting three components is strategic, mirroring the tripartite market states typically observed in financial markets: boom, sideways (or stable), and bust. This choice is grounded in both theoretical rationale and empirical evidence, as illustrated in Figure 3. The figure effectively shows that a GMM with three components, indicated by the red curve, aligns more accurately with the return distribution for the period analyzed than does a simplistic normal distribution assumption. Such alignment signals the model's superior capability in capturing the complex dynamics and variability inherent in financial market states. Consequently, the precise fit provided by the GMM underscores the potential benefits of incorporating a Gaussian HMM into our financial data analysis. The GMM's strength in reflecting the empirical distribution of returns lays a robust groundwork for a Gaussian HMM, which can leverage the temporal dynamics of financial states, thereby offering an advanced framework for modeling transitions among boom, sideways, and bust states over time.

The implementation of GMM for forecasting market states within diverse industry sectors has delivered encouraging initial outcomes. To enhance understanding, we present the analysis for the top eight industries in Figure 4. Historical analysis reveals that certain epochs, notably the 1930s and 1970s, were marked by pronounced volatility. Despite the consistency of this pattern across the industry spectrum, its manifestations varied within each sector, highlighting the complexity and diversity of market dynamics amid economic shifts. The adeptness of GMM in classifying these market states, as indicated by the green, purple, and yellow color codes, uncovers the hidden trends inherited in the change of volatility. The potential for an HMM to accurately model the transitions between these states could markedly add a dimension of understanding to market dynamics, where the analysis of existing market conditions and anticipated transitions is essential for active view in the B-L model.

4 METHODOLOGY

In the pursuit of optimizing investment portfolios, the integration of predictive models and advanced portfolio construction methods holds significant promise. This research employs the HMM in conjunction with the B-L to generate active views for maximizing portfolio performance metrics, such as the Sharpe ratio or mean-variance optimization. By leveraging the HMM's ability to detect underlying market states and the B-L's framework for incorporating investor views, this methodology aims to enhance the strategic allocation of assets. This section delineates the theoretical underpinnings, the procedural integration of HMM and B-L, and the extraction of active views, laying a foundation for empirical analysis that seeks to advance portfolio management practices.

4.1 Overview of HMM

The HMM represents a statistical framework based on the assumption that the modeled process is a Markov process [17] with unobserved states. Its application spans diverse fields that necessitate the recognition of temporal patterns, such as speech and handwriting recognition, gesture analysis, part-of-speech tagging, and bioinformatics. Fundamentally, the HMM comprises two critical elements: states and observations. Each state is associated with a distinct probability distribution that dictates possible output observations, and transitions between states are defined by a set of predetermined probabilities.

Central to the utility of the HMM are three foundational challenges: likelihood estimation, state sequence decoding, and model parameter learning. The likelihood of an observation sequence, given the model's parameters, is efficiently computed using the Forward Algorithm. Meanwhile, the Viterbi Algorithm [24] is used in unraveling the most probable sequence of hidden states based on observed data. Lastly, the Baum-Welch Algorithm is instrumental in refining the model by training both the state transition and observation likelihood density, also known as the emission density . The following sections introduce three algorithms that generate inputs for a simple trading algorithm, which in turn serves as input for the B-L model in the subsequent stage.

4.1.1 *Notations and Assumption for HMM.*

- $Q = \{q_i\}, i = 1, \ldots, N$: A set of N states of each industry portfolio return.
- $A = \{a_{ij}\}, i, j = 1, \ldots, N$: The transition probability matrix A, where each a_{ij} represents the probability of transition from state i to state j for each industry portfolio return s.t. $\sum_{j=1}^{n} a_{ij} = 1$.
- $O = \{o_t\}, t = 1, \ldots, T$: Sequence of observations, where o_t is the observed industry portfolio return at month t.
- $B = \{b_i(o_t)\}, i = 1, \ldots, N; t = 1, \ldots, T$: The emission densities, where $b_i(o_t)$ is the probability of observing the industry portfolio return at time t from state j.
- $\pi = \{\pi_i\}, i = 1, \ldots, N$: Initial state distribution, where π_i is the probability that the initial state is i s.t. $\sum_{i=1}^{n} \pi_i = 1$.
- Markov Assumption:

$$\mathbb{P}(q_i | q_{i-1}, q_{i-2} \ldots, q_1) = \mathbb{P}(q_i | q_{i-1}) \tag{1}$$

- Independence Assumption:

$$\mathbb{P}(o_i | q_T, \ldots, q_1, o_T, \ldots, o_1) = \mathbb{P}(o_i | q_i) \tag{2}$$

The HMM is arguably well-suited for analyzing stock market dynamics, based on two fundamental assumptions that offer significant applicability and realism in financial markets. The first, the Markov assumption, posits that the future state of the market depends solely on its current state, which is consistent with the volatile nature of financial markets. It acknowledges that while historical data provide background, the present market state is a primary indicator of near-term trends. The second, the independence assumption, suggests that observed market data at any given time is determined only by the current market state, capturing the direct influence of current events, economic announcements, or company news on market behavior, without reliance on historical data.

Market movements are indeed stochastic, driven by a myriad of factors, including economic indicators, investor sentiment, and global events, which can be effectively modeled as a sequence of discrete states within an HMM framework. In this model, each state can represent a distinct market regime—be it bullish, bearish, or sideways movement—providing a structured approach to decipher the market's complex behavior. The HMM utilizes the Markov assumption to navigate through these states, identifying hidden market regimes and predicting transitions based on observed returns.

4.1.2 *Likelihood Computation: The Forward Algorithm.* The forward algorithm employs a trellis, $\alpha_t(j)$, to denote the likelihood of occupying state j upon observing the initial t specific industry portfolio returns, contingent on the transition matrix A and emission distribution B.

Each cell represents the following probability:

$$\alpha_t(j) = P(o_1, \ldots, o_t, q_t = j | A, B) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(o_t) \tag{3}$$

The Forward Algorithm:

(1) Initialization:

$$\alpha_1(j) = \pi_j b_j(o_1); 1 \leq j \leq N$$

(2) Recursive Steps:

$$\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(o_t); 1 \leq j \leq N, 1 \leq t \leq T \tag{4}$$

(3) Result:

$$\mathbb{P}(O|A, B) = \sum_{i=1}^{N} \alpha_T(i)$$

4.1.3 Decoding: The Viterbi Algorithm. The objective of the decoding process is to ascertain the optimal sequence of hidden states based on the transition matrix A, emission distributions B, and observed data O. This involves determining the likelihood of the Hidden Markov Model being in state j after observing the initial t industry portfolio returns and transitioning through the most probable state sequence $q_1, \ldots, q_{t-1}$. To achieve this, the algorithm constructs a trellis, denoted as $v_t(j)$, which is formally defined as follows:

$$v_t(j) = \max_{q_i; 1 \leq i \leq t-1} P(q_1, \ldots, q_{t-1}, o_1, \ldots, o_t, q_t = j | A, B) = \max_{1 \leq i \leq N} v_{t-1}(i) a_{ij} b_j(o_t) \tag{5}$$

Also, define the best backtrace at time t and state j as $prev_t(j)$.

The Viterbi Algorithm:

(1) Initialization:

$$v_1(j) = \pi_j b_j(o_1); \qquad 1 \leq j \leq N$$

$$prev_1(j) = 0; \qquad 1 \leq j \leq N, 1 \leq t \leq T$$

(2) Recursive Steps:

$$\begin{aligned} v_t(j) &= \max_{1 \leq i \leq N} v_{t-1}(i) a_{ij} b_j(o_t); \quad 1 \leq j \leq N \\ prev_t(j) &= \arg\max_{1 \leq i \leq N} v_{t-1}(i) a_{ij} b_j(o_t); \quad 1 \leq j \leq N \end{aligned} \tag{6}$$

(3) Result:

$$\text{Probability of the Best Backtrace: } \max_{1 \leq i \leq N} v_T(i)$$

$$\text{The Start of the Best Backtrace: } \arg\max_{1 \leq i \leq N} v_T(i)$$

4.1.4 HMM Training: Baum-Welch Algorithm . The previous two stages were predicated on the assumption that the transition matrix A and the emission matrix B were known, which are precisely the parameters the Baum-Welch algorithm seeks to estimate based on observed specific industry portfolio returns.

The learning process involves estimating the parameters that maximize the likelihood of the observed data, given the model. For continuous distributions, the statistical properties, including mean (μ) and covariance matrix (Σ), associated with each hidden state q_i, are updated to reflect the distribution of the observed data most accurately. This process considers the posterior probabilities of the states given the observations, adjusting the means to be the weighted average of the observations O and the covariances to capture the variability and correlation of the feature set within each state. The transition probabilities A between states and the initial state probabilities π are also refined during training. The transition probabilities are updated to reflect the frequency of transitioning between states $\hat{a}_{ij}$ in the observed sequence, taking into account the likelihood of the sequence given the model parameters. The initial state probabilities are adjusted based on the likelihood of each state being the starting point of the observed sequences. The Baum-Welch algorithm is a specific implementation of the expectation–maximization algorithm for HMMs. It iteratively performs two main steps: the Expectation step, which calculates the expected value of the likelihood function with respect to the current estimate of the model parameters, and the Maximization step, which computes parameters that maximize this expected likelihood. This iterative process continues until convergence, defined as when the change in the log-likelihood of the observed data between iterations falls below a predefined threshold.

Define the backward probability β, the probability of observing a sequence of specific industry portfolio returns from time $t+1$ to T if landing in state j at t:

$$\beta_t(i) = \mathbb{P}(o_{t+1}, \ldots, o_T | q_t = i, A, B) \tag{7}$$

The Backward Algorithm:

(1) Initialization:

$$\beta_T(i) = 1; 1 \leq j \leq N$$

(2) Recursive Steps:

$$\beta_t(j) = \sum_{j=1}^{N} \beta_{t+1}(j) a_{ij} b_j(o_{t+1}); 1 \leq i \leq N, 1 \leq t \leq T \tag{8}$$

(3) Result:

$$\mathbb{P}(O|A, B) = \sum_{i=1}^{N} \pi_j b_j(o_1) \beta_1(j)$$

Define $\xi_t(i, j)$, the probability being in state i ate time t and j at time $t+1$ based in the observed sequence of returns while assuming A and B are given:

$$\xi_t(i, j) = \mathbb{P}(q_t = i, q_{t+1} = j | O, A, B) = \frac{\alpha_t(i) a_{ij} b_j(o(t+1)) \beta_{t+1}(j)}{\sum_{j=1}^{N} \alpha_t(j) \beta_{t+1}(j)} \tag{9}$$

Also, define $\gamma_t(j)$, the probability being in state j at time t based in the observed sequence of returns while assuming A and B are given:

$$\gamma_t(j) = \mathbb{P}(q_t = j | O, A, B) = \frac{\alpha_t(j) \beta_t(j)}{\mathbb{P}(O|A, B)} \tag{10}$$

The Baum-Welch Algorithm:

For any continuous distribution for the observations, update all parameters required to characterize the distribution in the maximization step after updating the essential probabilities in the expectation step, where $\varphi_j(o)$ is the density of the state j's realization of observation.

- Initialization A and B based on KMeans Algorithm and $\pi_i = \frac{1}{N} \forall i$.
- Iterate untile convergence:
  - E-Step:

$$\begin{aligned} \gamma_t(j) &= \frac{\alpha_t(j) \beta_t(j)}{\mathbb{P}(O|\hat{A}, \hat{B})} \\ \xi_t(i, j) &= \frac{\alpha_t(i) \hat{a}_{ij} \hat{b}_j(o(t+1)) \beta_{t+1}(j)}{\mathbb{P}(O|\hat{A}, \hat{B})} \end{aligned} \tag{11}$$

– M-Step:

$$
\begin{aligned}
\hat{a}_{ij} &= \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \sum_{k=1}^{N} \xi_t(i,k)} \\
\hat{\mu}_j &= \frac{\sum_{t=1}^{T} \gamma_t(j) o_t}{\sum_{t=1}^{T} \gamma_t(j)} \\
\hat{\Sigma}_j &= \frac{\sum_{t=1}^{T} \gamma_t(j)(o_t - \hat{\mu}_j)(o_t - \hat{\mu}_j)^\top}{\sum_{t=1}^{T} \gamma_t(j)} \\
&\vdots \\
\hat{b}_j(o_i) &= \hat{\varphi}_j(o_i) \\
\pi_i &= \gamma_1(i)
\end{aligned}
\tag{12}
$$

Following a preliminary data analysis, we have chosen to utilize a Gaussian HMM to analyze industry portfolio returns dynamics. The model extends the traditional HMM by assuming industry portfolio monthly returns are continuous variables following a Gaussian distribution, with each emission density modeled as such. The GaussianHMM is structured around a set number of hidden states, each defined by a distinct Gaussian distribution with specific mean μ and covariance Σ. This approach allows for a detailed modeling of return variances and covariance dynamics across various market conditions. Essentially, the model interprets returns at any time t as arising from a Gaussian mixture, directly correlating each mixture component to a market's latent state q_i, and determining the mixture's weights through transition probabilities a_{ij} and the initial state distribution π_i.

4.2 Overview of the B-L

The B-L model integrates market equilibrium assumptions with individual investor perspectives to determine optimal asset allocations. At the core of this model lie two critical assumptions: first, the normality of returns, facilitating the development of a closed-form Bayesian framework for the incorporation of active views; and second, the application of the CAPM to infer market-implied returns.

The methodology initiates by deriving market-implied returns through reverse optimization and combines investor's view to yield posterior expected returns, which, in conjunction with a covariance matrix for asset returns, underpin portfolio optimization.

4.2.1 Notations for B-L.

- N: Number of assets to consider.
- K: Number of active views.
- T: Number of time periods/data points to consider.
- δ: The investor's Risk Aversion coefficient.
- R_f: The risk free rate.
- τ: A scalar indicating the uncertainty of the CAPM prior.
- $\Pi \in \mathbb{R}^N$: the implied equilibrium return vector.
- $\Sigma \in \mathbb{R}^{N \times N}$: A covariance matrix of the assets.
- $w_{mkt} \in \mathbb{R}^N$: The market capitalization weight of the assets.
- $Q \in \mathbb{R}^{K \times 1}$: Views matrix.

- $P \in \mathbb{R}^{K \times N}$: The "projection" matrix identifies the assets involved in the views.
- $\Omega \in \mathbb{R}^{K \times K}$: Diagonal uncertainty matrix expressing covariance of the views.

4.2.2 Reverse Engineering. In the B-L model, reverse optimization is employed to infer implied market returns from the market capitalization weights of assets, reverse-engineering the expected returns that the market equilibrium prices suggest. The reverse-engineering process leverages the CAPM to rationalize how the collective market portfolio, through its asset weights, encapsulates investor expectations on future returns.

$$\Pi = \delta \Sigma w_{mkt} + R_f T \tag{13}$$

4.2.3 The B-L Master Equations. Subsequently, leveraging the Bayesian framework, posterior returns and covariances are derived using the B-L Master Formula [3], encapsulated by the following equations:

$$\mu^{BL} = [(\tau\Sigma)^{-1} + P^\top \Omega^{-1} P]^{-1} [(\tau\Sigma)^{-1}\Pi + P^\top \Omega^{-1} Q] \tag{14}$$

$$\Sigma^{BL} = \Sigma + [(\tau\Sigma)^{-1} + P^\top \Omega^{-1} P]^{-1} \tag{15}$$

The posterior returns and covariances serve as inputs for generating new portfolio weights. Thus, the B-L model enhances portfolio optimization by incorporating investor opinions, ensuring that portfolio construction is both performance-oriented and reflective of investor insights.

4.3 Combining HMM with the B-L

The subsequent section offers a comprehensive walkthrough of the entire methodology displayed in Figure 7, detailing the adaptations and automation introduced to refine the portfolio selection and construction within the B-L framework.

After preprocessing the dataset, each industry's monthly portfolio returns are analyzed through an HMM to distill active views and ascertain their associated confidence levels. The preliminary phase of the model involves selecting the optimal number of latent states via the Bayesian Information Criterion (BIC), with three states often emerging as the most representative—symbolically capturing the market's bullish, bearish, and sideways movements. This trichotomy offers a conceptual framework that aligns with the qualitative understanding of market phases.

For each industry, approximately two-thirds of the data serve as a 'burn-in' set, establishing the initial conditions for the model. Subsequent predictions employ a rolling window w of 200 months to dynamically update the model. At each predictive time step t, the HMM is calibrated over the window $[t - w - 1, t - 1]$ using the Baum-Welch Algorithm to refine the parameters and the Viterbi algorithm to decode the most likely sequence of hidden states q_t.

The confidence metric ω_{t+1} is defined as the probability of observing the current return o_t given the preceding returns and the estimated mixture of Gaussian distributions, which quantifies an investor's conviction by aligning with patterns resonant with past experiences, thereby offering a statistically inferred, rather than subjectively judged, basis for confidence.

Building on this foundation, a simple trading algorithm is proposed to determine the absolute active view of the industry. The algorithm employs the inferred state probabilities and transitions to inform trading decisions, encapsulating the investor's perspective on the market's future trajectory.

Simple Trading Algorithm:

Input: Historical hidden states $\{q_1, \dots, q_{t-1}\}$, historical observations $\{o_1, \dots, o_{t-1}\}$,

transition matrix $A^{(1,t-1)}$, risk aversion thresholds ω, μ

Ensure: Simple absolute active view, decision

ExtractActiveViews($\{q_i\}, \{o_i\}, A, \omega, \mu$)

Compute long_win_rate:

$$\text{long\_win\_rate} \leftarrow \frac{\sum_{i=t-w-1}^{t-1} \mathbb{1}_{[o_i>0]} \cdot \mathbb{1}_{[q_i=q_{t-1}]}}{\sum_{i=t-w-1}^{t-1} \mathbb{1}_{[q_i=q_{t-1}]}}$$

Compute short_win_rate:

$$\text{short\_win\_rate} \leftarrow \frac{\sum_{i=t-w-1}^{t-1} \mathbb{1}_{[o_i<0]} \cdot \mathbb{1}_{[q_i=q_{t-1}]}}{\sum_{i=t-w-1}^{t-1} \mathbb{1}_{[q_i=q_{t-1}]}}$$

$T \doteq \{i \mid q_{i-1} = q_t\}$

$\bar{s}_t^{simple} \leftarrow \frac{\sum_{i \in T} o_i}{|T|}$

Compute Simple Absolute View:

Simple Absolute View$_t$ $\leftarrow \exp(\bar{s}_t^{simple}) - 1$

if long_win_rate $> \omega$ and $\bar{s}_t^{simple} > 0$ then

decision $\leftarrow 1$

else if short_win_rate $> \mu$ and $\bar{s}_t^{simple} < 0$ then

decision $\leftarrow -1$

else

decision $\leftarrow 0$

end if

Return: Simple Absolute View$_t$, decision$_t$

For weighted absolute active view, simply replace the line:

$$\bar{s}_t^{simple} \text{ to } \bar{s}_t^{weighted} \leftarrow \frac{\sum_{i \in T} A_{q_t, q_i} \cdot o_i}{\sum_{i \in T} A_{q_t, q_i}} \tag{16}$$

We introduced two critical parameters, μ and ω, as thresholds for reflecting an investor's risk aversion, calibrated against historical success rates of long and short positions. These parameters crucially influence the willingness to trade, with a higher μ indicating a cautious approach to short selling, predicated on significant historical evidence of negative returns, and a higher ω reflecting a conservative attitude towards buying, contingent upon a strong historical record of positive outcomes. Setting $\mu = 0.5$ and $\omega = 0.6$ as downsides over a month is less common, and an average investor makes quite cautious decisions to join the market. Since we model investors who do not short-sell, a lower threshold for "sell" only implies the rare negative average return from the data set caused by the granularity.

For the second stage of the modeling, I employ the B-L model to analyze the extracted active views. Central to this process is the determination of δ, τ and Ω.

Leveraging the S&P 500 index as a proxy for the hypothetical market portfolio, our fully automated model adopts market-implied risk aversion based on mean-variance analysis from Modern Portfolio Theory to gauge investors' risk tolerance through market data. To inform portfolio management decisions, the method estimates the market-implied risk aversion coefficient δ via the Sharpe ratio, which represents the excess return of the market portfolio (R_m) over the risk-free rate (R_f) divided by the standard deviation of the market portfolio returns (σ_m). This reflects the principle that investors seek higher returns for increased risk, illustrating the risk-return trade-off.

$$\delta = \frac{R_m - R_f}{\sigma_m^2} \tag{17}$$

However, the empirical δ exhibits a wide range, varying from -12.821880 to 121.460412, indicating significant variability shown in Figure 8. While numerous methodologies exist for measuring risk aversion, there remains no universally agreed-upon estimate. Generally, the coefficient of relative risk aversion is considered to fall within the range of 1 to 3, reflecting common consensus. However, the literature presents a broad spectrum of estimates, spanning from as low as 0.2 to values exceeding 10 [6]. Consequently, in the subsequent model, we explore two sets of values: one constrained within the range $\delta \in [1, 3]$, and another spanning $\delta \in [0.01, 10]$.

In keeping with tradition, we adhere to He and Litterman's [7] approach of setting $\tau = \frac{1}{T}$ inversely proportional to the number of available observations, which accounts for the inherent uncertainty stemming from limited data when estimating asset returns and covariance matrices. As the number of observations diminishes, the reliability of these estimates tends to decrease, necessitating a more cautious weighting of prior beliefs in the B-L model.

In our implementation, we offer two methods to determine the uncertainty matrix (Ω). The first method, referred to as the Idzorek method [8], involves computing Ω based on user-specified percentage confidences for each view, where our approach utilizes a closed-form solution described by Jay Walters [25]. The formula to calculate Ω using the Idzorek method is given as:

$$\Omega = \mathrm{diag}(\omega_1, \omega_2, ..., \omega_K) \tag{18}$$

where ω_i for each view i is computed as:

$$\alpha = \frac{1 - \kappa_i}{\kappa_i} \tag{19}$$

$$\omega_i = \tau \alpha P_i \Sigma P_i^T \tag{20}$$

Here, κ_i := confidence of the active view Q_i.

The second method, known as the default omega method, is based on the approach proposed by He and Litterman [7] if the uncertainty matrix Ω is not provided explicitly and recognizing that the ratio $\frac{\Omega}{\tau}$ is proportional to the variance of the view portfolio. Mathematically, the uncertainty matrix Ω is computed similar to equation 18, where ω_i for each view i is determined as:

$$\omega_i = \tau P_i \Sigma P_i^T \tag{21}$$

In both methods, Ω is a diagonal matrix, where each diagonal element corresponds to the uncertainty associated with the respective view.

After computing the B-L posterior return and covariance matrix utilizing equations 14 and 15, respectively, we proceed to employ two classical metrics for the maximization problem: the standard mean-variance optimization method and the maximization of Sharpe ratio. These measures serve to generate posterior sets of portfolio weights for the 30 industry portfolios.

Mean-Variance Optimization:

$$w^{BL} = \arg\max_{w} \left(w^T \mu^{BL} - \frac{\delta}{2} w^T \Sigma^{BL} w \right) \tag{22}$$

Max Sharpe Ratio Optimization:

$$w^{BL} = \arg\max_{w} \left(\frac{w^T \mu^{BL} - r_f}{\sqrt{w^T \Sigma^{BL} w}} \right) \tag{23}$$

Mean-Variance Optimization and Sharpe ratio maximization both aim to maximize return per unit of risk through different mechanisms: the former optimizes the trade-off between expected return and risk (variance/standard deviation), while the latter seeks to maximize excess return over the risk-free rate per unit of risk (standard deviation). Both strategies strive to construct portfolios with the optimal return-risk balance.

In addition to maximizing returns while minimizing risks, it's crucial to impose constraints on portfolio weights to align with real-world investor behavior. We introduce two categories of constraints aimed at simulating typical investor practices. Type 1 constraints enforce that portfolio weights $\{w^{BL} | w_i^{BL} \geq 0, i = 1, \ldots 30; \mathbb{1}^T w^{BL} = 1\}$ to emulate investors who refrain from short selling and allocate all available funds to the market. Type 2 constraints ensure that portfolio weights $\{w^{BL} | w_i^{BL} \geq 0, i = 1, \ldots 30; \mathbb{1}^T w^{BL} \leq 1\}$ to represents investors who avoid short selling and adjust their investment allocation based on market conditions. For instance, during market upswings, they may invest a larger portion of their total wealth, whereas during downturns, they may reduce their exposure. These constraints mirror real-world investor behavior and help prevent unrealistic portfolio compositions, fostering more sensible and practical investment strategies.

4.4 Grid-Search for Best Hyperparameter

Once all the necessary components for the B-L model are obtained, a grid search technique is employed to identify the optimal set of hyperparameters in Table 1 for final evaluation. The selection process is guided by three key evaluation metrics: cumulative return, Sharpe ratio, and maximum drawdown. The evaluation period spans from July 1993 to July 2023, trained on the training window of a selected choice below, providing a comprehensive assessment of the performance of the HMM-based B-L portfolios.

4.5 Evaluation Metrics and Baseline

The following section defines the evaluation metrics used to assess the performance of the constructed portfolios and introduce the baseline models.

4.5.1 Evaluation Metrics. We evaluate the performance of the constructed portfolios using the following metrics:

Table 1. Hyperparameters

Hyperparameter	Choices
Window (month) Size	6, 12, 18, 24, 30, 36
Adjustment Choices	[1, 3], [0.2, 10]
Active View	Simple view, Weighted view
Uncertainty Matrix Method	Idzorek, Default
Maximization Objective	Mean-Variance (mv), Max Sharpe Ratio (msr)
Weights Constraints	Sum to 1, Between 0 to 1

- Win Rate: The proportion of periods in which the portfolio receives greater than the market return.

$$\text{Win Rate} = \frac{\sum_{i=1}^{T} \mathbb{1}_{R^{\text{portfolio}} >} > R^{\text{S\&P 500}}}{T} \tag{24}$$

- Annualized Return: The average annual return of the portfolio.

$$\text{Annualized Return} = \left(\frac{C}{P}\right)^{\frac{1}{T}} - 1 \tag{25}$$

where C is the final portfolio value, P is the initial portfolio value, and T is the total number of periods.
- Annualized Volatility: The standard deviation of the portfolio's returns over a year, adjusted for the number of months in a year.

$$\text{Annualized Volatility} = \sqrt{12} \times \sigma_p \tag{26}$$

where σ_p is the portfolio's standard deviation of returns.
- Cumulative Return: The total return of the portfolio over the entire evaluation period.

$$\text{Cumulative Return} = \frac{C}{P} - 1 \tag{27}$$

- Sharpe Ratio: The ratio of portfolio excess return to its volatility, adjusted for risk-free rate.

$$\text{Sharpe Ratio} = \frac{R_p - R_f}{\sigma_p} \tag{28}$$

where R_p is the portfolio's average return.
- Max Drawdown: The maximum loss from peak to trough during a specific period of investment.

$$\text{Max Drawdown} = \max\left(\frac{P_i - P_j}{P_i}\right) \tag{29}$$

where P_i is the portfolio value at the peak and P_j is the portfolio value at the trough.

4.5.2 Baseline Models. We compare the performance of the constructed portfolios against three baseline models:

- Equally Weighted Portfolio: All 30 industry portfolios in the aggregate portfolio are allocated equal weights.

$$w_i = \frac{1}{N}$$

where N is the number of assets in the portfolio.

- Value Weighted Portfolio: The 30 industry portfolios are weighted based on their market capitalization.

$$w_i = \frac{\text{Market Cap}_i}{\sum_{j=1}^{N} \text{Market Cap}_j}$$

- Global Minimum Variance Portfolio: Portfolio constructed to minimize volatility while maintaining diversification.

$$\text{Minimize} \quad \frac{1}{2} w^T \Sigma w$$

subject to:

$$w^T \mathbb{1} = 1$$

$$w_i \geq 0$$

5 RESULTS

Table 2. Best Hyperparameter Selection for HMM-BL Portfolios

Strategy	Window Size	Risk Aversion	View	Uncertainty	Objective	Constraints
Best Cumulative Return	6	[1, 3]	Simple	Default	msr	Between 0 and 1
Best Sharpe Ratio	36	[0.2, 10]	Simple	Idzorek	msr	Between 0 and 1
Best Max Drawdown	36	[1, 3]	Simple	Default	msr	Sum to 1

In the Results section, we evaluate the performance metrics of portfolios derived from the HMM-BL framework and benchmark them against traditional investment strategies: EW, VW, and GMV. Our investigation is specifically tuned to a 15% volatility standard, adhering to a counter-cyclical investment philosophy that advocates adjusting the proportion of risky assets in response to fluctuating market volatilities.

Table 3. Portfolio Performance Metrics - Part 1

Portfolio Strategy	Winrate	Cumulative Return	Annualized Return	Annualized Vol	Sharpe Ratio
EW	0.5429	41.7243	0.1365	0.1586	0.6533
VW	0.5886	60.8586	0.1513	0.1605	0.7351
GMV (Sample Cov.)	0.5486	31.6353	0.1257	0.1561	0.5969
HMM-BL Best Cum. Return	0.4800	37.2306	0.1320	0.1547	0.6420
HMM-BL Best Sharpe	0.5514	101.7787	0.1717	0.1532	0.9002
HMM-BL Best Max Drawdown	0.5857	41.9756	0.1367	0.1592	0.6521

The analysis begins with an observation from Figure 9 showing the adjusted cumulative returns, where HMM-BL configured portfolios, aligned with a 15% volatility threshold, outperform baseline models. Before adjustment, the HMM-BL Best Cumulative Return strategy yields a cumulative return of 995.5678 per dollar invested from July 1993 to July 2023. Yet, when tempered to the volatility standard, the strategy's profitability moderates, underscoring a previously masked higher risk quotient.

Tables 3 and 4 present the backtesting performance of HMM-BL portfolios against benchmarks. The HMM-BL Best Sharpe Ratio portfolio outperforms the VW strategy in annualized return (0.1717 vs. 0.1513) after adjusting for volatility,

Table 4. Portfolio Performance Metrics - Part 2

Portfolio Strategy	Skewness	Kurtosis	Cornish-Fisher VaR (5%)	Historic CVaR (5%)	Max Drawdown
EW	-0.3200	2.9212	0.0676	0.0896	-0.4271
VW	-0.3413	2.8632	0.0678	0.0871	-0.4612
GMV (Sample Cov.)	-0.3382	2.7189	0.0675	0.0871	-0.3885
HMM-BL Best Cum. Return	0.1780	3.8130	0.0589	0.0856	-0.2707
HMM-BL Best Sharpe	0.0463	3.3930	0.0575	0.0788	-0.3479
HMM-BL Best Max Drawdown	-0.3452	2.8393	0.0683	0.0899	-0.4369

demonstrating efficiency with reduced volatility (0.1532). The GMV strategy is identified as the most conservative, with a lower annualized return (0.1257), illustrating the classic risk-return trade-off.

Win rates, cumulative returns, and max drawdown are key metrics for assessing a portfolio's ability to yield consistent gains with minimal risks involved. While the VW strategy shines with the highest win rate (0.5886), the HMM-BL Best Sharpe Ratio portfolio, after adjusting for volatility, achieves the highest cumulative return (101.7787), underscoring the HMM-BL model's effectiveness in capturing market trends. The superior risk-adjusted return of the HMM-BL Best Sharpe Ratio portfolio is highlighted by its Sharpe ratio of 0.9002 compared to 0.7351 for VW. In juxtaposition, the Max Drawdown conveys the HMM-BL Best Cumulative Return portfolio's robust capital preservation potential during market troughs, with the lowest recorded drawdown (-0.2707).

Analyzing skewness and kurtosis offers insights into the tails of return distributions. The HMM-BL Best Cumulative Return portfolio, with a positive skew (0.1780) and high kurtosis (3.8130), indicates a likelihood of substantial positive returns but with a risk of negative extremes. Conversely, the VW and GMV portfolios' negative skew suggests a tendency towards negative outcomes. Moreover, the HMM-BL Best Sharpe ratio portfolio exhibits the most symmetrical distribution, mirroring a near-normal distribution, and setting a benchmark for stability.

Similarly, lower Cornish-Fisher VaR and CVaR metrics in HMM-BL portfolios relative to benchmarks intimate a reduced risk of extreme loss under typical market conditions. The HMM-BL Best Sharpe Ratio portfolio particularly exemplifies the resilience to extreme negative tail risks, which exhibits the lowest CVaR (0.0788), indicating heightened resilience during downturns. This resilience is supported by distribution plots in Figure 10.

The HMM-BL model's portfolio outcomes, demonstrating adaptability across investor profiles, ranging from the high-risk tolerance of the Best Cumulative Return strategy to the conservative bent of the Best Max Drawdown approach. The overall best performer, the HMM-BL Best Sharpe Ratio portfolio, reinforces the investment ethos of prioritizing the highest Sharpe ratio over time. The efficacy of the HMM-BL model can be ascribed to its dynamic acclimation to shifting market climates, as opposed to the static assumption of market behavior by traditional models. It proactively recalibrates portfolio compositions to leverage or mitigate the evolving market states.

Contrasting these portfolios with the S&P 500 in Figure 10 further illustrates the HMM-BL models' elaborate risk comprehension and their potential to surpass the market index in delivering risk-adjusted returns and managing drawdowns. This compelling evidence suggests that investors might realize superior diversification and performance with the HMM-BL strategies over broad market investments. Thus, the HMM-BL models emerge not only as sophisticated analytical tools but also as practical vehicles for investors aiming to optimize their portfolios in line with dynamic market realities.

6 CONCLUSION AND DISCUSSION

This study has ventured into the confluence of HMM with the B-L framework to enhance portfolio optimization. The HMM component provides a systematic approach to extracting active views from a basic trading strategy, embedding investor risk preferences via parameters μ and ω, and determining confidence levels from emission probabilities. Empirical results solidify the premise that HMM-BL portfolios, when fine-tuned to a specified volatility target, offer a notable advancement in performance metrics compared to conventional models which is evidenced by their heightened annualized returns (e.g. HMM-BL Best Sharpe: 0.1717 v.s. VW: 0.1513) and Sharpe ratios (e.g. HMM-BL Best Sharpe: 0.9002 v.s. VW: 0.7351). The study brings to light the adaptability of the HMM-BL approach, capable of aligning with diverse investor objectives—be it minimizing downside risk or pursuing consistent volatility profiles.

The findings herein augment the potential of the HMM-BL model as a robust tool that encapsulates both market dynamics and investor sentiment, resulting in portfolios that are responsive to shifting market conditions through a data-centric paradigm. In an era where contradictory market signals are common, an automated portfolio adjustment methodology can serve as a strategic aid for retail investors and a sophisticated addition to the arsenal of institutional portfolio managers.

Reflecting on the methodology, our study opens avenues for potential improvements that resonate with practical market scenarios. The monthly data, while useful in mitigating volatility and discerning long-term trends, might mask the intricate movements of the market on a shorter timescale. Transitioning to higher-frequency data could empower the HMM to capture the latent market states and transitions with greater detail, offering a refined outlook of market dynamics.

Furthermore, the Markov assumption's applicability to the financial market, which traditionally considers only the present state's influence on the future, may overlook the autocorrelation present across multiple periods. Incorporating higher-order Markov models could uncover extended temporal dependencies, potentially enhancing the model's ability to anticipate market momentum.

Moreover, the consideration of multi-dimensional data inputs, encompassing industry-specific and macroeconomic indicators, could significantly enrich the HMM's analysis. This would enable a more layered representation of the market's state, informed by a broader spectrum of economic factors.

Lastly, the study recognizes the limitations imposed by the normal distribution assumptions of the B-L model within real-world financial markets, which are frequently characterized by anomalies and non-normal distributions. Advancing towards a Bayesian framework and integrating non-parametric methods could prove instrumental in formulating empirical distributions that more accurately embody market conditions, leading to a portfolio optimization model that is more reflective of real-world behaviors and resilient in its decision-making efficacy.

In summary, this research introduces the HMM-BL model as a novel approach in portfolio management, with particular relevance to automated financial advisory services, such as Robo-Advisors. It establishes a foundation for further studies to enhance and implement this methodology in the complex and ever-evolving environment of financial markets.

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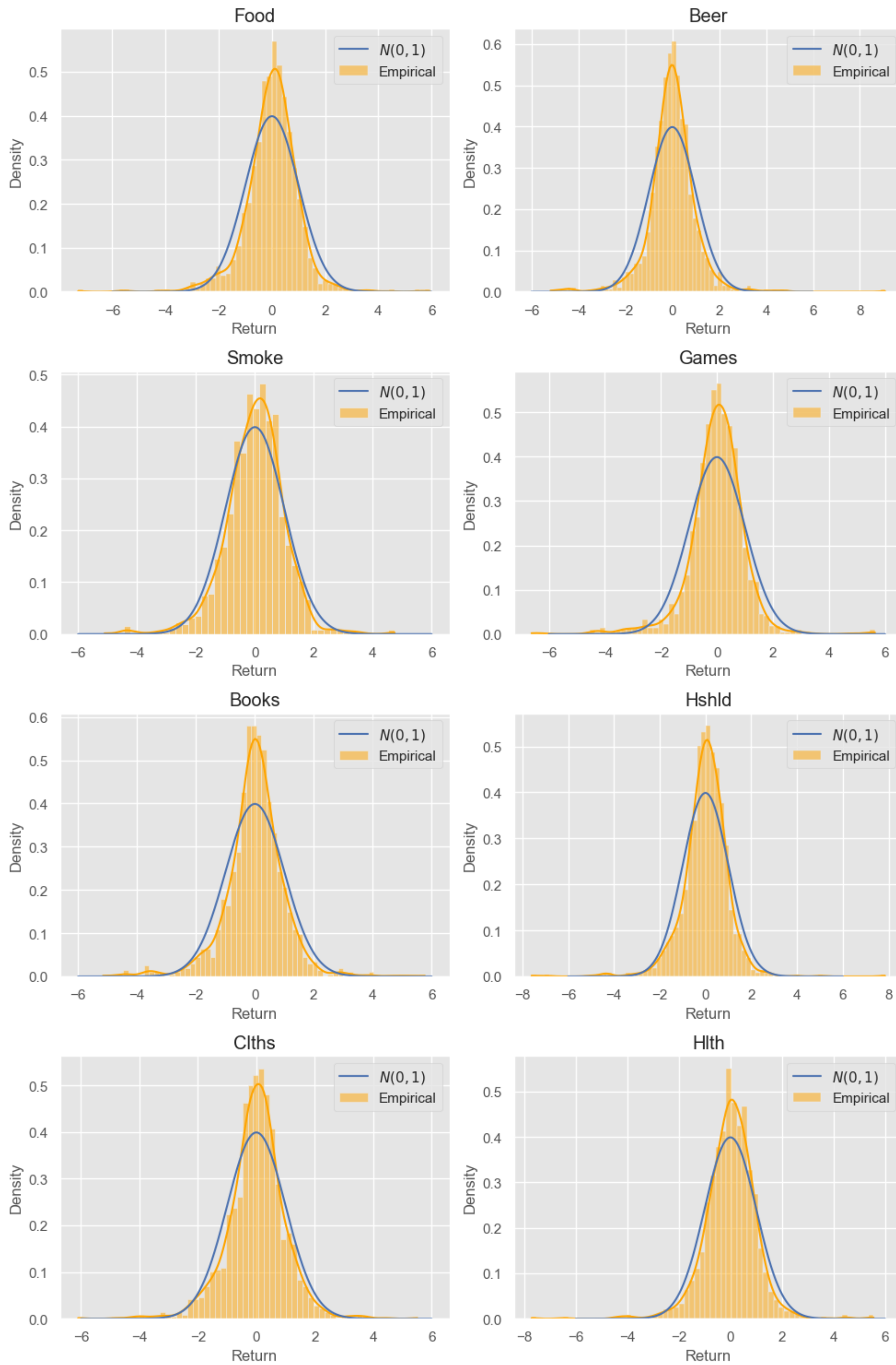

Fig. 1. Empirical Distributions of the First 8 Portfolio Returns

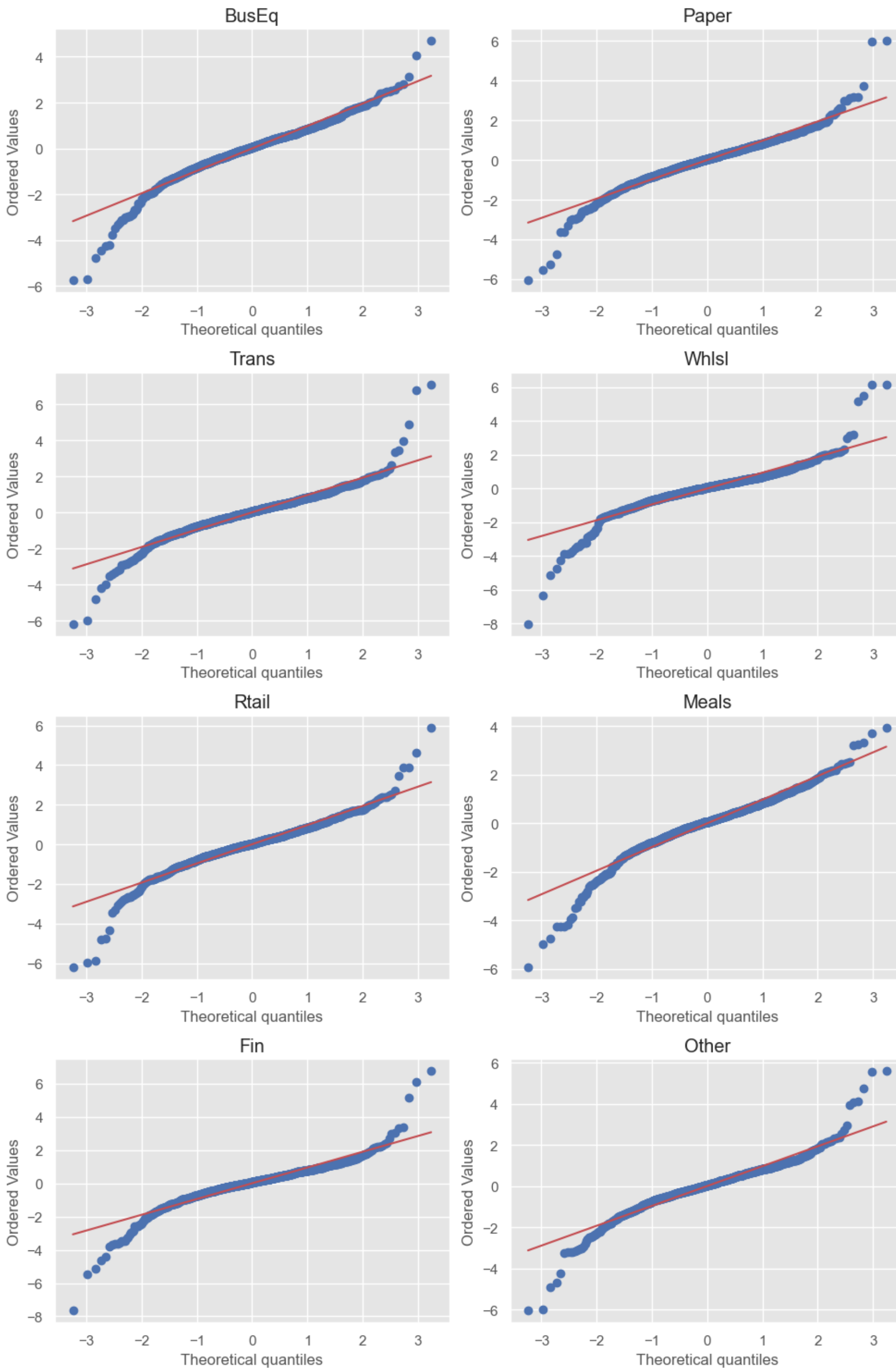

Fig. 2. QQ Plots of the Last 8 Industry Portfolios

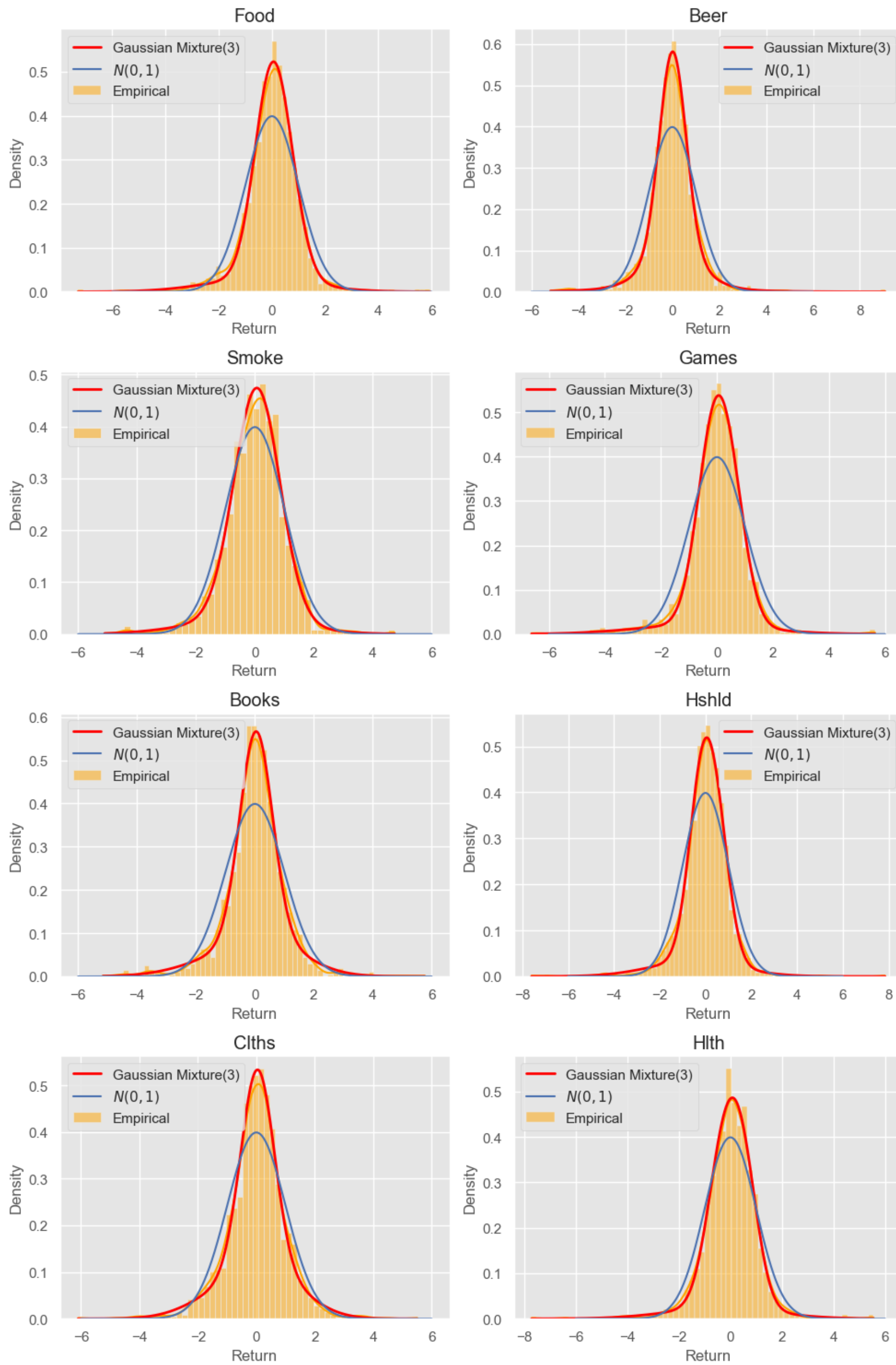

Fig. 3. Gaussian Mixture Models with 3 Components Fitted to the First 8 Industry Portfolios

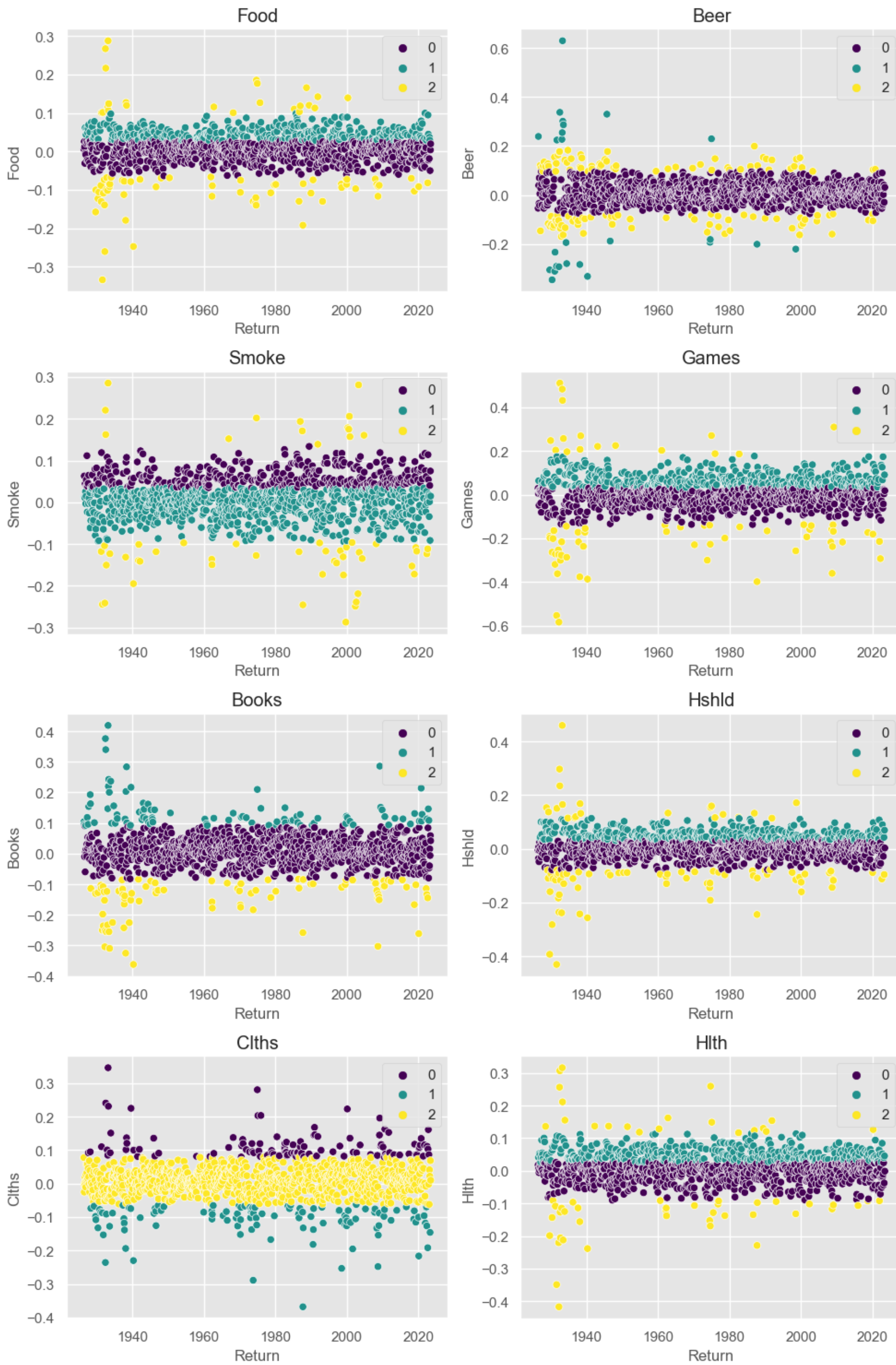

Fig. 4. Predicted States of Gaussian Mixture Models with 3 Components Fitted to the First 8 Industry Portfolios

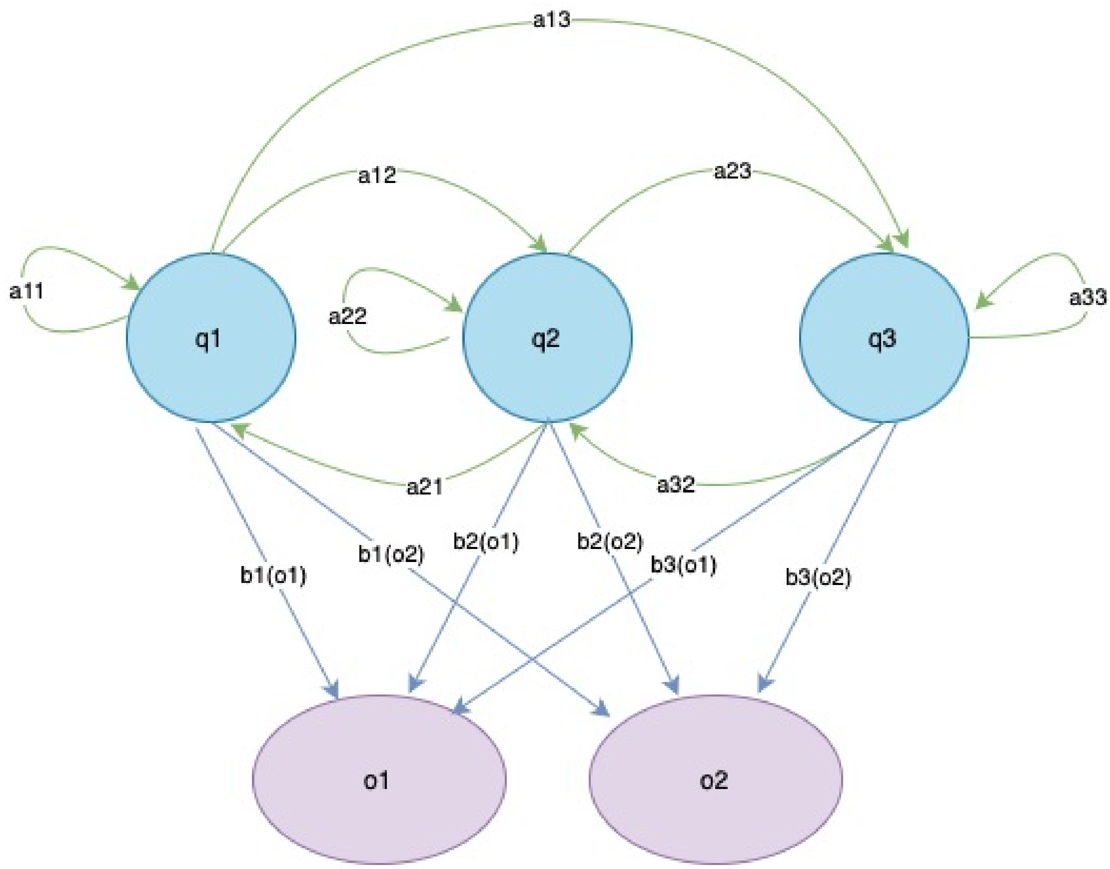

Fig. 5. A simplified diagram of 3 states and 2 observations Hidden Markov Model diagram

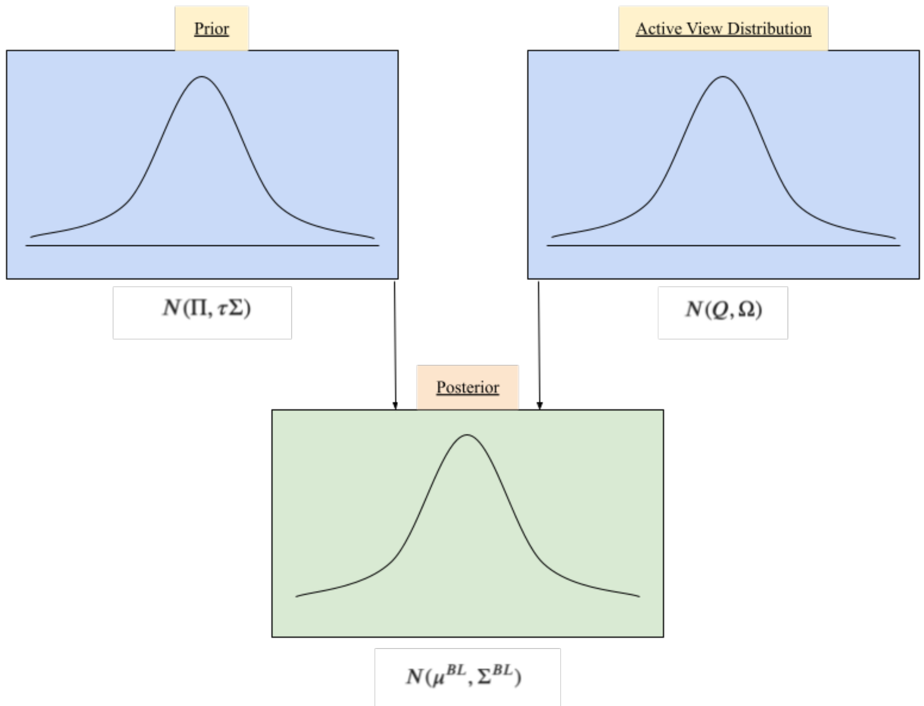

Fig. 6. Bayesian Updating in the Black-Litterman Model

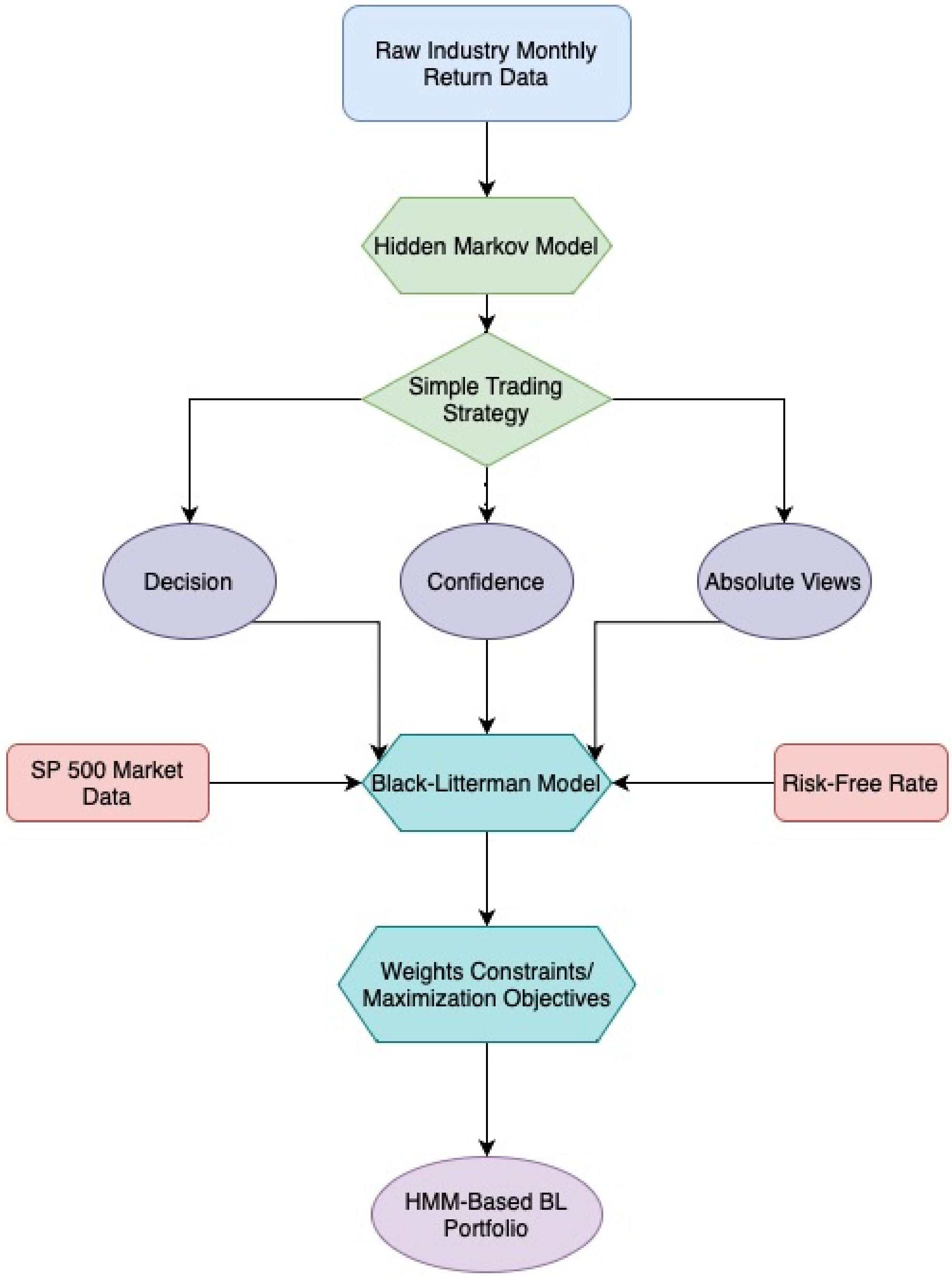

Fig. 7. HMM Based BL Model Workflow

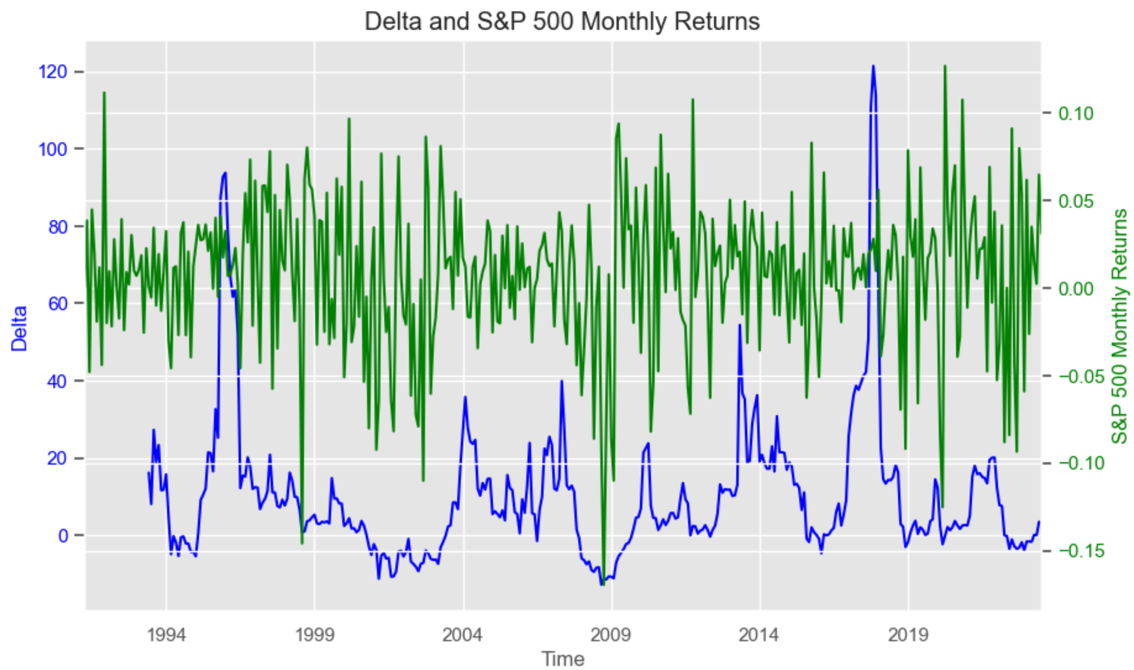

Fig. 8. Estimated Market Implied Return and S&P 500 Monthly Returns

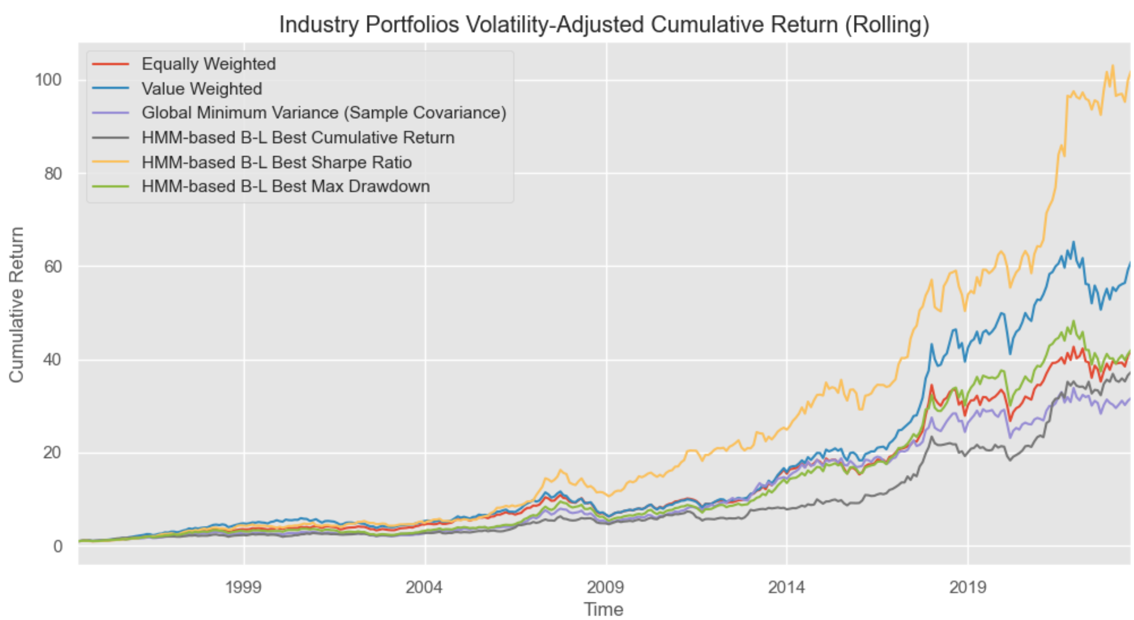

Fig. 9. Performance of the HMM-based B-L Portfolios v.s. Baseline Models (Volatility Scaling to 15%)

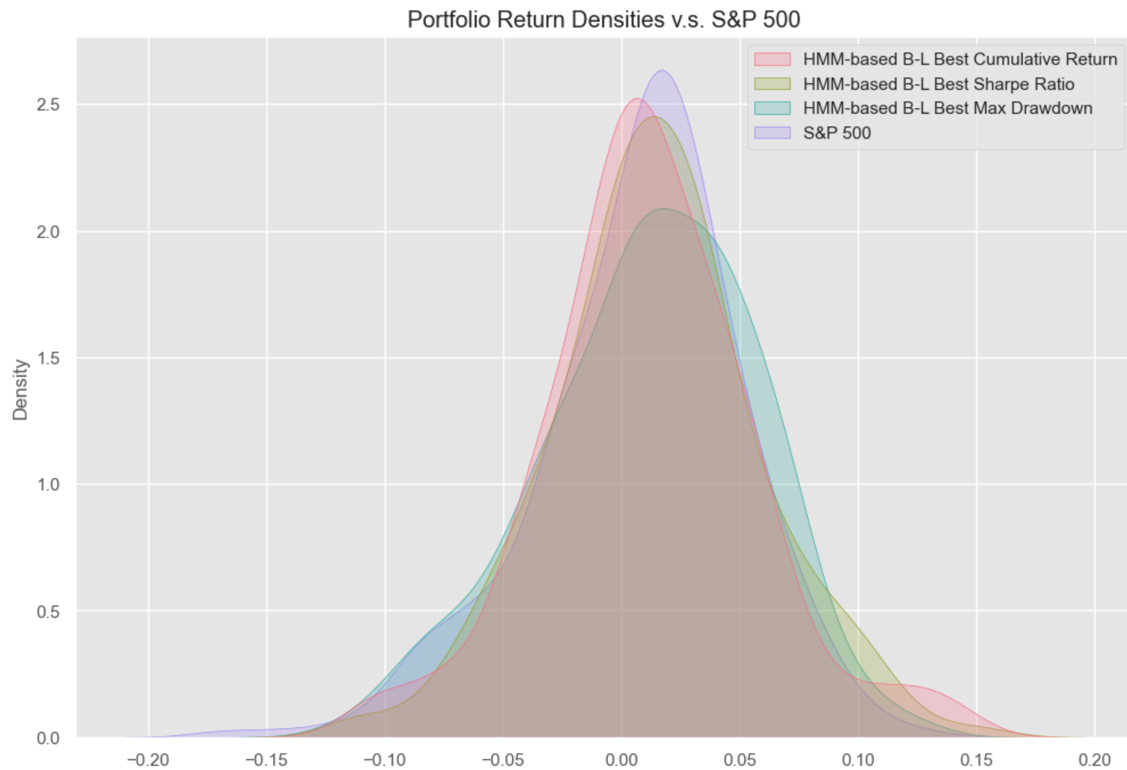

Fig. 10. HMM-based B-L Portfolios Return Densities v.s. S&P 500

Reproducing the Hidden Markov Based Black-Litterman Model Project

This guide provides detailed instructions for reproducing the analysis and results of the Hidden Markov Based Black-Litterman Model project.

Getting Started

Cloning the Repository

To get started with the project, first clone or download the repository directly from the GitHub page:

Hidden Markov Based Black-Litterman Model Repository

Structure of the Repository

The repository is structured into two primary directories:

- **`Data` Directory:** Contains all the input and output data necessary for the project.
- **`Code` Directory:** Includes Jupyter notebooks for running the model, with detailed comments in markdown cells, and custom libraries for model construction.

Detailed File Descriptions

Code Directory

- `EDA.ipynb`: An exploratory data analysis notebook for analyzing the distribution of the 30 industry portfolios. Includes additional tests for stationarity not covered in the main paper.

- `HMM.ipynb`: Demonstrates applying the Hidden Markov Model to the food industry, extending the algorithm to generate confidence and active views for all 30 industries, and implementing a simple trading strategy.

- `BlackLitterman.ipynb`: Walks through the construction, determination of hyperparameters, and training of the model on 30 industry portfolios. It includes a detailed example of a one-period portfolio trained without constraints on weights, using only a mean variance objective. The latter part of the notebook focuses on analyzing the returns (HMM-BL) of selected hyperparameter combinations, an essential step in the study's trial and error process not included in the final paper results.

- `BlackLitterman_with-Constraint.ipynb`: Contains similar components as the `BlackLitterman` notebook but adds constraints to the output HMM-BL portfolio weights and optimization over different objectives. This notebook concludes with an analysis of the best selected hyperparameter portfolios compared to baseline portfolios, presenting summary statistics included in the thesis.

- `HMM.py`: A Python class designed to support the `HMM` notebook, including functions for visualizing the results on each industry.

- `black_litterman.py` and `black_litterman_constrained.py`: Python classes to support Bayesian framework implementation in the `BlackLitterman` notebooks, with the latter extending the former to include weight constraints and optimization function selection.

- `toolkit.py`: A utility library offering basic and helper functions for analyzing financial data, focusing on data transformation and summary statistical analysis.

Data Directory

Contains datasets such as `RF.csv` for US Treasury yields, `all_hyper_combo_portfolio_returns.csv` for HMM-BL portfolio returns across various hyperparameter combinations, and several other datasets detailing confidence, decisions, and expected returns based on simple and weighted trading algorithms from July 1993 to July 2023. It also includes the `Fama French 6 factors` dataset and 30 industry monthly returns datasets.

Package Dependencies and Installation

Before running the models, ensure the following packages are installed with their specified versions:

- pandas 2.0.3
- numpy 1.24.3
- yfinance 0.2.35
- seaborn 0.12.2
- matplotlib 3.7.2
- scipy 1.11.1
- python-dateutil 2.8.2
- scikit-learn 1.3.0
- hmmlearn 0.3.2
- statsmodels 0.14.0

Use Conda or Pip to install these packages.

Reproducing the Analysis

For those interested in running the models and reproducing the paper's findings without engaging in the full breadth of additional analysis, the following streamlined process is recommended:

Environment Setup

Ensure that all necessary packages are installed using either Conda or Pip. Reference the list of packages and their specific versions provided earlier in this document.

Data Extraction and Preprocessing

1. **Begin with the `HMM.ipynb` Notebook:**
    - Open `HMM.ipynb` and import the required packages as outlined in the initial cell.
    - Navigate to the *Extract Values into CSV* section.
    - Execute the `extract_data` function with your chosen parameters for train size, window, omega, mu, and metrics. This step is crucial for generating the decision, view, and confidence

files needed for subsequent analyses. For replication consistent with the paper, execute the following commands:

```
decisions_simple, views_simple, confidence_simple =
extract_data(train_size=2/3, window=200, omega=0.6, mu=0.5,
metrics='simple')
decisions_weighted, views_weighted, confidence_weighted =
extract_data(train_size=2/3, window=200, omega=0.6, mu=0.5,
metrics='weighted')

# Save the extracted data to CSV files for further use.
confidence_simple.to_csv('../Data/confidence_simple.csv')
decisions_simple.to_csv('../Data/decisions_simple.csv')
views_simple.to_csv('../Data/views_simple.csv')
confidence_weighted.to_csv('../Data/confidence_weighted.csv')
decisions_weighted.to_csv('../Data/decisions_weighted.csv')
views_weighted.to_csv('../Data/views_weighted.csv')
```

2. **Proceed to the `BlackLitterman_with-Constraint.ipynb` Notebook:**
    - Import the necessary packages and datasets as instructed at the beginning of the notebook.
    - Follow the steps outlined in the *Building Trailing Window* section to set up your analysis environment.
    - Continue through the notebook, executing all cells under the *Hyperparameter Selection* section to generate portfolio returns across all hyperparameter combinations.
    - Analyze and compare the best hyperparameter portfolios with baseline models in the *Visualization on Best Hyperparameter Models* section.
    - In the *Adjusting the Composition of a Portfolio in Graphing* section, generate backtesting results and metrics for both baseline portfolios and HMM-BL portfolios.

Reproducing Paper Figures and Tables

To recreate specific elements from the paper:

- **Tables 3 and 4:** These can be generated by running the relevant code in the `BlackLitterman_with-Constraint.ipynb` notebook, specifically within the *Adjusting the Composition of a Portfolio in Graphing* section.
- **Figures 1-4:** Execute all cells in `EDA.ipynb` up to the point marked “Trying Food First” to generate these figures.
- **Figure 8:** In `BlackLitterman_with-Constraint.ipynb`, reproduce this figure by executing cells up to the *Construct a Class for Each Single Period* section.
- **Figures 9 and 10:** These figures can be reproduced by following the instructions in the *Adjusting the Composition of a Portfolio in Graphing* section of the `BlackLitterman_with-Constraint.ipynb` notebook.