NMC Problem Set #56

OneShot Math Group

Oct. 15, 2023

Welcome!

Problem set delivered to you by the great Niko (in exchange for pancakes). Harder problems are marked with chilies (🌶), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. Hazard Duty Pay!
Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate $\lim_{n\to\infty} \frac{M(n)}{n}$.

§2 Combinatorics

C1. (...?) **End Credits!**
Let $X = \{1, 2, \ldots, n\}$, and let $k \in X$. Show that there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$, there is a $j \geq 0$ such that $f^j(x) \leq k$. (Note that f^j is the jth iterate of f).

C2. ($\times$ Open) **Trust!**
Is it necessarily true that, when played perfectly, $5 \times 5 \times 5$ tic-tac-toe ends in a draw? (5 in a row to win)

§3 Geometry

G1. () **NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO! NO!**[1,2]

A line in the plane of a triangle T is called an equalizer if it divides T into two regions having equal area and equal perimeter. Find positive integers $a > b > c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two equalizers.

G2. Dam! Dam! Dam!

What is the maximum number of rational points that can be on a circle in $\mathbb{R}^2$ whose center is not a rational point?

[1] i'm sorry

[2] *clearly* i dislike geo hehe

§4 Number Theory

N1. () **Kissy Face Emoji!**

Let $B(n)$ be the number of ones in the base two expression of the positive integer n. For example, $B(6) = B(110_2) = 2$ and $B(15) = B(1111_2) = 4$. Determine whether or not

$$k = \exp\left(\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)}\right)$$

is rational.

a) Compute e^k.

N2. () **God Don't Like Ugly!**

Let $\delta(x)$ be the greatest odd divisor of the positive integer x. Show that, for all positive integers x,

$$\left|\sum_{n=1}^{x} \frac{\delta(n)}{n} - \frac{2x}{3}\right| < 1.$$