Network Modeling
Solutions for 09/07/2007

Problem 1

Distribution of X_1 is the first row of P.
Distribution of X_2 is the first row of P^2.
Distribution of X_{500} is the first row of P^{500}. Obviously it is not required to compute the $500-th$ power of P: we can assume the chain is in long run behavior, so we can use the steady-state distribution. $\pi = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right]$.
For the average first passage times we need to solve the usual system:
$\begin{cases} \hat{\theta}_{02} = 1 + p_{00}\hat{\theta}_{02} + p_{01}\hat{\theta}_{12} \\ \hat{\theta}_{12} = 1 + p_{10}\hat{\theta}_{02} + p_{11}\hat{\theta}_{12} \end{cases} \Rightarrow \begin{cases} \hat{\theta}_{02} = 3 \\ \hat{\theta}_{12} = 2 \end{cases}$. Then last average first passage time is $\bar{\theta}_{22} = \frac{1}{\pi_2} = 4$.

$$\Pr[X_1 = 1, X_3 = 1 | X_2 = 1] = \frac{\Pr[X_1 = 1, X_2 = 1, X_3 = 1 | X_0 = 0]}{\Pr[X_2 = 1 | X_0 = 0]} = \frac{p_{01}p_{11}^2}{p_{01}^{(2)}}$$

$$\Pr[X_2 = 1 | X_1 = 1, X_3 = 1] = \frac{\Pr[X_2 = 1, X_3 = 1 | X_1 = 1]}{\Pr[X_3 = 1 | X_1 = 1]} = \frac{p_{11}^2}{p_{11}^{(2)}}$$

Problem 2

$$\Pr[\text{no disposal}] = \left(\frac{\frac{1}{\beta}}{\frac{1}{\alpha} + \frac{1}{\beta}}\right)^2 = \left(\frac{\alpha}{\alpha + \beta}\right)^2 = \left(\frac{1}{10}\right)^2 = 0.01.$$

Average pieces produced per hour are:

$$\begin{aligned} T &= 2 \cdot 12 \cdot \Pr[\text{2 p. working}] + 12 \cdot \Pr[\text{1 p. working}] \\ &= 24 \cdot \left(1 - \frac{1}{10}\right)^2 + 12 \cdot \left(1 - \left(1 - \frac{1}{10}\right)^2 - \left(\frac{1}{10}\right)^2\right) = 21.6 \text{ pieces per h.} \end{aligned}$$

Last point is straightforward:

$$\begin{aligned} T &= 30 \cdot \Pr[\text{2 p. working}] + 12 \cdot \Pr[\text{1 p. working}] \\ &= 30 \cdot \left(1 - \frac{1}{10}\right)^2 + 12 \cdot \left(1 - \left(1 - \frac{1}{10}\right)^2 - \left(\frac{1}{10}\right)^2\right) = 26.46 \text{ pieces per h.} \end{aligned}$$

Problem 3

Embedded Markov chain is $P = \begin{bmatrix} 0 & 1 & 0 \\ 1-\alpha & 0 & \alpha \\ 1 & 0 & 0 \end{bmatrix}$. Average times matrix is $\mathbf{T} = \begin{bmatrix} - & T & - \\ \beta T & - & \frac{\beta T}{2} \\ \gamma T & - & - \end{bmatrix}$.

Average times associated to the visits are given by $\mu = \begin{bmatrix} \mu_S \\ \mu_L \\ \mu_R \end{bmatrix} = \begin{bmatrix} T \\ (1-\alpha)\beta T + \alpha\frac{\beta T}{2} \\ \gamma T \end{bmatrix}$.

Fraction of time spent in each state is given by $P_i = \dfrac{\mu_i \pi_i}{\sum_j \mu_j \pi_j}$.

We now solve $\begin{cases} \pi_L = \pi_S \\ \pi_R = \alpha \pi_L \\ \pi_S + \pi_L + \pi_R = 1 \end{cases} \Rightarrow \begin{cases} \pi_L = \frac{1}{2+\alpha} \\ \pi_S = \frac{1}{2+\alpha} \\ \pi_R = \frac{\alpha}{2+\alpha} \end{cases}$.

Finally $P_i = \dfrac{(2+\alpha)(\pi_i \mu_i)}{T\left(1+\beta-\frac{\alpha\beta}{2}+\alpha\gamma\right)}$.

Problem 4

Notice that service time Y is deterministic, hence $G_Y(x) = \begin{cases} 1 & x \geq 6 \\ 0 & x < 6 \end{cases}$.

Let $M(t)$ be the random variable counting users in the system at time t.

$\Pr[M(0.5) = 0] = \dfrac{e^{-\Lambda}(\Lambda)^0}{0!}$, where $\Lambda = \lambda \int\limits_0^{0.5} [1 - G_Y(z)]dz = \dfrac{\lambda}{10} = 2$.

Now recall that $M(t)$ conditioned on $X(t)$ is a Binomial random variable:

$$\Pr[M(t) = m | X(t) = n] = \binom{n}{m} p^m (1-p)^{n-m}$$

where

$$p = \frac{1}{t} \int\limits_0^t [1 - G_Y(z)]dz$$

Finally $\Pr[M(0.5) = 0 | X(0.5) = 10] \binom{10}{0} (0.2)^0 (0.8)^{10}$