Network Modeling
Solutions for 22/09/2005

Problem 1

For the first question we need to compute π and $\theta_{11}, \theta_{22}, \theta_{33}$.

First we need to solve the usual system $\pi = \pi P = \begin{cases} \pi_2 = 0.3\pi_1 + 0.2\pi_2 \\ \pi_3 = 0.2\pi_1 + 0.6\pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 0 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{5}{9} \\ \pi_2 = \frac{5}{24} \\ \pi_3 = \frac{17}{72} \end{cases}$.

Then by the Basic Limit Theorem we have $\begin{bmatrix} \overline{\theta}_{11} = m_1 = \frac{1}{\pi_1} \\ \overline{\theta}_{22} = m_2 = \frac{1}{\pi_2} \\ \overline{\theta}_{33} = m_3 = \frac{1}{\pi_3} \end{bmatrix}$.

For the second point we need to solve $\overline{\theta}_{ij} = 1 + \sum\limits_{k \neq j} p_{ik}\overline{\theta}_{kj}$ for two values.

$\begin{cases} \overline{\theta}_{31} = 1 + p_{32}\overline{\theta}_{21} + p_{33}\overline{\theta}_{31} = 1 \\ \overline{\theta}_{12} = 1 + p_{22}\overline{\theta}_{21} + p_{23}\overline{\theta}_{31} = \frac{p_{23}}{1-p_{22}} = \frac{3}{4} \end{cases}$ $\begin{cases} \overline{\theta}_{13} = 1 + p_{11}\overline{\theta}_{13} + p_{12}\overline{\theta}_{23} = 5 \\ \overline{\theta}_{23} = \frac{3}{2} \end{cases}$

For the variance we need to compute the second moments according to $\overline{\theta^2}_{ij} = 2\overline{\theta}_{ij} - 1 + \sum\limits_{k \neq j} p_{ik}\left(1 + \overline{\theta^2}_{kj}\right)$.

Then variance is $var(\overline{\theta}_{ij}) = \overline{\theta^2}_{ij} - (\overline{\theta}_{ij})^2$.
Notice that $var(\overline{\theta}_{31}) = 0$ since this step is deterministic.

$$\mathrm{P}\left[X(1) = 1, X(3) = 1 | X(2) = 2\right] = \frac{\mathrm{P}\left[X(1) = 1, X(2) = 2, X(3) = 1 | X(0) = 3\right]}{\mathrm{P}\left[X(2) = 2 | X(0) = 3\right]} = \frac{p_{31}p_{12}p_{21}}{p^{(2)}_{32}}$$

$$\mathrm{P}\left[X(2) = 2 | X(1) = 1, X(3) = 1\right] = \frac{\mathrm{P}\left[X(1) = 1, X(2) = 2, X(3) = 1 | X(0) = 3\right]}{\mathrm{P}\left[X(1) = 1, X(3) = 1 | X(0) = 3\right]} = \frac{p_{31}p_{12}p_{21}}{p_{31}p^{(2)}_{11}}$$

Problem 2

The distribution of the first arrival is exponential, hence $E[\text{empty}] = \dfrac{1}{\lambda}$.

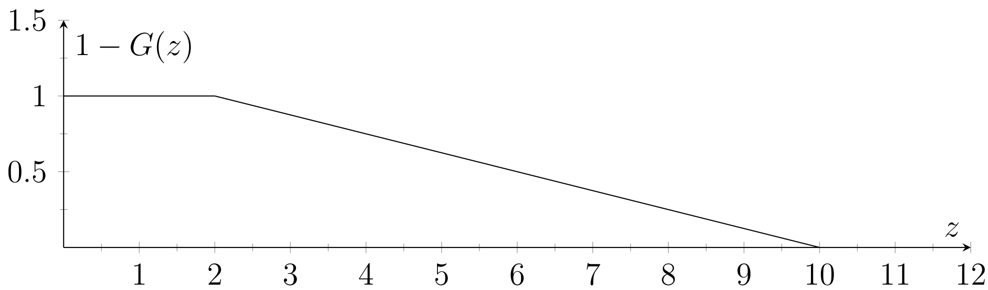
After the first arrival, we wait until another arrival or up to 2 seconds, then we send. The distribution is a truncated exponential: $E[\text{busy}] = \int\limits_0^2 e^{-\lambda t}dt = \dfrac{1 - e^{-2\lambda}}{\lambda} = 1 - e^{-2} \simeq 0.864$.

Fraction of time spent empty is $P_{empty} = \dfrac{E[\text{empty}]}{E[\text{empty}] + E[\text{busy}]} = 0.536$.

If a packet finds the queue non empty, then transmission is immediate. Otherwise it has to wait $1 - e^{-2}$ on average. By law of total probability we have:
$E[\text{delay}] = E[\text{delay}|\text{empty}]P_{empty} + E[\text{delay}|\text{busy}]P_{busy} = 0.864 \cdot 0.536 = 0.463$.

Problem 3

This scheme is clearly a $M/G/\infty$ queue.
Let $X(t)$ be the number of occupied channels at time t. $X(t) \sim \mathcal{P}(\Lambda)$.



So $E[X(t)] = \Lambda$, where $\Lambda = \lambda \int\limits_0^t [1 - G(z)]dz = \lambda \int\limits_0^t e^{-\mu t} = \frac{\lambda}{\mu}(1 - e^{-\mu t}) = \begin{cases} 10(1 - e^{-1}) & t = 6 \\ 10(1 - e^{-\frac{10}{6}}) & t = 10 \\ 10 & t = \infty \end{cases}$.

For the second point $\mathrm{P}[X(t) = 10] = \dfrac{\Lambda^1 0 e^{-\Lambda}}{10!} = \begin{cases} 0.05 & t = 6 \\ 0.125 & t = \infty \end{cases}$.

For the last point we have to compute Λ again, now for an uniform duration of Y between 2 and 10 minutes:

$\Lambda = \lambda \int\limits_0^t [1 - G(z)]dz = \begin{cases} 6\frac{100}{60} & t = 10, \infty \\ \frac{100}{60}(6 - 1) & t = 6 \end{cases}$.

These results come from the graphical interpretation of $1 - G(z)$. It is 1 between 0 and 2, then it linearly goes to 0 in 10, then it remains 0. The three integrals can be easily computed by analyzing the area under the function.

Problem 4

Transition matrix is $P = \begin{bmatrix} 0.99 & 0.01 \\ 0.1 & 0.9 \end{bmatrix}$. The protocol matrix is $C = \begin{bmatrix} p_{00} & p_{01} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix}$.

Reward and time vectors are $\mathbf{R} = \begin{bmatrix} R_G \\ R_B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\quad \mathbf{T} = \begin{bmatrix} T_G \\ T_B \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Throughput is then $\mathcal{T} = \dfrac{\sum_i \pi_i R_i}{\sum_i \pi_i T_i} = \dfrac{\pi_G}{\pi_G + 2\pi_B} = \dfrac{p_{10}^{(2)}}{p_{10}^{(2)} + 2p_{01}}$.

For the last question we have $\mathcal{T} = \dfrac{(1-\delta)p_{10}^{(2)}}{(1-\delta)p_{10}^{(2)} + 2\left((1-\delta)p_{01} + \delta p_{01}^{(2)} + \delta p_{10}^{(2)}\right)}$, where $\delta = 0.1$.