1) Prove that a Markov chain with a finite number of states must have at least one positive recurrent state.

Assume no positive recurrent states.

$N = |E| < +\infty$ numb. Of states

$\Rightarrow \sum_{j=1}^{N} P_{ij}^{(n)} = 1 \ \forall i \in E, n \geq 0$

$\Rightarrow 1 = \lim_{n\to\infty} \sum_{j=1}^{N} P_{ij}^{(n)} = \sum_{j=1}^{N} \lim_{n\to\infty} P_{ij}^{(n)} = 0$

Since $|E|$ is

The prob. are positive only for pos. recurrent

$\Rightarrow Contradiction$

1.1) Prove that a Markov chain with a finite number of states cannot have any null recurrent state.

[prove (1)] *

Then, suppose there's one null recurrent stat which will then belong to a finite null recur. Class.

Since a recurrent class is a MC by itself, this isn't possible from *

3) Prove that for a Markov chain the n-steo transition probabilities, $P_{ij}^{(n)}$, satisfy the relationship

$$P_{ij}^{(n)} = \sum_m P_{im}^{(k)} P_{mj}^{(n-k)}, k = 0,1,\dots,n$$

$$\begin{aligned} P_{ij}^{(n)} &= P[X_n = j | X_0 = i] = \sum_m^{\infty} P[X_n = j, X_k = m | X_0 = i] = \\ &= \sum_m P[X_n = j | X_k = m, X_0 = i] P[X_k = m | X_0 = i] = \\ &= \sum_m P[X_n = j | X_k = m] P[X_k = m | X_0 = i] = \\ &= \sum_m P_{mj}^{(n-k)} P_{im}^{(k)} \end{aligned}$$

5) Prove that in a Markov chain the period is a class property.

This means: $i, j \in E \ s.t. \ i \leftrightarrow j \Rightarrow d(i) = d(j)$

Proof:

$i \leftrightarrow j$ means $\exists m, n > 0 \ s.t. \ \begin{matrix} P_{ij}^m > 0 \\ P_{ji}^m > 0 \end{matrix}$

$P_{jj}^{m+n} \geq P_{ij}^m \cdot P_{ji}^n > 0 \Rightarrow d(j) | m + n$

Let $s \in \{n \geq 1: P_{ii}^n > 0\} = D_i$

$P_{jj}^{n+m+s} \geq P_{ji} \cdot P_{ii}^S \cdot P_{ij}^m > 0 \Rightarrow d(j) | n + m + s$

$\Rightarrow d(j) | (n + m + s) - (n + m) = s$

$d(j) | s \ \forall s \in D_i$

$\Rightarrow d(j) | d(i)$

By the same argument $d(i) | d(j)$

$\Rightarrow d(j) = d(i)$

6) Prove that for Poisson process $X(t)$ the statistics of $X(s)$ conditioned on $X(t), \ s < t$ is binomial, and provide the expression of $P[X(s) = k | X(t) = n]$

$P[X(s) = k | X(t) = n] = \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$

$Proof$:

Since $X(t) = n$, the n events are i.i.d. $\sim U[0, t]$, the prob. that each fall in $[0, s]$ is $\frac{s}{t}$ therefore $X(s)$ is Binomial with parameters $\left(n, \frac{s}{t}\right)$

7) Prove that if states i and j of a Markov vhain communicate and i is recurrent, then j is also recurrent.

$Proof$:

$i \leftrightarrow j \Rightarrow \exists n, m: P_{ij}^{(n)}, P_{ji}^{(m)} > 0$

Let $k > 0$

$P_{jj}^{(n+m+k)} \geq P_{ji}^{(m)} \cdot P_{ii}^{(k)} \cdot P_{ij}^{(n)}$

$\Rightarrow \sum_k^{\infty} P_{jj}^{(k)} \geq \sum_k^{\infty} P_{jj}^{(n+m+k)} \geq \sum_k^{\infty} P_{ji}^{(m)} \cdot P_{ii}^{(k)} \cdot P_{ij}^{(n)} = P_{ji}^{(m)} \cdot P_{ij}^{(n)} \cdot \sum_k^{\infty} P_{ii}^{(k)}$

$\Rightarrow If \ \sum_k^{\infty} P_{ii}^{(k)}$ diverges, $\sum_k^{\infty} P_{jj}^{(k)}$ diverges

$\Rightarrow if \ i$ is recurrent,

j is recurrent

8) For a Poisson process of rate λ, prove that the interarrival times are iid exponential with mean $1/\lambda$

Let $S_n =$ time between $(n-1)$st and n^{th} event

(1) $P[S_0 > t] = P[no \ arr. \ in \ [0, t)] = e^{-\lambda t}$

$\Rightarrow S_1 \sim \exp(\lambda) \ mean = \frac{1}{\lambda}$

(2) $P[S_1 > t | S_0 = s] = P[no \ arr. \ in \ (s, s + t] | \cancel{S_0 = 0}] = e^{-\lambda t}$

Indep. and stationary increments

$\Rightarrow S_0 \sim \exp(\lambda) \ mean = \frac{1}{\lambda}$

and independent of S_0

(3) $P[S_n > t | S_i = s_i, i = 0, \dots, n-1] =$

$= P[no \ arr. \ in \ (s_0 + \dots + s_{n-1}, s_0 + \dots + s_{n-1} + t] | S_i \cancel{= s_i, i} = 0, \dots, n-1]$

$= e^{-\lambda t}$

Indep. and stat increments

$\Rightarrow S_n \sim \exp(\lambda) \ mean = \frac{1}{\lambda}$

and indep. of $S_0, \dots, S_{n-1}$

9) Consider a random walk over the non-negative integers with the following transition probabilities: $P_{01} = 1, P_{i,i+1} = p, P_{i,i-1} = p - 1, i > 0$, with $p + q = 1$. Study its behaviour, and in particular characterize its recurrence or transiency and derive the steady-state distribution.

$0 \ \begin{matrix} \rightarrow 1 \rightarrow \\ \leftarrow 1 - p \end{matrix} \ 1 \ \begin{matrix} \rightarrow p \rightarrow \\ \leftarrow 1 - p \end{matrix} \ 2 \ \begin{matrix} \rightarrow p \rightarrow \\ \leftarrow 1 - p \end{matrix} \ \cdots$

$$p = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & 0 & \\ 0 & 1-p & 0 & p & 0 & \\ \vdots & & & & & \ddots \end{pmatrix} \end{array}$$

To derive steady state distrib. We need to solve

$x_i = \sum_{j=0}^{\infty} x_j P_{ij} = p \, x_{i-1} + q x_{i+1}$ where $\sum_{k=0}^{\infty} x_k = 1$ **(2)**

Firstly

$x_0 = q x_1 \quad x_1 = \frac{x_0}{q}$

$x_1 = x_0 + q x_2 \quad x_2 = \frac{x_1 - x_0}{q} = \frac{x - qx}{q^2} = \frac{(1-q)x}{q^2} = \frac{p x_0}{q^2}$

$\Rightarrow Generally, x_i = \frac{1}{p} \left(\frac{p}{q}\right)^i x_0$

So using **(2)**

$1 = \sum_k^{\infty} x_k = \frac{1}{p} x_0 \sum_k^{\infty} \left(\frac{p}{q}\right)^k \Rightarrow x_0 = \frac{P}{\sum_k \left(\frac{p}{q}\right)^k}$ **(3)**

$\Rightarrow$ From **(3)** we can conclude that:

- $If \ p < q$, sum converge so chain is positive recurrent
- $If \ p \geq q$, sum diverges so chain is transient

11) For a Poisson process $X(t)$ of rate λ, state and derive the expression of $P[X(u) = k | X(t) = n]$ fir the two cases $(i) \ 0 < u < t, 0 \leq k \leq n$ and $(ii) \ 0 < t < u, 0 \leq n \leq k$

Disjoint intervals $\Rightarrow$ independent

Binomial theorem

$i) \quad o < u < t, 0 \leq k \leq n$

$= \frac{P[X(u)=k, X(t)=n]}{P[X(t)=n]} = \frac{P[X(u)=k, X(t)-X(u)=n-k]}{P[X(t)=n]} =$

$= \frac{\left(\cancel{e^{-\lambda u}} \frac{(\lambda u)^k}{k!}\right)\left(\cancel{e^{-\lambda(t-u)}} \frac{(\lambda(t-u))^{(t-u)}}{(t-u)!}\right)}{\left(\cancel{e^{-2t}} \frac{(\lambda t)^n}{n!}\right)} =$

$= \binom{n}{k} \frac{u^k}{t^n} (t - u)^{n-k} = \binom{n}{k} \left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k}$

$i) \quad o < t < u, 0 \leq k \leq n$

$= \frac{P[X(u)=k, X(t)=n]}{P[X(t)=n]} = \frac{P[X(t)=n, X(u)-X(t)=k-n]}{P[X(t)=n]} =$

$\begin{bmatrix} a = X(t) = n \\ b = X(u) - X(t) = k - n \end{bmatrix}$

$= \frac{\cancel{P[a]} P[b]}{\cancel{P[X(t)=n]}} = P[X(u - t) = k - n] =$

Stationary increments

$= e^{-\lambda(u-t)} \frac{(\lambda(u-t))^{k-n}}{(k-n)!}$

12) For a renewal process, state precisely (also providing a formal proof) what is the value of $\lim_{t\to\infty} \frac{N(t)}{t}$

$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \ \ w.p.1$

Proof:

$S_{N(t)} \leq t < S_{N(t)+1}$

$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$

$\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} = \lim_{t\to\infty} \frac{S_n}{n} = \mu \ \ w.p.1$

$N(t) \to \infty$ LLN

$as \ t \to \infty$

$\lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} = \mu \cdot 1 \ \ w.p.1$

LLN

$\Rightarrow \mu \leq \lim_{t\to\infty} \frac{t}{N(t)} \leq \mu \ \ w.p.1$