NMC Problem Set #48

OneShot Math Group

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Welcome!

This is a selection of interesting problems derived from curious thoughts, curated so you can nibble on them throughout the week! The point of this document is to introduce you to fun puzzles that require thinking. We recommend you try the ones that you find interesting! Feel free to work on them with others (even us teachers!). Harder problems are marked with chilies (), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. Resistant Prime

Does there exist a collection of 15 integers $n_1, n_2, \ldots, n_{15}$ such that

$$\sum_{k=1}^{15} n_k \cdot \arctan(k) = \arctan(16)?$$

A2. () The Red Furry on Sesame Street

Let $a_1, a_2, \ldots, a_m$ be a sequence of positive integers. Show that there exists three positive integers b, c, N such that

$$\left\lfloor \sum_{i=1}^{m} \sqrt{n + a_i} \right\rfloor = \left\lfloor \sqrt{bn + c} \right\rfloor.$$

§2 Combinatorics

C1. Vandalism

Niko and Arky are playing a simple game where they take turns placing pennies flat on a circular table. If a player has no space to place a penny when it is their turn, they lose the game. Given that Niko starts first and Arky places second, who has the winning strategy and what is it?

§3 Geometry

G1. (🌶) **Fr*nch Duo Solve**

Let us have a cyclic quadrilateral $ABCD$. Let $AD \cap BC = E$, $AB \cap CD = F$, $AC \cap EF = R$, and $BD \cap EF = Q$. If M, N are midpoints of BD, AC respectively, show that $MNRQ$ are concyclic.

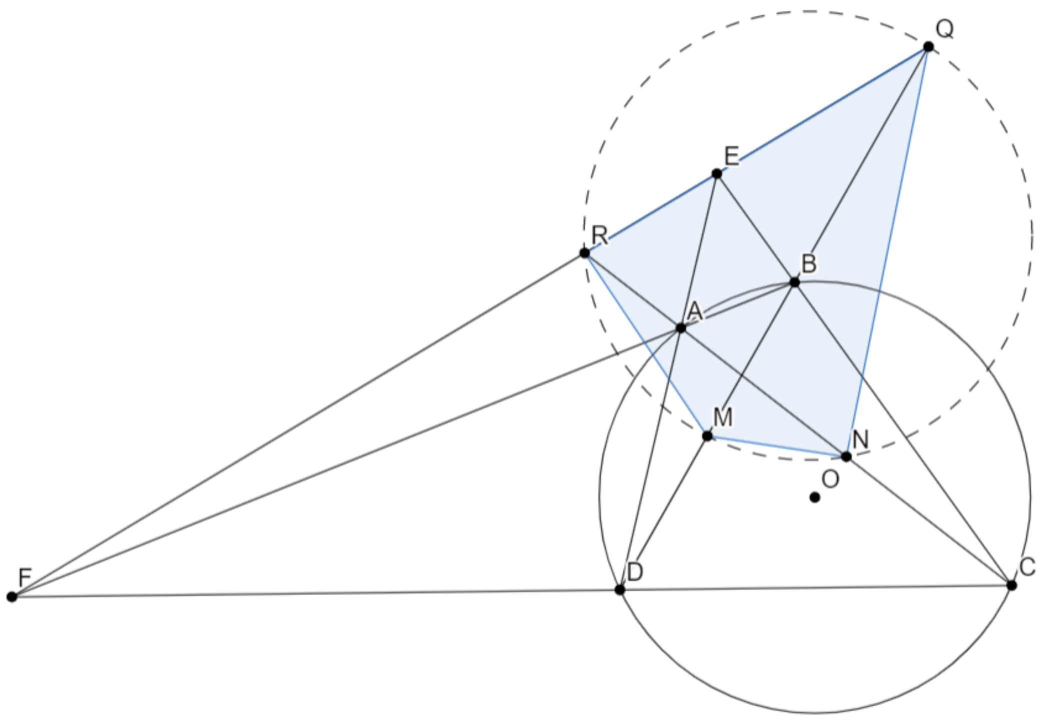

Figure 1: wow we have a diagram again x2

G2. (🌶 × 2) **Tetra-Pak**

In a tetrahedron $ABCD$ we denote α, β, γ to be the measures, in radians, of the angles of the three pairs of opposite edges. Let r, R be the radii of the inscribed and circumscribed sphere respectively. Show that

$$\left(\frac{3r}{R}\right)^3 \leq \sin\frac{\alpha+\beta+\gamma}{3}.$$

§4 Number Theory

N1. (🌶 $\times$ 1.5) **Unsuitable for Competition?**
Let n be a positive integer, and $p(x)$ be a polynomial degree n with integer coefficients. Show that

$$\max_{x\in[0,1]} |p(x)| > \frac{1}{e^n}.$$

a) (🌶 $\times$ 2+) Let $p \in \mathbb{Z}[x]$ as above with n as its degree. Prove that

$$\int_0^1 \log|p(x)|\,dx \geq -n.$$

N2. Iwai Munehisa[1]
We wish to construct a set of geometric series $A_1, A_2, \ldots, A_n$ of real numbers such that

$$\{1, 2, 3, \ldots, 100\} \subseteq A_1 \cup A_2 \cup \cdots \cup A_n.$$

Show that $n \geq 31$ by considering squarefree terms in series.

a) Without using Wolfram to bash it all out, improve the previous bound and determine the minimum possible n.

[1] i think i used the phrase layer cake for a problem before so here's an easter egg for all you p5r gamers