Acquisition and Analysis of Neural Data

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Analytical Problem Set 1

CONVOLUTION AND DIRAC DELTA

The convolution of two functions $u(t)$ and $v(t)$ is defined as

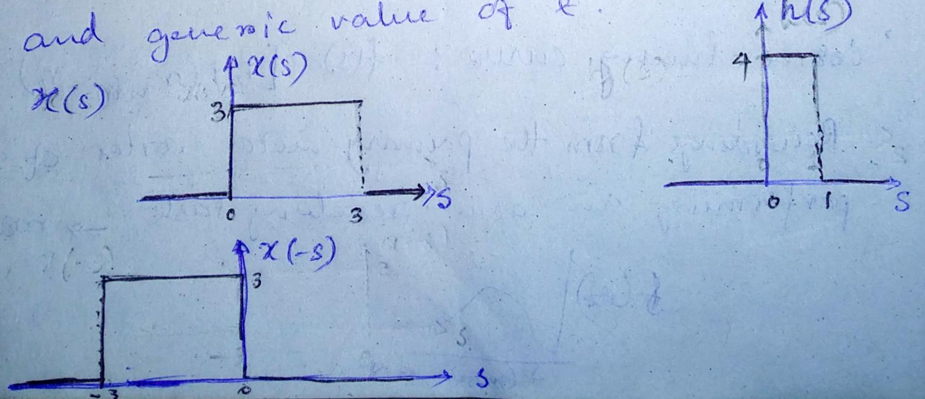
$$(u * v)(t) = \int_{-\infty}^{\infty} u(t-s)\, v(s)\, ds$$

1. Graphical convolution

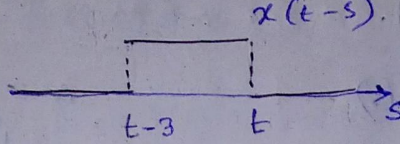
The response $A = x * h$ of a ~~lin~~ linear, time invariant, system is given by the convolution of the input signal $x(t)$ and system's impulse response $h(t)$. Here, we perform a "graphical" convolution ~~b/w x & h~~ between x & h by computing the area under the product of the two ~~functions~~ functions. Consider the following functions:—

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ 3 & \text{for } 0 \le t \le 3 \\ 0 & \text{for } t > 3 \end{cases} \qquad h(t) = \begin{cases} 0, & t < 0 \\ 4, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

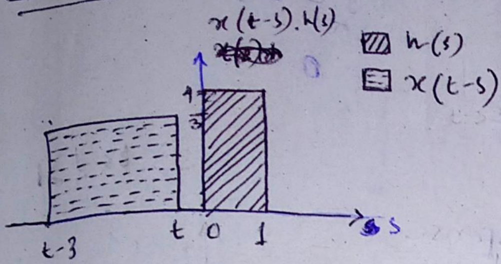
(a) $x(t-s)$ & $h(s)$ as a function of 's' for a fixed and generic value of t.



$x(t-s) \equiv$



Case 1 :- $\boxed{t < 0}$

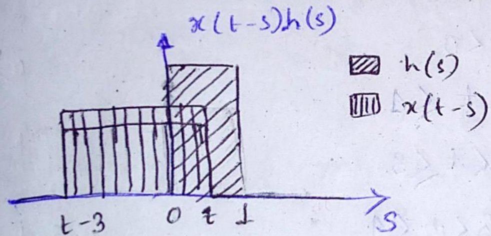


$A = x * h$

$= \int_{-\infty}^{\infty} x(t-s).\,h(s)\,ds$

$= \int_{-\infty}^{\infty} 0\,ds = 0$

Case 2 :- $\boxed{0 \leq t \leq 1}$



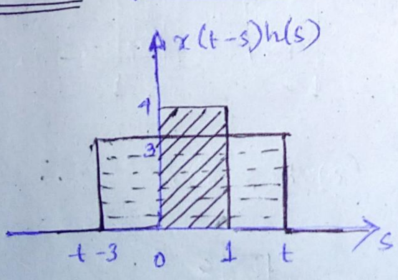
$A = x * h$

$= \int_{-\infty}^{\infty} x(t-s).\,h(s)\,ds$

$= \int_{0}^{t} x(t-s)\,h(s)\,ds$

$= [12s]_0^t = 12t$

Case 3 :- $\boxed{1 < t \leq 3}$



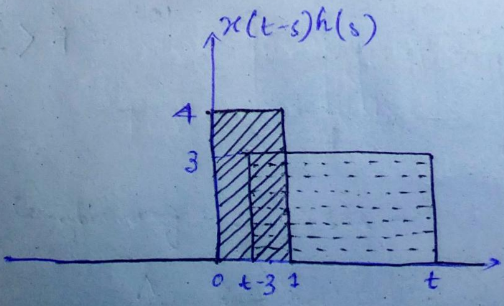
$A = x * h$

$= \int_{-\infty}^{\infty} x(t-s)\,h(s)\,ds$

$= \int_{0}^{1} x(t-s).\,h(s)\,ds$

$= [12s]_0^1 = 12.$

Case 4 :- $\boxed{3 < t \leq 4}$



$A = x * h$

$= \int_{-\infty}^{\infty} x(t-s)\,h(s)\,ds$

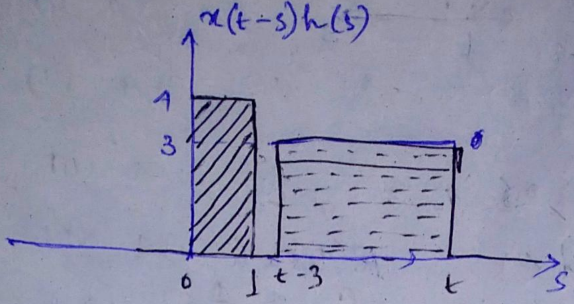
$= \int_{t-3}^{1} x(t-s)\,h(s)\,ds$

$= [12s]_{t-3}^{1}$

$= 12 - 12t + 12 \times 3$

$= 4 \times 12 - 12t = (4-t)12$

Case 5 :— $\boxed{t > 4}$



$$A = x * h$$
$$= \int_{-\infty}^{\infty} x(t-s)\,h(s)\,ds$$
$$= \int_{-\infty}^{\infty} 0 \, ds$$
$$= 0$$

(b). Compute the system's response

$$A(t) = \int_{-\infty}^{\infty} x(t-s)\,h(s)\,ds.$$ and sketch the result ~~result~~.

From the above derivations we can write

$$A(t) = \begin{cases} 0 & \text{for } t < 0 \\ 12t & \text{for } 0 \le t \le 1 \\ 12 & \text{for } 1 < t \le 3 \\ (4-t)12 & \text{for } 3 < t \le 4 \\ 0 & \text{for } t > 4 \end{cases}$$

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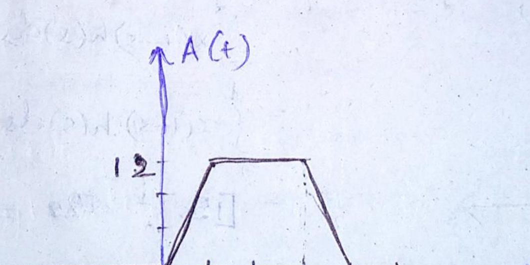


2. Properties of convolution

(i) Commutative: $(u * v)(t) = (v * u)(t)$

L.H.S $= (u * v)(t) = \int_{-\infty}^{\infty} u(t-s)\, v(s)\, ds$

$= \int_{\infty}^{-\infty} u(m)\, v(t-m)\, dm$

Considering
$t - s = m$
$\therefore s = t - m$
$\therefore s = +\infty \Rightarrow m = -\infty$
$s = -\infty \Rightarrow m = \infty$

$= \int_{-\infty}^{\infty} v(t-m)\, u(m)\, dm$

$= (v * u)(t)$

$=$ R.H.S. (proved)

(ii) Distributive: $(u * (\alpha v + \beta w))(t) = \alpha (u * v)(t) + \beta (u * w)(t)$

L.H.S. $= (u * (\alpha v + \beta w))(t)$

$= \int_{-\infty}^{\infty} u(t-s)\, (\alpha v(s) + \beta w(s))\, ds$

$= \int_{-\infty}^{\infty} u(t-s)\, \alpha . v(s)\, ds + \int_{-\infty}^{\infty} u(t-s)\, \beta . w(s)\, ds$

$= \alpha \int_{-\infty}^{\infty} u(t-s)\, v(s)\, ds + \beta \int_{-\infty}^{\infty} u(t-s)\, w(s)\, ds$

$= \alpha . (u * v)(t) + \beta (u * w)(t)$

$=$ R.H.S proved.

(iii) Associative: $(u * (v * w))(t) = ((u * v) * w)(t)$

L.H.S $= (u * (v * w))(t)$ ~~$= \int_{-\infty}^{\infty} u(t-s) \int_{-\infty}^{\infty} v(t-s)$~~

$= \int_{-\infty}^{\infty} u(t-s) (v(s) * w(s))\, ds$

$= \int_{-\infty}^{\infty} u(t-s) \int_{-\infty}^{\infty} v(s-m) . w(m)\, dm\, ds$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(t-s)\, v(s-m)\, w(m)\, dm\, ds$

(iii) ~~Associative~~ : $(u * (v * w))(t) = ((u * v) * w)(t)$

~~L.H.S = (u * (v *~~

$$= \int_{-\infty}^{\infty} u(t-s) \int_{-\infty}^{\infty} v(s-m)\, w(m)\, dm\, ds$$

$$\cancel{= \int_{-\infty}^{\infty} u(t-s)}$$

[considering, $s - m = r$ or $s = r + m$ and $s = \infty \Rightarrow r = \infty$; $s = -\infty \Rightarrow r = -\infty$]

$$= \int_{-\infty}^{\infty} u(t-r-m) \int_{-\infty}^{\infty} v(r)\, w(m)\, dm\, dr$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [u(t-r-m)\, v(r)]\, w(m)\, dm\, dr$$

$$= \int_{-\infty}^{\infty} (u * v)(t-m) \cdot w(m)\, dm$$

[~~$A(t) = (u * w)(t)$~~ ~~$= \int u(t-r)\, v(r)\, dr$~~ ~~$A(t-m) = \int_{-\infty} u(t-m-r)\, v(r)$~~]

$$= ((u * v) * w)(t)$$

= R.H.S
(proved)

[$\therefore A(t) = (u * w)(t)$
$= \int_{-\infty}^{\infty} u(t-r)\, v(r)\, dr$
$\therefore A(t-m) = \int_{-\infty}^{\infty} u(t-m-r) \cdot v(r)\, dr$]

3. The Dirac Delta

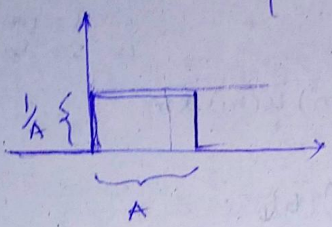
The dirac delta is often used to describe analytically the spike times of a neuron.

Derivative of a function $\Rightarrow \dfrac{d}{dt} x(t) = \lim_{h \to 0} \dfrac{x(t+h) - x(t)}{h}$.

The dirac function $\delta(t)$ can be constructed from a rectangular function with infinitely large height.

3(a) Define a rectangle $R(t)$ with width A and height $1/A$ using a difference of two heaviside step function. Use only ~~used~~ rescaled and shifted versions of the Heaviside step function.

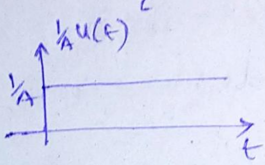
$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0. \end{cases}$$



$$R(t) = \begin{cases} 0 ; & t < 0 \\ \frac{1}{A} ; & 0 \leq t \leq A \\ 0 ; & t > A \end{cases}$$

$$\therefore \boxed{R(t) = \frac{1}{A}\left[u(t) - u(t-A)\right]}$$

$$= \frac{1}{A}u(t) - \frac{1}{A}u(t-A)$$



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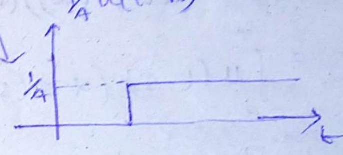
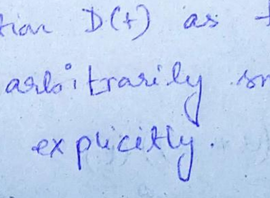


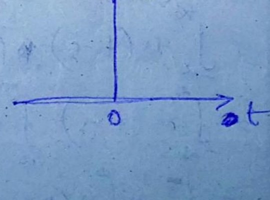
$\frac{1}{A}u(t) - \frac{1}{A}u(t-A)$



(b) Define a new function $D(t)$ as the limit of $R(t)$ when the width A is arbitrarily small. Write down the function $D(t)$ explicitly.

$$D(t) = \lim_{A\to 0} R(t)$$

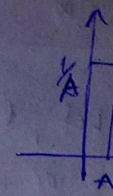
$$= \lim_{A \to 0} \begin{cases} 0 ; & t < 0 \\ \frac{1}{A} ; & 0 \leq t \leq A \\ 0 ; & t > A \end{cases}$$



(c) Show that for $D(t)$ the following properties hold:

(i) $\boxed{D(t) = 0 \text{ for } t \neq 0}$

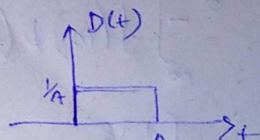
we know, $D(t) = \lim_{A \to 0} \begin{cases} 0 & ; t < 0 \\ \frac{1}{A} & ; 0 \leq t \leq A \\ 0 & ; t > A \end{cases}$



$\therefore$ if $A = 0$, $D(t) = \begin{cases} \infty & ; t = 0 \\ 0 & ; t \neq 0. \end{cases}$ (proved)

when the $A = 0$, the amplitude of the dirac function $= \frac{1}{A} = \frac{1}{0} = \infty$; and thus the rest of place the value becomes 0.

(ii) $\boxed{\int_{-\infty}^{\infty} D(t)\, dt = 1}$



$$\text{L.H.S.} = \int_{-\infty}^{\infty} D(t)\, dt$$

$$= \int_{-\infty}^{\infty} \lim_{A \to 0} R(t)\, dt$$

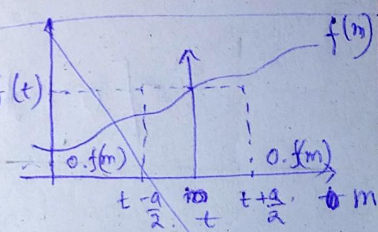
$$= \lim_{A \to 0} \left[A \cdot \frac{1}{A} \right]$$

Area under the curve

$$= \lim_{A \to 0} 1$$

$$= 1 = \text{R.H.S}$$

(iii) $\boxed{\int_{-\infty}^{\infty} f(t-s)\, D(s)\, ds = f(t).}$



~~L.H.S = $\int_{-\infty}^{\infty} f(t-s)\, D(s)\, ds$~~

let, $t - s = m$

$\therefore s = t - m$

$ds = -dm$

~~D.R. $\frac{d}{ds} f(t) = \lim_{t \to 0} f(s+t)$~~

$$\text{L.H.S} = \int_{-\infty}^{\infty} f(t-s)\, D(s)\, ds$$

$$= \int_{\infty}^{-\infty} f(m)\, D(t-m)\, dm = \int_{\infty}^{-\infty} f(m)\, D(m-t)\, dm$$

[$\because D(s)$ = even function]

~~$= f(m) \int_{\infty}^{-\infty} D(t-m)\, dm$~~ $\int_{\infty}^{-\infty}$

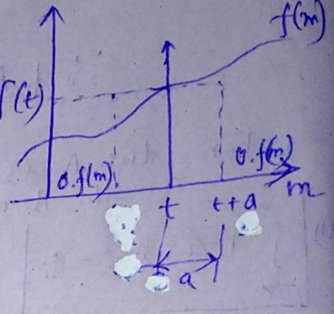
$$= \int_{\infty}^{t+a/2} f(m) \cdot 0\, d$$

(iii) $\boxed{\int_{-\infty}^{\infty} f(t-s)\, D(s)\, ds = f(t)}$

Let, $t - s = m$

or $s = t - m \Rightarrow s = \infty,\ m = -\infty$
$\qquad\qquad\qquad\quad s = -\infty,\ m = \infty$

$\therefore ds = -dm$



$$\therefore \text{L.H.S.} = \int_{-\infty}^{\infty} f(t-s)\, D(s)\, ds$$

$$= -\int_{\infty}^{-\infty} f(m)\, D(t-m)\, dm$$

$$= -\left[\int_{\infty}^{t+a} f(m).0\, dm + \int_{t+a}^{t} f(m).\, D(t-m)\, dm + \int_{t}^{-\infty} f(m).0\, dm\right]$$

$$= -\left[0 + \int_{t+a}^{t} f(m).\, D(t-m)\, dm + 0\right]$$

$$= -\left[\lim_{a\to 0} \int_{t+\frac{a}{2}}^{t} f(m).\, \frac{1}{a}\, dm\right]$$

$$= -\lim_{a\to 0} \frac{1}{a} \int_{t+\frac{a}{2}}^{t} f(m)\, dm$$

$$= -\lim_{a\to 0} \frac{1}{a} F(m)\Big|_{t+\frac{a}{2}}^{t}$$

$\left[\text{let } \int f(m)\, dm = F(m) \text{ and } F'(m) = f(m)\right]$

$$= -\lim_{a\to 0} \frac{F(t) - F(t+a)}{a}$$

$$= \lim_{a \to 0} \frac{F(t+a) - F(t)}{a}$$

$= F'(t)$ [By definition of derivative]

$= f(t)$

$=$ R.H.S. (proved)

3(d) Can you think of other ways of constructing a function that fulfills the three properties in (c)?

The dirac ~~function~~ delta function can as also be constructed by using gaussian distribution.

$$\delta(x) = \lim_{a \to 0} \frac{1}{|a|\sqrt{\pi}} e^{-(x/a)^2}$$