NMC Problem Set #39

OneShot Math Group

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Welcome!

This is a selection of interesting problems derived from curious thoughts, curated so you can nibble on them throughout the week! The point of this document is to introduce you to fun puzzles that require thinking. We recommend you try the ones that you find interesting! Feel free to work on them with others (even us teachers!). Harder problems are marked with chilies (🌶), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. Overcomplicating Much?
For any six irrational numbers, show that there is a subset of at least three of them where the sum of any pair is irrational.

§2 Combinatorics

C1. (🌶) **Triangulating Bigly**

Let A be the area of a triangle formed by three points selected at random and independently from a convex set $D \subset \mathbb{R}^2$ of area 1. Show that for $0 \le a \le 1$, we have $\mathbb{P}(A \le a) \ge a$.

a) (🌶 $\times 2.5$) Let A be the area of a triangle formed by three points independently picked at random from a unit disc (of area π). Prove that for $a > 0$, we have $\mathbb{P}(A \le a) \le 4a$.

b) (🌶 $\times 3$) Let's prove a sharper inequality: first, denote x, y, z as the three points, and show that

$$\mathbb{P}(A \le a \mid d(x,y) = t) \le \frac{4a}{t\pi}.$$

Deduce from $\mathbb{P}(d(x,y) \le t) \le t^2$ that

$$\mathbb{P}(A \le a) \le \int_0^1 2t \left(\frac{4a}{t\pi}\right) dt + \frac{4a}{\pi}.$$

c) (🌶 $\times 3$) Prove that, for every $n \ge 3$, you can find n points in the unit disc such that no three points form a triangle of area less than $1/6n^2$.

§3 Geometry

G1. Equilateral Condition
Let the incircle of triangle ABC be tangent to BC at D, CA at E, and AB at F. Prove that if the centroids of triangles ABC and DEF coincide, then ABC must be equilateral.

a) Let DEF be a cevian triangle of point P relative to ABC. Show that if the centroids of ABC and DEF coincide, then P is the centroid of ABC.

G2. (🌶️) Polygonating Small-ly[1]
Show that every convex set of area 1 can be contained in a rectangle of area 2.

a) (🌶️ $\times\, 3$) Same as the above, but with a triangle: show that every convex set of area 1 can be contained within a triangle of area 2.

[1] this ain't even a triangulation

§4 Number Theory

N1. Mouthful of Ones
The digits of a number contain 81 ones. Show that it's divisible by 81.

N2. (🌶) "Geometric Solution" (they never found it)[2]
For all $n > 1$, show that

$$\left\lfloor \frac{n^k}{k} \right\rfloor$$

is odd for infinitely many choices of k.

[2] story goes that arky thought this was doable with geometry until they solved it w/o lol. probably was too tired to think when that happened