NMC Problem Set #50

OneShot Math Group

Aug. 27

Welcome!

This is a selection of interesting problems derived from curious thoughts, curated so you can nibble on them throughout the week! The point of this document is to introduce you to fun puzzles that require thinking. We recommend you try the ones that you find interesting! Feel free to work on them with others (even us teachers!). Harder problems are marked with chilies (🌶️), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. Conversion

Prove the following conversion formula,

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor .$$

A2. (🌶️) Choicy Choices

Find a closed form for

$$\sum_{k \geq 1} \binom{n}{\lfloor \log_m k \rfloor}$$

with m, n positive integers.

A3. (🌶️) Choicier Choices

Let p be a prime. Then, show that

$$\binom{n}{m} \equiv \binom{\lfloor n/p \rfloor}{\lfloor m/p \rfloor} \binom{n \bmod p}{m \bmod p} \pmod{p}$$

for all nonnegative integers m and n.

§2 Combinatorics

C1. Coinflip Gamble
Does there exist patterns A and B of heads and tails such that A is longer than B, yet A appears before B does more than half the time when a fair coin is being flipped?

C2. () Ellipticity
Given a certain number of balls, of which some are blue, pick 5 at random. The probability that all 5 are blue is 1/2. Determine the original number of balls and decide how many were blue, where $m, n < 15$ reasonably.

a) ($\times$ Open...?) If we do not consider $m, n < 15$, is there more than one solution to
$$\binom{n}{5} = 2\binom{m}{5}?$$

C3. () Frog has the Hard Problems this time
Given positive integers $a_1, a_2, \ldots, a_k$, let $S(n)$ denote the number of solutions to $a_1x_1 + a_2x_2 + \cdots + a_kx_k = n$ for a nonnegative integer sequence (x_k). It is known that $S(n) \neq 0$ for all large enough n. Prove that $S(n+1) < 2S(n)$ for all large enough n.

a) Can the factor of 2 be improved to $1 + \varepsilon\,(\varepsilon > 0)$ above, as in
$$S(n+1) < (1+\varepsilon)S(n)?$$

§3 Geometry

G1. () **Trivium for "all UGs"**
Find the mathematical expectation of the area of the projection of a cube with edge of length 1 onto a plane with an isotropically distributed random direction of projection.

§4 Number Theory

N1. Fibonacci Avoiding Sequence[1]

Given an integer n, show that $n^7 - 77$ is never a Fiboancci number by considering the sequence mod 29.

N2. () Composite Sum[2]

Let a, b, c, d be naturals such that $ad = bc$. Show that $a+b+c+d$ cannot be prime.

[1] jeez we really hate this fibonacci dude

[2] idk we tend to just say not prime these days i haven't heard the word composite in the past while