**Models of Neural Systems, WS 2020/21**
Computer Practical 1
Solution due on November, 23rd, 2020, at 10 am

McCulloch–Pitts Neurons and Perceptron Learning

In this exercise we introduce a simple neuron model proposed by McCulloch and Pitts (1943). Historically, this model provided the first simplification of biological neurons as computational units. Here, we will show how such neurons can be trained with supervision to perform real-world computations such as classification. Finally, we will explore its severe limitations.

1. **McCulloch–Pitts neuron**
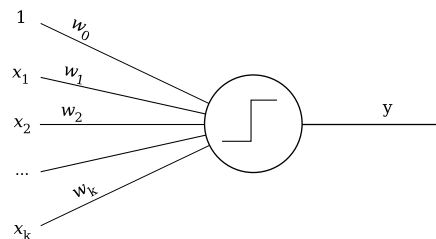
   (a) Implement a McCulloch–Pitts neuron (see the diagram):

$$y(\mathbf{x}) = \mathrm{sgn}\left(\mathbf{w}^T \mathbf{x}\right),$$

   where $\mathbf{x} = [-1, x_1, x_2, \ldots, x_k]$ is a vector of inputs, $\mathbf{w} = [w_0, w_1, w_2, \ldots, w_k]$ is a vector of weights and $y(\mathbf{x})$ is the output.

   (b) Take weights $\mathbf{w} = [3, 2, 2]$ and two binary inputs $x_1, x_2 \in \{-1, +1\}$. Show that the neuron performs a logical AND operation.

2. **Activation functions**

   Often, the step function (as used in the McCulloch–Pitts neuron) is too crude for estimating the activity of a neuron as a function of its input. Plot the following functions for three reasonable choices of the free parameter $a > 0$. What is the meaning of the parameter a?

   (a) Sigmoid function:

$$f(x) = \frac{2}{1 + \exp(-ax)} - 1$$

   (b) Hyperbolic tangent function:

$$g(x) = \tanh(ax).$$

   (c) Piecewise linear function:

$$l(x) = \begin{cases} 1 & \text{if } x \geq \frac{1}{a} \\ ax & \text{if } -\frac{1}{a} < x < \frac{1}{a} \\ -1 & \text{if } x \leq -\frac{1}{a}. \end{cases}$$

How can you choose a such that you obtain the step function from any of the three functions?

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

3. **Rosenblatt's perceptron**

   (a) Prepare a training set $\{\mathbf{x}_i, d(\mathbf{x}_i)\}$ for $i = 1, 2, \ldots, 1000$, where
   $\mathbf{x}_i = [-1, x_{i,1}, x_{i,2}]$ is an input vector with random values $x_{i,1}$ and $x_{i,2}$ drawn from a standard normal distribution. $d(\mathbf{x})$ is the desired response. Let the desired response be a comparison between the coordinates x_1 and x_2:

   $$d(\mathbf{x}) = \begin{cases} 1, & \text{if } x_2 \geq 0.5 - x_1, \\ -1, & \text{if } x_2 < 0.5 - x_1. \end{cases}$$

   (b) Train a McCulloch–Pitts neuron on the training set by presenting all examples iteratively and, after each example $\mathbf{x}_i$, updating the weights using an error-correction update rule:

   $$\mathbf{w}_{\text{new}} = \mathbf{w}_{\text{old}} + \eta(d(\mathbf{x}_i) - y(\mathbf{x}_i))\mathbf{x}_i,$$

   where $\eta > 0$ is the learning rate and $y(\mathbf{x})$ is the response of a McCulloch–Pitts neuron, as defined in exercise 1:

   $$y(\mathbf{x}) = \text{sgn}\left(\mathbf{w}^T \mathbf{x}\right).$$

   One training "epoch" is completed, when all examples have been presented once. At the end of each epoch, check whether, over this past epoch, any weights have been changed. Train the neuron over multiple epochs until no more weight changes occur. What values for η do you use, and why?

   (c) Test on a new dataset (validation set) that the neuron can indeed perform the trained comparison function.

   (d) Plot the training set and label each input vector according to its response class. Superimpose the weight vector on the same plot (think about the bias term w_0). Explain in what sense the weight vector is optimal.

4. **Linear separability**

   (a) Train a perceptron to perform the XOR function on binary inputs. As a reminder, the truth table of XOR is:

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

   (b) Show that the learning algorithm does not converge, i.e., the weights do not settle on fixed values.

   (c) Plot the XOR classification problem on an Euclidean plane. Explain why the problem is not linearly separable.

   Upload your solution to Moodle (see also submission checklist in the general information section). Please, comment your code and provide short answers to the questions in each task.

CONTACT	LOCATION	PHONE	EMAIL
GREGORY KNOLL	HAUS 2	2093 6247	GREGORY@BCCN-BERLIN.DE
PAULA KUOKKANEN	ITB, HAUS 4, ROOM 107	2093 98407	PAULA.KUOKKANEN@HU-BERLIN.DE
ERIC REIFENSTEIN	ITB, HAUS 4, ROOM 013	2093 98413	E.REIFENSTEIN@GMAIL.COM
NATALIE SCHIEFERSTEIN	ITB, HAUS 4, ROOM 106	2093 98406	NATALIE.SCHIEFERSTEIN@BCCN-BERLIN.DE