NMC Problem Set #52

OneShot Math Group

Sep. 17, 2023

Welcome!

Problem set delivered to you by the great Niko (in exchange for pancakes). Harder problems are marked with chilies (🌶), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. Field Addition (Arky Homework Moment)
Show that if $1+1+1+1=0$ is true in a field, then $1+1=0$ as well.

A2. (🌶 $\times 2$) **Not Necessarily Lagrange**
Let $a_1, a_2, \ldots, a_{n+1}$ be sequence of distinct nonzero reals, where

$$\sum_{j=1}^{n+1} a_j^2 = 1, \qquad \sum_{j=1}^{n+1} a_j = 0.$$

Show that

$$0 < \sum_{k=1}^{n+1} \frac{1}{|a_k|} \prod_{\substack{j=1 \\ j \neq k}}^{n+1} \frac{a_k}{a_k - a_j} \leq \sqrt{2}.$$

A3. (🌶) **Arky Practice Exam Moment**[1]
Show that for every positive integer n,

$${\frac{2n-1}{e}}^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) < {\frac{2n+1}{e}}^{\frac{2n+1}{2}}.$$

[1] yeah im an idiot i tried to stirling approximation it lol

§2 Combinatorics

C1. Mixing the Functions

How many functions are necessary to generate all functions $f : [n] \to [n]$ through composition? Note $[n] = \{1, 2, \ldots, n\}$.

C2. (🌶 × 2) **Partial Summation**[2]

Given any Riemann integrable f, any sequence of complex (a_n), and any $x \geq 1$, show that

$$\int_1^x f'(t) \left(\sum_{n \leq t} a_n \right) dt = \sum_{n \leq x} a_n \left(f(x) - f(n) \right).$$

Then, show that

$$\sum_{n \leq x} a_n f(n) = \int_{1-\varepsilon}^x f(t) \, d \left(\sum_{n \leq t} a_n \right),$$

with $\varepsilon \to 0^+$.

C3. (🌶 × Open) **Cycles**

Let x, y, z be nonnegative integers. How many solutions are there to

$$xy + yz + zx = N,$$

for N is an integer? Is it true that the upper bound for the number of solutions is $9\sqrt{N}$?

[2] i spotted this on a blackboard lol

§3 Geometry

G1. "Annulus" not "Donut"

Suppose we have circles C_1, C_2 on a plane. Find the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

G2. () **Elliptic Polygon**

Let n be an even number. Given an n-gon circumscribed about an ellipse, such that its vertices lie on another ellipse, show that its principle diagonals coincide at one point.

§4 Number Theory

N1. Alternating Primes

How many primes, written in base 10, have alternating digits of 1's and $0's$ (starting and ending with 1)?