NMC Problem Set #57

OneShot Math Group

Oct. 22, 2023

Welcome!

Problem set delivered to you by the great Niko (in exchange for pancakes). Harder problems are marked with chilies (🌶), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. Tron!

Let n be a positive integer, and let

$$f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}.$$

Prove that f_n has no roots in the closed unit disc $\{z \in \mathbb{C} \mid |z| \leq 1\}$.

A2. (🌶) One or Six

Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials in the form $ax + by + cz$, where a, b, c are integers.

A3. (🌶) Gamble

For each $\alpha \in \mathbb{R}$, define $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let $n \in \mathbb{N}$; show that there exists a real α such that for $1 \leq \ell \leq n$:

$$f_\alpha^\ell(n^2) = n^2 - \ell = f_{\alpha^\ell}(n^2).$$

Here, $f_\alpha^\ell = (f_\alpha \circ f_\alpha \circ \cdots \circ f_\alpha)(x)$ where the composition is carried out ℓ times.

§2 Combinatorics

C1. () **Halfway Home**

Let $f_1, f_2, \ldots, f_{10}$ be bijections $\mathbb{Z} \to \mathbb{Z}$ such that for each integer n, there is some composition $f_{\ell_1} \circ f_{\ell_2} \circ \cdots \circ f_{\ell_m}$ (allowing repetition) mapping 0 to n. Consider the set of 1024 functions,

$$\mathcal{F} = \{f_1^{\varepsilon_1} \circ f_2^{\varepsilon_2} \circ \cdots \circ f_{10}^{\varepsilon_{10}} \mid e_i = 0 \text{ or } 1 \text{ for } 1 \le i \le 10\}.$$

Show that if A is a finite set of integers then at most 512 of the functions in $\mathcal{F}$ map A into itself.

C2. Homework...?

For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \le m$. Show that $f(m, n) = f(n, m)$.

§3 Geometry

G1. Rare Sighting
Given A, B, C non-collinear points in the integer lattice plane such that AB, BC, CA are of integer length, what's the smallest possible value of AB?

G2. Hit or Miss
Let $\mathfrak{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E \subset \mathbb{R}^2$.[1] Show that there is a pairwise disjoint subcollection $D_1, D_2, \ldots, D_n$ in $\mathfrak{F}$ such that

$$\bigcup_{j=1}^{n} 3D_j \supset E.$$

Here, if D is a disc of radius r and center P, then $3D$ is a disc of radius $3r$ and center P.

[1] note that I use $\subset$ and $\subsetneq$, sorry- habits

§4 Number Theory

N1. (🌶) **"Square Plus One"**

Let p be an odd prime and let $\mathbb{Z}_p$ denote the field of integers modulo p. How many elements are in the set

$$\{x^2 \mid x \in \mathbb{Z}_p\} \cap \{y^2 + 1 \mid y \in \mathbb{Z}_p\}?$$