1,1.1,3,5

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1) T1 Prove that a Markov chain with a finite number of states must have at least one positive recurrent state

Assume no positive recurrent states.

$N = |E| < +\infty$ numb. of states

$$\Rightarrow \sum_{j=1}^{N} P_{ij}^{(n)} = 1 \quad \forall i \in E,\ n \geq 0$$

$$\Rightarrow 1 = \lim_{n \to +\infty} \sum_{j=1}^{N} P_{ij}^{(n)} = \sum_{j=1}^{N} \lim_{n \to +\infty} P_{ij}^{(n)} = 0$$

since $|E|$ is finite

the prob. are positive only for pos. recurrent states

$\Rightarrow$ Contradiction

$*$

1.1) T1 Prove that a Markov chain with a finite number of states cannot have any null recurrent state.

Prove this first

Then, suppose there's one null recurrent state which will then belong to a finite null recurr. class.

Since a recurrent class is a MC by itself, this isn't possible from $*$

3) T3 Prove that for a Markov chain the n-step transition probabilities, $P_{ij}^{(n)}$, satisfy the relationship

$$P_{ij}^{(n)} = \sum_m P_{im}^{(k)} P_{mj}^{(n-k)}, k = 0, 1, \ldots, n$$

$$P_{ij}^{(n)} = P[X_n = j \mid X_0 = i] = \sum_{m}^{+\infty} P[X_n = j, X_k = m \mid X_0 = i] =$$

$$= \sum_m P[X_n = j \mid X_k = m, X_0 = i]\, P[X_k = m \mid X_0 = i] =$$

$$= \sum_m P[X_n = j \mid X_k = m]\, P[X_k = m \mid X_0 = i] =$$

$$= \sum_m P_{mj}^{(n-k)} P_{im}^{(k)}$$

5) T2 Prove that in a Markov chain the period is a class property.

Theorem: Let $i, j \in E$ s.t. $j \leftrightarrow i$, then $\mathrm{d}(i) = \mathrm{d}(j)$.

Proof.

By $j \leftrightarrow i$, $\exists m, n > 0$ s.t. $P_{ij}^m > 0$ and $P_{ji}^n > 0$.

Call $\mathcal{D}_j = \{n \geq 1 : P_{jj}^n > 0\}$

Then

$$P_{jj}^{n+m} \geq P_{ji}^n P_{ij}^m > 0 \Rightarrow \mathrm{d}(j) \text{ divides } n + m.$$

Take any $k \in \mathcal{D}_i$, then

$$P_{jj}^{n+k+m} \geq P_{ji}^n P_{ii}^k P_{ij}^m > 0 \Rightarrow \mathrm{d}(j) \text{ divides } (n+m) + k$$

$$\Rightarrow \mathrm{d}(j) \text{ divides } k, \forall k \in \mathcal{D}_i \Rightarrow \mathrm{d}(j) \text{ divides } \mathrm{d}(i).$$

By the same argument $\mathrm{d}(i)$ divides $\mathrm{d}(j)$, therefore $\mathrm{d}(i) = \mathrm{d}(j)$.

This means: $i, j \in E$ s.t. $i \leftrightarrow j \Rightarrow d(i) = d(j)$

Proof:

$i \leftrightarrow j$ means $\exists\, m, n > 0$ s.t. $P_{ij}^m > 0$, $P_{ji}^n > 0$

$$P_{jj}^{m+n} \geq P_{ij}^m \cdot P_{ji}^n > 0 \Rightarrow d(j) \mid m+n$$

Let $s \in \{n \geq 1 : P_{ii}^n > 0\} = D_i$

$$P_{jj}^{n+m+s} \geq P_{ji}^n \cdot P_{ii}^s \cdot P_{ij}^m > 0 \Rightarrow d(j) \mid n+m+s$$

$$\Rightarrow d(j) \mid (n+m+s) - (n+m) = s$$

$$d(j) \mid s \quad \forall s \in D_i$$

$$\Rightarrow d(j) \mid d(i)$$

By the same argument $d(i) \mid d(j)$

$$\Rightarrow d(i) = d(j) \ \blacksquare$$