# NMC Problem Set #49

OneShot Math Group

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## Welcome!

This is a selection of interesting problems derived from curious thoughts, curated so you can nibble on them throughout the week! The point of this document is to introduce you to fun puzzles that require thinking. We recommend you try the ones that you find interesting! Feel free to work on them with others (even us teachers!). Harder problems are marked with chilies (🌶), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

## §1 Algebra

**A1. "Yearlong Contest"**[1]

Let $a, b, c$ be the legs of a right angled triangle. Show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 5+3\sqrt{2}.$$

a) (🌶 $\times 1.5$) In any triangle $ABC$ where $\max\{A, B, C\} \geq \phi \geq 90^\circ$, show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 5+\frac{3-2\cos\phi}{\sin\frac{\phi}{2}}.$$

**A2. (🌶) Out of Nowhere**

Let $(a_n)$ be a sequence of nonnegative reals. Then,

$$\sum_{n=1}^{\infty}(a_1a_2\ldots a_n)^{1/n} \leq e\sum_{n=1}^{\infty}a_n,$$

where equality holds if and only if $(a_n) = 0$.

[1] yozsef wildt thingy

# §2 Combinatorics

**C1. Domino Collector**

An eccentric collector of $2 \times n$ domino tilings pays \$4 for each vertical domino and \$1 for each horizontal domino. Note that each domino is of size $2 \times 1$! How many tilings are worth exactly \$$m$ by this criterion for some value $m$?

a) Let $S_n$ denote the collection of all dominoes used in a $2 \times n$ domino tiling. Show that the probability a randomly selected tile from $S_n$ is vertical approaches $\frac{1}{\sqrt{5}}$ as $n \to \infty$.

b) (🌶️ $\times 2$) Show that the following expression,

$$2^{mn/2} \prod_{\substack{1 \le j \le m \\ 1 \le k \le n}} \left( \left( \cos^2 \frac{j\pi}{m+1} \right) A^2 + \left( \cos^2 \frac{k\pi}{n+1} \right) B^2 \right),$$

is the generating function for the number of tilings of an $m \times n$ board with dominoes. Note that the coefficient of $A^j B^k$ is the number of ways to tile with $j$ vertical and $k$ horizontal dominoes.

**C2.** (🌶️ $\times 2$) **Polygon Decomposition**[2]

Show that the number of ways to dissect a convex $n$-gon into $m$ polygons by non-intersecting diagonals is expressed by the factor of $w^m z^{n-2}$ in the following generating function,

$$F(z, w) = \frac{1 + z - \sqrt{1 - (4w + 2)z + z^2}}{2(1 + w)z}.$$

Start by proving that the coefficient of $z^{n-2}$ in the generating function

$$Q = 1 + zQ^2 + z^2Q^3 + z^3Q^4 + \cdots = \frac{1 + z - \sqrt{1 - 6z + z^2}}{4z},$$

is the number of ways to place non-overlapping diagonals in a convex $n$-gon.

a) (🌶️) Let $e_n$ and $o_n$ be the number of dissections of a convex $n$-gon by non-intersecting diagonals into an even or odd number of regions, respectively. Show that $e_n - o_n = (-1)^n$.

[2] decomposable plastic

## §3 Geometry

**G1.** () **Triangle Inequality Again**

Let $\alpha, \beta, \gamma$ be the angle measures of a non-degenerate triangle. Show that

$$\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos \alpha + \cos \beta + \cos \gamma} \geq -2,$$

with equality occuring if and only if the triangle is equilateral.

**G2.** ($\times 2$) **'19 ISL G6**

Let $I$ be the incenter of acute triangle $ABC$. Let the incircle meet $BC, CA$, and $AB$ at $D, E$, and $F$ respectively. Let line $EF$ intersect the circumcircle of the triangle at $P$ and $Q$, such that $F$ lies between $E$ and $P$. Prove that $\angle DPA + \angle AQD = \angle QIP$.

## §4 Number Theory

**N1.** () **Elephant Test (of a description)**
Let $n, x$ be positive integers such that $x$ has no divisors less than or equal to $n$ aside from 1. If $p$ is a prime, show that at least $\lfloor n/p \rfloor$ of the numbers $\{x-1, x^2-1, x^3-1, \ldots, x^{n-1}-1\}$ are multiples of $p$.

**N2. Kind Server Overlord**
Let $p$ be a prime. Show that $p^p - (p-1)^{p-1}$ cannot be a square.