

NMC Problem Set #28

ONESHOT MATH GROUP

Mar. 12, 2023

Welcome!

This is a selection of interesting problems derived from curious thoughts, curated so you can nibble on them throughout the week! The point of this document is to introduce you to fun puzzles that require thinking. We recommend you try the ones that you find interesting! Feel free to work on them with others (even us teachers!). Harder problems are marked with chilies (🌶️), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. This Somehow Works

Show that the equations,

$$\begin{aligned}x &= u + v + w, \\y &= u^2 + v^2 + w^2, \\z &= u^3 + v^3 + w^3,\end{aligned}$$

with $uvw = 1$ define a cubic surface.

§2 Combinatorics

C1. Probability of a Non-singularity

Given a random square matrix M of size $n \times n$ in the field \mathbb{F}_2 , what is the probability that $\det(M) = 1$?

§3 Geometry

G1. (👉) Generalized Power of a Point

Fix a point P inside an n -dimensional sphere with center O . Show that the sum of the squares of the lengths of any n mutually perpendicular chords is invariant (that is, the value depends only on the radius of the n -sphere and the distance \overline{OP}).

§4 Number Theory

N1. (👉) Sylvester's Theorem

Let a and b be coprime positive integers. Prove that $ab - a - b$ is the largest positive integer c for which the equation

$$ax + by = c$$

is not solvable in nonnegative integers.

- a) This time, let a and b be positive integers (not necessarily coprime). For a nonnegative integer n let $s(n)$ be the number of nonnegative integer solutions to the equation $ax + by = n$. Prove that the generating function of the sequence $(s(n))_n$ is

$$f(x) = \frac{1}{(1 - x^a)(1 - x^b)}.$$

- b) Given a positive integer $n > 6$, show that the equation $x + y = n$ admits a solution with coprime $x, y > 1$. ¹

¹funny but easy result