NMC Problem Set #19

OneShot Math Group

Dec. 25, 2022

Welcome!

This is a selection of interesting problems derived from curious thoughts, curated so you can nibble on them throughout the week! The point of this document is to introduce you to fun puzzles that require thinking. We recommend you try the ones that you find interesting! Feel free to work on them with others (even us teachers!). Harder problems are marked with chilies (🌶), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. **Detour (Calculus)**
Suppose we have a real polynomial, $P(x)$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(f(x)) = P(x)$.

a) Suppose we have $P(x) = 2x^3 - 2x^2 + 2x - 1$. Prove that f is not differentiable.

b) Suppose $f(f(x)) = x$ has the unique solution a. Prove that $f(x) = x$ also has the unique solution a, then conclude that if $P'(a)$ is negative, f is not differentiable. [1]

c) (🌶 × Open) For which choices of polynomial $P(x)$ will we have a non-differentiable f?

A2. **French Functions**
Call a continuous function *French* if $f(f(x)) = 2f(x) - x$ for all x. Determine all such *French* functions.

[1] note that this is a sufficient but definitely not a necessary condition by any means

§2 Combinatorics

C1. Snake Game

Suppose we are playing the snake on an $m \times n$ board. Here's a link to how said game works!

a) Define a win position to be a board where the snake covers all mn squares. How many possible different win positions are there for $m, n = 3, 4, 5$?

b) (🌶 × Open) Determine the exact number of win positions for any choice of naturals m, n. This is the same as asking for the number of Hamiltonian paths on a rectangular lattice of the same dimensions.

C2. Selection

A group of friends contains n people. If the number of ways of selecting 6 friends from the group is the same as the number of ways to select 9, find the number of ways to select 4 friends.

§3 Geometry

G1. (🌶) **Hypersphere**
Suppose we have a d-dimensional unit hypersphere. Provide an intuitive argument on why, as $d \to \infty$, the volume of said hypersphere approaches 0.

G2. **Staring at Polyhedra**
Prove that for any polyhedron P with n faces, there is always at least one way to "look" at it so that you can see $n/2$ faces. Also prove the maximum possible number of faces you can see at once is $n - 1$. Here's an example below:

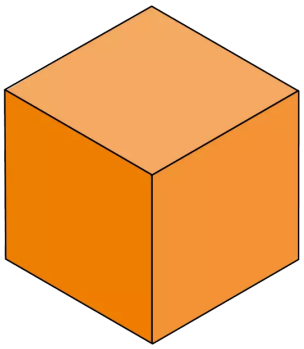

Figure 1: Here, we can see 3 visible sides of the cube.

§4 Number Theory

N1. **Nice Dates**

It's nearly 2023! We call a day *nice* if, when the date is written out in the form $YYYY/DD/MM$, the number

$$\frac{YYYY}{DD/MM}$$

is an integer. How many nice days does 2023 have and has there been a nicer year in the past century? The "niceness" of a year is defined by how many nice days it has. [2]

N2. **How Santa Determines if a Kid is Naughty or Nice**

Suppose we have a sequence (a_n), with $a_1 = 1$. Let a_n be defined as the least positive integer not equal to a_i or $a_i + i$ for all $i < n$.

a) Show that $a_n = \lfloor n\varphi \rfloor$.

b) (🌶) Should Santa only deliver gifts to children whose address contains some number from (a_n)? Provide a 3000 word philosophical essay detailing the ethics of selective gift-giving and the ramifications of civil lawsuits on jealousy and selfishness. [3]

[2] DMO Daily

[3] what did i do here idk