NMC Problem Set #30

OneShot Math Group

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Welcome!

This is a selection of interesting problems derived from curious thoughts, curated so you can nibble on them throughout the week! The point of this document is to introduce you to fun puzzles that require thinking. We recommend you try the ones that you find interesting! Feel free to work on them with others (even us teachers!). Harder problems are marked with chilies (🌶), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. (🌶 × 2) **Superficial Similarities Spawn Spurious Statements** [1]
Let n be a positive integer. Prove or disprove,

$$\left\lceil \frac{2}{2^{1/n}-1} \right\rceil = \left\lfloor \frac{2n}{\log 2} \right\rfloor.$$

A2. (🌶) **The Sophomore's Dream** [2]
Prove that

$$\int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n}.$$

[1] if you haven't yet, you should read Richard K. Guy's *Strong Law of Small Numbers*! figured i'd put this problem bc it's a pretty famous example... can br proven with alg or combi, but i prefer alg

[2] are u guys fine with calculus in the alg section? i mean, it's done with alg techniques ig... lmk!

§2 Combinatorics

C1. Collection

Let $S_1, S_2, S_3, \ldots, S_m$ be distinct subsets of $\{1, 2, 3, \ldots, n\}$ such that $|S_i \cap S_j| = 1$ for all $i \neq j$. Prove that $m \leq n$.

§3 Geometry

G1. Sharing a Donut

Suppose we have a torus. What is the most number of pieces (no moving the pieces around!) into which said torus can be cut with three straight cuts?

a) Prove that with n straight cuts, you can slice this donut into

$$\frac{n^3 + 3n^2 + 8n}{6}$$

pieces. For a bit of intuition, consider the "cake numbers," which are the number of pieces a cake (with no holes) can be sliced into with n planar cuts,

$$C_n = \frac{n^3 + 5n + 6}{6}.$$

b) (🌶) Suppose you're now allowed to rearrange your mess of shredded donut that you are about to create (as in, you may rearrange each piece after each slice). What is the maximum number of pieces you can obtain with 3 slices?

c) (🌶 × Open) What about n slices while allowing for rearrangement?

G2. (🌶) Concave Quadrilaterals

Can a convex polygon be dissected entirely into concave quadrilaterals?

§4 Number Theory

N1. Arrangements

Let (a_n) be real numbers. Show that

$$\min_{i<j}(a_i - a_j)^2 \leq M^2(a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2),$$

where

$$M^2 = \frac{12}{n(n^2-1)}.$$