Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2006/2007
prova scritta – 09 luglio 2007 – parte A

E1 Consider a Markov chain X_n with the following transition matrix (states are numbered from 0 to 2), and initial state $X_0 = 0$:

$$P = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.6 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Draw the transition diagram, and find the probability distributions of X_1, X_2 and X_{500}

(b) Compute the average first passage times from states 0, 1 and 2 to state 2.

(c) Compute $P[X_1 = 1, X_3 = 1|X_2 = 1]$ and $P[X_2 = 1|X_1 = 1, X_3 = 1]$.

E2 Consider a factory with two identical machines. Each machine alternates periods of time in which it is working or not working, of exponential duration with mean $1/\alpha = 27$ days (working) and $1/\beta = 1/(9\alpha)$ (not working). Each machine, whose operation is independent of the other, can produce 12 pieces per hour when it is working.

(a) Compute the fraction of time in which there is no production (i.e., both machines are not working).

(b) Compute the average number of pieces per hour produced by the factory.

(c) Compute the average number of pieces per hour produced by the factory if the number of pieces produced per hour is 12 when only one machine is working, and 30 when they are both working.

E3 Consider a network node that works as follows. If there is no traffic, the node alternates between a sleep state for an exponential duration of average T and an awake state for a fixed duration βT. When in the awake state, the node can receive, whereas in the sleep state it cannot. If while the node is awake a packet is transmitted, the node receives it entirely (even if this requires it to remain awake for a total time longer than βT), and goes to sleep immediately after. If instead while the node is awake there is no transmission, the node goes back to sleep after being awake for βT. The probability that a packet is transmitted while the node is awake is α, the time at which such transmission starts is uniformly distributed in $[0, \beta T]$, and the average packet transmission time is γT. Develop and solve a semi-Markov model for the node, and in particular:

(a) Consider the three states sleep (S), listening (L) and receiving (R), determine the matrix of the transition probabilities of the embedded Markov chain, and draw its transition diagram.

(b) Determine the matrix of the average times associated to each transition, $\mathbf{T}$, and the average times associated to the visits to each state, μ_S, μ_L, μ_R.

(c) Find an expression for the fraction of time the node spends in each of the three states, and find its numerical value for $\alpha = 0.5, \beta = 0.1, \gamma = 0.2$.

E4 Consider a system which receives service requests according to a Poisson process of rate $\lambda = 20$ requests per hour. Each request remains in the system for a service time equal to 6 minutes, and there is no limit to the number of requests simultaneously in service. Assume that the system started its operation at time $t = 0$.

(a) Compute the probability that the system is empty at time $t = 30$ minutes

(b) Compute the probability that the system is empty at time $t = 30$ minutes, conditioned on the fact that there were 10 arrivals between 0 and t.

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T1 State and prove the elementary renewal theorem.

T2 Prove that in a Markov chain the period is a class property.

T3 Prove that for a Poisson process $X(t)$ the statistics of $X(s)$ conditioned on $X(t), s < t$, is binomial, and provide the espression of $P[X(s) = k|X(t) = n]$.