Network Modeling
Solutions for 14/07/2006

Problem 1

Distribution of X_1 is the first row of P.
Distribution of X_2 is the first row of P^2.
Distribution of X_{500} is the first row of P^{500}. Obviously it is not required to compute the $500-th$ power of P: we can assume the chain is in long run behavior, so we can use the steady-state distribution.
$\pi = \left[\frac{10}{27}, \frac{4}{9}, \frac{5}{27}\right]$.

Average first passage time from 0 to 0 is $\bar{\theta}_{00} = \frac{1}{\pi_0} = \frac{27}{10}$.

For the other states we need to solve the usual systems:
$\begin{cases} \hat{\theta}_{01} = 1 + p_{00}\hat{\theta}_{01} + p_{02}\hat{\theta}_{21} \\ \hat{\theta}_{21} = 1 + p_{20}\hat{\theta}_{01} + p_{22}\hat{\theta}_{21} \end{cases}$ and $\begin{cases} \hat{\theta}_{02} = 1 + p_{00}\hat{\theta}_{02} + p_{01}\hat{\theta}_{12} \\ \hat{\theta}_{12} = 1 + p_{10}\hat{\theta}_{02} + p_{11}\hat{\theta}_{22} \end{cases}$

Last point requires to compute $W_{ij}^{(n)} = E\left[\sum_{k=0}^{n-1} \chi\{X_k = j\} | X_0 = i\right] = \sum_{k=0}^{n-1} p_{ij}^{(k)}$.

In vector form we can write $W_{0j}^{(3)} = p_{0j}^{(0)} + p_{0j}^{(1)} + p_{0j}^{(2)} = \begin{bmatrix} 1.6 \\ 0.84 \\ 0.56 \end{bmatrix}$.

For $W_{0j}^{(5000)}$ we can just use the steady state distribution multiplied by 5000:
$W_{0j}^{(5000)} \simeq 5000\pi_j = \begin{bmatrix} 1852 \\ 2222 \\ 926 \end{bmatrix}$.

Problem 2

This system can be modeled as an alternating process, where the renewal instant is the first arrival since the link is empty.
A transmission requires exactly $E[\text{tx time}] = \frac{|\text{packet}|}{L} = 1$ ms.
Average waiting time is $E[\text{tx time}] = \frac{100}{\lambda} = 200$ ms.
Fraction of time the link is empty is distributed as an exponential of mean $\frac{1}{\lambda} = 2$ ms.
Now we can compute the average cycle duration: $E[\text{cycle time}] = E[\text{tx time}] + E[\text{tempty time}] = 3$ ms.
Throughput is then $T = L\frac{E[\text{tx time}]}{E[\text{cycle time}]} = 0.333$ Mbps.
Let β be the probability of finding the system in the busy state: $\beta = \frac{E[\text{tx time}]}{E[\text{cycle time}]} = \frac{1}{3}$.
Let N be the number of consecutive failed attempts before a successful transmission: clearly N is a geometric random variable.
We have $\mathrm{P}[N \geq k] = \beta^k$ and so $E[N] = \sum_{k=1}^{\infty} \beta^k = \frac{\beta}{1-\beta} = \frac{1}{2}$.
Then the average delay is $E[\text{delay}] = E[N]\, E[\text{tx time}] = \frac{1}{2}\frac{100}{\lambda} = 100$ ms.
For the last point: $\frac{E[\text{gain in a cycle}]}{E[\text{cycle time}]} = \frac{1 - 0.2E[\text{arrivals in 1 ms}]}{E[\text{cycle time}]} = \frac{1 - 0.2\lambda 10^{-3}}{E[\text{cycle time}]} = 300$.

Problem 3

Clearly this system can be modeled as an $M/G/\infty$ queue.
Arrivals are Poisson of parameter $\lambda = 10$ customers per hour.
Service time Y is uniformly distributed between 20 and 30 minutes. Converting in hours, it is between 0.33 and 0.5 hours.
Let $M(t)$ be the number of users in the whole system at time t.
For first question, we have

$$\begin{aligned} \mathrm{P}\left[X(0.5) < 3\right] &= \mathrm{P}\left[X(0.5) = 0\right] + \mathrm{P}\left[X(0.5) = 1\right] + \mathrm{P}\left[X(0.5) = 2\right] \\ &= \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^0}{0!} + \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^1}{1!} + \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^2}{2!} = e^{-5}\left(1 + 5 + \frac{25}{2}\right) \simeq 0.1246. \end{aligned}$$

Second point is straightforward, since $\mathrm{P}\left[M(0.25) = 1\right] = \dfrac{e^{-\lambda pt}\left(\lambda pt\right)^1}{1!}$ where now $\lambda pt = \int\limits_0^t \left[1 - G(z)\right] dz$.

Computed in $t = 0.25$, $\lambda pt = \lambda\left(0.25\right) = 2.5$. Finally $\mathrm{P}\left[M(0.25) = 1\right] = \dfrac{e^{-2.5}\left(2.5\right)^1}{1!} \simeq 0.2052$.

For the last point we can simply use the approximation for $t \to \infty$: $\mathrm{P}\left[M(10) = 0\right] = \dfrac{e^{-\Lambda}\left(\Lambda\right)^0}{0!}$ where $\Lambda = \lambda E\left[Y\right]$.

Problem 4

Transition matrix is $P = \begin{bmatrix} 0.98 & 0.02 \\ 0.1 & 0.9 \end{bmatrix}$. Protocol matrix is $C = \begin{bmatrix} p_{00} & p_{01} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix}$.

First question is straightforward: $\mathcal{T} = \dfrac{p_{10}^{(m)}}{p_{10}^{(m)} + mp_{01}}$, where $m = 2$.

We do not need to compute all C^2 but only $p_{10}^{(2)} = p_{10}p_{00} + p_{11}p_{10} = 0.188$. Then $T = 0.8245$.
For the second point we can model the system as an alternating process: in the first phase we have a Markov behavior already discussed, while in the second one we have the forward IID errors behavior.
For the latter we have $\mathcal{T}_{iid} = \dfrac{1-\epsilon}{1-\epsilon+m\epsilon} = 0.98$.

Then $\mathcal{T} = \mathcal{T}_{Markov}\dfrac{E\left[\text{Markov behavior}\right]}{E\left[\text{cycle duration}\right]} + \mathcal{T}_{iid}\dfrac{E\left[\text{IID behavior}\right]}{E\left[\text{cycle duration}\right]} = \mathcal{T}_{Markov}\dfrac{1}{3} + \mathcal{T}_{iid}\dfrac{2}{3} = 0.9281$.