E2. Consider a network node that under normal conditions can handle a traffic equal to 1Gbps. This node works normally for an exponential time with mean 99 T, and then enters an alarm state during which its capacity is reduced to 250 Mbps. After being for a time T in the alarm state, the node is instantaneously repaired and starts again to work normally.

a) Compute the fraction of time the node spends in the alarm state, and the average traffic handled (suppose that the queues are always full i.e. there always packets to transmit.

$$P[\text{alarm}] = \frac{T}{99T + T} = 0.01$$

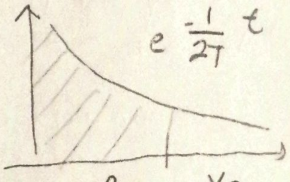
$$\frac{1\,Gb \cdot 99T + 0.25\,Gbps \cdot 1T}{100T} = 0.01$$

Throughput $= 0.99 \cdot 1 + 0.01 \cdot 0.25 = 992.5$ Mbs

b) Suppose now that once entering the alarm state the node completely stops working after an exponential time of mean 2T. Unless it is repaired earlier (as before the repaires requires exactly a time T from when the node enters the alarm state). If the node stops working, it needs to be replaced and this tires a time 20 T, during which the node cannot handle any traffic (Note that this replacement is different from the simple repaired considered in the previous case).

i) Compute the average time between 2 subsequent substitutions.

$E = \int 1 - T$



$B = e^{-1/2}$

| NORMAL | ALARM | Replacement |
|---|---|---|
| 99 T | ~exp(2T) | 20 T |

$$Exp\left(\tfrac{1}{2T}\right) < T] = \int_0^T 1 - e^{-\frac{t}{2T}}\,dt = \frac{1-e^{-1/2}}{1/2T} = 2T(1-e^{-1/2})$$

Average time between substitutions $= 99T + 2T(1-e^{-1/2}) + 20T\cdot(1-e^{-1/2})$

ii) The fraction of time in which the node is not working

$$P[\text{NOT working}] = \frac{E[\text{NOT WORK}]}{E[\text{CYCLE}]} = \frac{20T\cdot(1-e^{-1/2})}{99T + 2T(1-e^{-1/2}) + 20T\cdot(1-e^{-1/2})} = 0.0731$$

iii) ~~[illegible]~~

The average system througput.

$$\text{Throgput} = \frac{99T \cdot 1\,Gbps + 2T(1-B)\,0.25 + 20(1-B)\cdot 0}{99T + 2T(1-B) + 20(1-B)}$$

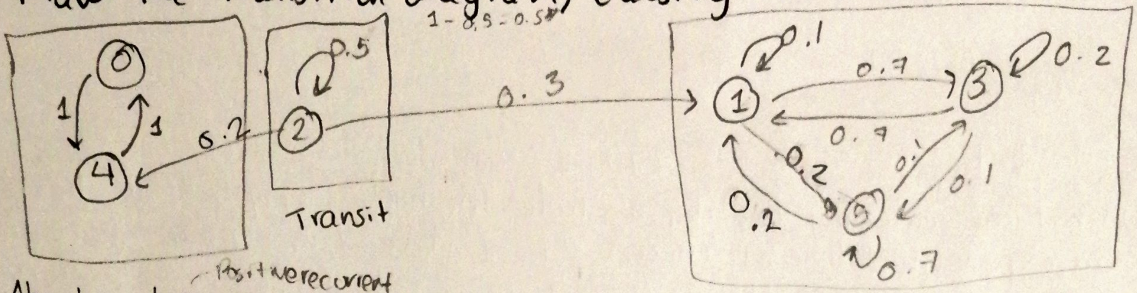
$= 0.9214$ Gbps

12/12/2006 $\boxed{14}$ ①

E1 Consider a Markov Chain with the following transition matrix (states are from 0 to 5)

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0.1 & 0 & 0.7 & 0 & 0.2 \\ 2 & 0 & 0.3 & 0.5 & 0 & 0.2 & 0 \\ 3 & 0 & 0.7 & 0 & 0.2 & 0 & 0.1 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0.2 & 0 & 0.1 & 0 & 0.7 \end{array}$$

a) Draw the transition diagram, classify the states and identify the classes




Absorbing class period d=2

Positive recurrent class?

Positive recurrent aperiodic (d=1)

b) Compute $\lim_{n\to\infty} P^n$

Since absorbing class has a pimpong behavior, the limiting distribution is not defined for the whole chain.

And $\pi_2 = 0$ because it is transient.

The submatrix induced by class $\{1,3,5\}$ is doubly stochastic, meaning that each row and column sums one. Then, $\pi_1 = \pi_3 = \pi_5 = 1/3$

$\pi_3(\{0,4\}) = \frac{0.2}{0.5^{**}} = 0.4$ $\quad \pi_3(\{1,3,5\}) = = \frac{0.3}{0.5} = 0.6 = 3/5$

Prob. of being absorbed starting from 3

$$\lim_{n\to\infty} \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & X & 0 & 0 & 0 & X & 0 \\ 1 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 2 & X & \frac{1}{3}\cdot\frac{3}{5} & 0 & \frac{1}{3}\cdot\frac{3}{5} & X & \frac{1}{3}\cdot\frac{3}{5} \\ 3 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 4 & X & 0 & 0 & 0 & X & 0 \\ 5 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \end{array}$$

$$\lim_{n\to\infty} \frac{1}{m}\sum_{K=1}^{\infty} P_K = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 1 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 2 & \frac{1}{2}\cdot\frac{2}{5} & \frac{1}{3}\cdot\frac{3}{5} & 0 & \frac{1}{3}\cdot\frac{3}{5} & \frac{1}{2}\cdot\frac{2}{5} & \frac{1}{3}\cdot\frac{3}{5} \\ 3 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 4 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 5 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \end{array}$$

c) Compute $\lim_{n\to\infty} \frac{1}{n}\sum_{K=1}^{n} P^K$ In periodic states is the average fraction time the chain spend in those states

d) Compute $P[X_4 = 5, X_2 = 3 \mid X_3 = 1, X_1 = 3] = \frac{P_{33} P_{31} P_{15}}{P_{31}^{(2)}}$

$$P[X_4 = 5, X_2 = 3 \mid X_3 = 1, X_1 = 3] = \frac{P[X_4 = 5, X_3 = 1, X_2 = 3 \mid X_1 = 3]}{P[X_3 = 1 \mid X_1 = 3]} = \frac{P_{33} P_{31} P_{15}}{P_{31}^{(2)}}$$

E2 Consider a link with capacity 1 Mbps, shared among many users who collectively produce packets according to a Poisson process with rate $\lambda = 500$ packets per second. All packets are of the same length equal to 1000 bits. The access protocol is an ideal CSMA, where a packet generated when the channel is idle gets immediate access, whereas a packet that finds the channel busy is rescheduled after an exponential time of average $100/\lambda$. (If this new access attempt again finds a busy channel the protocol keeps trying after random times until success.) Assume that the total traffic (new packets plus all retransmissions) can be approximated as Poisson with rate $\lambda$.

a) Compute the throughput (average traffic handled) on the link.

This system can be modeled as an alternating process, where the renewal instant is the first arrival since the link is empty

Idle | Busy
2ms | 1ms

$E[\text{Tx time}] = \frac{\text{Packet}}{L_{\text{capacity}}} = \frac{1000\text{bits}}{1 \times 10^6} = 1\text{ ms}.$

Average waiting time is $= \frac{100}{\lambda} = 200\text{ms}$

Fraction of time the link is empty is distributed as an exp. of mean $\frac{1}{\lambda} = 2\text{ms}$.

$E[\text{cycle time}] = E[\text{tx time}] + E[\text{empty time}] = 3\text{ms}$

Throughput is $T = L \frac{E[\text{tx time}]}{E[\text{cycle time}]} = (1 \times 10^6) \frac{1\text{ms}}{3\text{ms}} = 0.333 [M] bps$

(b) Compute the average access delay from when a packet is generated to when it finally gets access to the channel.

Let $\beta$ be the prob. of finding the system busy $\beta = \frac{E[\text{tx time}]}{E[\text{cycle time}]} = \frac{1}{3}$

N, the number of consecutive failed attemps before a succesful transmission, where is a r.v. with geometric distribution

$P[N \geq k] = \beta^k$ and $E[N] = \sum_{k=1} \beta^k = \frac{\beta}{1-\beta} = \frac{1}{2}$

Average delay $E[\text{delay}] = E[N]\, E[\text{waiting time}] = \frac{1}{2} \frac{100}{\lambda} = 100\text{ms}$

c) If a transmission corresponds to a gain of 1 unit and each failed attempt (packet finding the channel busy) corresponds to a cost of 0.2 units, compute the total gain of the system in units per second.

PASTA

$E[\text{gain per arrival}] = 1 \cdot P[\text{idle}] - 0.2 P[\text{Busy}] = 1 \cdot \frac{2}{3} - 0.2 \cdot \frac{1}{3} = 0.6$

$E[\text{gain per unit time}] = \lambda \cdot E[\text{gain per arrival}] = 300$ unit/sec

ó

$\frac{E[\text{gain in a cycle}]}{E[\text{cycle time}]} = \frac{1 - 0.2\, E[\text{arrivals in 1ms}]}{E[\text{cycle time}]} = \frac{1 - 0.2 \lambda 10^{-3}}{E[\text{cycle time}]} = 300$ units/sec

E1 Consider a web server which receives download requests according to a Poisson process with rate $\lambda = 20$ requests per second. Each requests, after a fixed processing time of 20ms, triggers the transfer of a file with size uniformly distributed between 1 and 2 MBytes. A request is said to be active from when it arrives to when the corresponding file transfer is completed. Assume that the server capacity, in terms of how many simultaneous request it can handle, is infinite, and that the transfer data rate for each file is 100 Mbit/s, regardless of the number of files that are being transferred at any given time.

(a) Assuming that the server is switched on at time = 0, when does the statistics of the number of active requests in the system reach its steady-state condition? In such condition express P[k active requests]

Let processing time be $T_{proc} = 0.02$ s.
And the packet length be $L \sim U(8 \cdot 10^6, 16 \cdot 10^6)$ bits.
Server capacity $C = 100 \cdot 10^6$ bits

Time required for a transmission is $\boxed{T_{Tx} = T_{proc} + \frac{L}{C}}$ r.v. with $\sim U(0.1, 0.18)$

using $8 \cdot 10^6$ for the minimum Tx and $16 \cdot 10^6$ for the max Tx.

$E[T_{Tx}] = 0.14$ s $\quad \frac{a+b}{2} = \frac{0.1+0.18}{2} = 0.14$

Problem asks the first t for which the system can be considered in steady-state condition. Clearly t is the first instant from which the function $1-G(z)$ becomes zero, by inspection $t = 0.18$s.

$\Lambda = \lambda E[T_{Tx}] = 2.8$ ; $Pr[k \text{ requests at } t > 0.18] = \frac{e^{-\Lambda} (\Lambda)^k}{k!}$

(b) Given that in an interval of duration T the system received N requests, find the probability that at the end of such interval there are no active requests in the 2 cases (b1) $T = 0.1$ s, $N = 2$.

As $X(t)$ is a Binomial r.v. $P_r[M(T) = m | X(T) = N] = \binom{N}{m} p^m (1-p)^{N-m}$

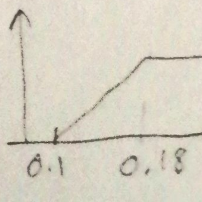
where $\quad p = \frac{1}{T} \int_0^T [1 - G_{Tx}(z)] dz$.

In case a is impossible to have these conditions, then $p = 0$.

$Pr[M(0.1) = 0 | X(0.1) = 2] = \binom{2}{1} 1^0 0^{2-0} = 0$

(b2) $T = 1$ s, $N = 20$.

$P[M(1) = 0 | X(1) = 20] = \binom{20}{0} (0.14)^0 (0.86)^{20-0} = 0.0489$



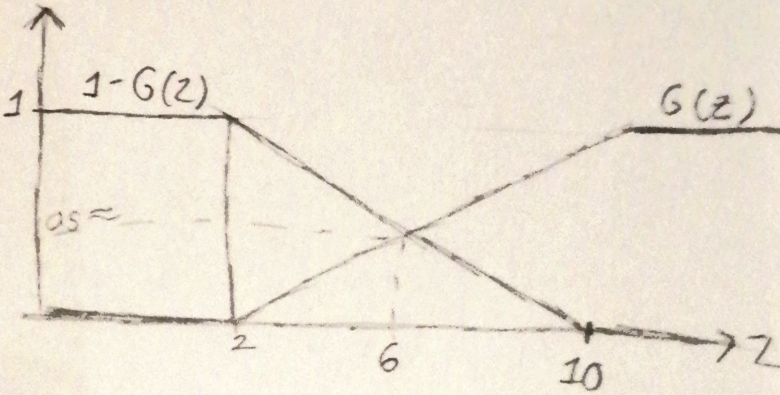
22/09/2005 

c) Repeat the previous calculations assuming that the call duration is uniformly distributed in [2, 10] minutes.

The part of the integral is computed graphically

$t=6 \rightarrow A=5=2+(4-1)$

$t=10 \rightarrow A=6=2+4$




$$\Lambda = \begin{cases} \frac{100}{60}(5) = 8.33 & t=6 \\ \frac{100}{6}(6) = 10 & t=10, \infty \end{cases}$$

$$P[x(t)=10] = \frac{\Lambda^{10} e^{-\Lambda}}{10} = \begin{cases} 0.107 & t=6 \\ 0.125 & t \geq 10 \end{cases}$$

E4 Consider a Go-Back-N Protocol over a two-state Markov channel with transition prob. 0.99 (from the good state to itself) and 0.1 (from the bad state to the good state). The packet error prob. is 1 for a bad slot and 0 for a good slot, respectively. The round-trip time is $m=2$ slots, i.e packet that is erroneous in slot $t$ will be retransmitted in slot $t+2$.

a) Compute the throughput of the protocol for an error-free feedback channel

Transition matrix is $P = \begin{bmatrix} 0.99 & 0.01 \\ 0.1 & 0.9 \end{bmatrix}$ The protocol matrix is $C = \begin{bmatrix} P_{00} & P_{01} \\ P_{10}^{(m)} & P_{11}^{(m)} \end{bmatrix}$

Reward vector $R = \begin{bmatrix} R_G \\ R_B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Time vector $T = \begin{bmatrix} T_G \\ T_B \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\text{Throughput} = \frac{\sum_i \pi_i R_i}{\sum_i \pi_i R_i} = \frac{\pi_G}{\pi_G + 2\pi_B} = \frac{P_{10}^{(2)}}{P_{10}^{(2)} + 2p_{01}}$$

b) Compute the throughput of the protocol for a feedback channel subject to iid errors with prob. $0.1 = \delta$

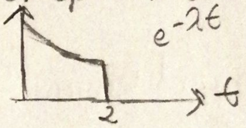
$$\text{throughput} = \frac{(1-\delta)P_{10}^{(2)}}{(1-\delta)P_{10}^{(2)} + 2((1-\delta)P_{01} + \delta P_{01}^{(2)}) + \delta P_{10}^{(2)}}$$

E2 Consider a queue where packets arrive according to a Poisson process with rate $\lambda = 1$ packet per second. All packets in the queue are transmitted when either of the following events occurs: (i) there are two packets in the queue, or (ii) there is one packet in the queue and its waiting time reaches two seconds. Transmissions is instantaneous, i.e. the queue empties every time there is a packet arrival when one packet is already in the queue, or when the only packet in the queue has been there for enough time. $\boxed{10}$

(a) Compute the fraction of time which the queue is empty.

The distribution of the first arrival is exponential, $E[empty] = 1/\lambda$

After the first arrival we wait until another arrival or up to 2 seconds, then send. The distribution is a truncated exponential.




$$E[busy] = \int_0^2 e^{-\lambda t} dt = \frac{1-e^{-2\lambda}}{\lambda} = 0.8646$$

Fraction of time spent empty is $P_{empty} = \dfrac{E[empty]}{E[empty] + E[busy]} = 0.536$

b) Compute the average delay (i.e. the average time a packet spend in the queue).

If a packet finds the queue non empty, the transmission is immediate. Otherwise, it has to wait $\frac{1-e^{-2\lambda}}{\lambda}$ on average. By the law of total prob. we have:

$$E[delay] = E[delay|empty]\, P_{empty} + E[delay|busy]\, P_{busy}$$
$$= (0.864)(0.536) = 0.463$$

E3 Consider a frequency division transmission system in which the number of channels is so large that the probability they are all occupied is negligible. Such system receives connection requests according to a Poisson process with rate $\lambda = 100$ calls per hour, and the duration of each call is exp. with mean 6 minutes. Let $X(t)$ be the number of occupied channels at time $t$.

a) Compute the average of $X(t)$ at $t = 6, 10$ minutes and for $t = \infty$.

$\lambda = 100 \frac{calls}{h} \frac{1\,h}{60\,min} = \frac{10}{6} \frac{call}{min}$ $\mu = \frac{1}{6}$

This scheme is $M/G/\infty$ queue.

Let $X(t)$ be the number of occupied channel at time $t$. $X(t) \sim P(\Lambda)$

$$E[X(t)] = \Lambda, \text{ where } \Lambda = \lambda \int_0^t [1 - G(z)] dz = \lambda \int_0^t e^{-\mu t} = \frac{\lambda}{\mu}(1 - e^{-\mu t}) = \begin{cases} 10(1-e^{-1}) & t=6 \\ 10(1-e^{-\frac{10}{6}}) & t=10 \\ 10 & t=\infty \end{cases}$$

b) Compute $P[X(t) = 10]$ for $t = 6$ and $t = \infty$

Very important: using the values from the previous point

$$P[X(t) = 10] = \frac{\Lambda^{10} e^{-\Lambda}}{10} = \begin{cases} 0.05 & t=6 \\ 0.102 & t=10 \\ 0.125 & t=\infty \end{cases}$$

22/09/05

$\boxed{9}$ ①

**E1** Consider a Markov chain $X_n$, with states 1, 2 and 3. $X(0)=3$ and transition matrix

$$P = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.2 & 0.6 \\ 1 & 0 & 0 \end{pmatrix}$$

a) Compute the steady-state prob. and the average recurrence time of all states.
The question is asking for $\pi$ and $\theta_{11}, \theta_{22}, \theta_{33}$.

$\rightarrow \pi_1\, 0.5 + \pi_2\, 0.2 + \pi_3\, 1 = \pi_1$ $\qquad \pi_1 = \frac{5}{9}$

$\rightarrow \pi_1\, 0.3 + \pi_2\, 0.2 + \pi_3\, 0 = \pi_2$ $\qquad \pi_2 = 5/24$

$\pi_1\, 0.2 + \pi_2\, 0.6 + \pi_3\, 0 = \pi_3$ $\qquad \pi_3 = 17/72$

$\rightarrow \pi_1 + \pi_2 + \pi_3 = 1$

with the basic limit theorem we have

$\theta_{11} = m_1 = \frac{1}{\pi_1} = \frac{9}{5}$

$\theta_{22} = m_2 = \frac{1}{\pi_2} = \frac{24}{5}$

$\theta_{33} = m_3 = \frac{1}{\pi_3} = \frac{72}{17}$

b) Compute mean and variance of the first passage time from state 3 to state 1

$\boxed{\theta_{31}} = 1 + P_{32}\theta_{21} + P_{33}\theta_{31} = 1$ ①

$\theta_{12} = 1 + P_{22}\theta_{21} + P_{23}\theta_{31};$ ②

$$\bar{\theta}_{ij} = 1 + \sum_{k \neq j} P_{ik}\, \bar{\theta}_{kj}$$

① in ② $\rightarrow \boxed{\theta_{12}} = \frac{P_{23}}{1-P_{22}} = \frac{0.6}{1-0.2} = \frac{3}{4}$

For the variance we need to compute the second moment $\bar{\theta^2}_{ij} = 2\bar{\theta}_{ij} - 1 + \sum_{k\neq j} P_{ik}(1+\bar{\theta}^2_{kj})$

Then the variance is $var(\bar{\theta}_{ij}) = \bar{\theta}^2_{ij} - (\theta_{ij})^2$

The variance $var(\bar{\theta}_{31}) = 0$ since this step is deterministic.

c) Compute mean and variance of the first passage time from state 1 to state 3

$$\begin{cases} \bar{\theta}_{13} = 1 + P_{11}\bar{\theta}_{13} + P_{12}\bar{\theta}_{23} \\ \bar{\theta}_{23} = 1 + P_{21}\theta_{13} + P_{22}\theta_{23} \end{cases} \rightarrow \begin{cases} -1 = (0.5-1)\bar{\theta}_{13} + 0.3\bar{\theta}_{23} \\ -1 = 0.2\,\bar{\theta}_{13} + (0.2-1)\theta_{23} \end{cases}$$

$X = \frac{55}{17}$ ? $Y = \frac{35}{17}$ ?

d) Compute $P[X(1)=1, X(3)=1 \mid X(2)=2]$ and $P[X(2)=2 \mid X(1)=1, X(3)=1]$

$$P[X(1)=1, X(3)=1 \mid X(2)=2] = \frac{P[X(1)=1, X(2)=2, X(3)=1 \mid X(0)=3]}{P[X(2)=2 \mid X(0)=3]} = \frac{P_{31}P_{12}P_{21}}{P_{32}^{(2)}}$$

$$P[X(2)=2 \mid X(1)=1, X(3)=1] = \frac{P[X(1)=1, X(2)=3, X(3)=1 \mid X(0)=3]}{P[X(1)=1, X(3)=1 \mid X(0)=3]} = \frac{P_{31}P_{12}P_{21}}{P_{31}P_{11}^{(2)}}$$

E2 Consider a queue where packets arrive according to a Poisson process with rate $\lambda = 1$ packet per second. All packets in the queue are transmitted when either of the following events occurs: (i) there are two packets in the queue, or (ii) there is one packet in the queue and its waiting time reaches two seconds. $\boxed{10}$

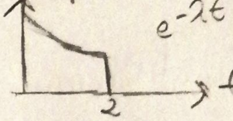
Transmissions is instantaneous, i.e. the queue empties every time there is a packet arrival when one packet is already in the queue, or when the only packet in the queue has been there for enough time.

(a) Compute the fraction of time which the queue is empty.

The distribution of the first arrival is exponential, $E[empty] = 1/\lambda$

After the first arrival we wait until another arrival or up to 2 seconds, then send.

The distribution is a truncated exponential.




$$E[busy] = \int_0^2 e^{-\lambda t} dt = \frac{1 - e^{-2\lambda}}{\lambda} = 0.8646$$

Fraction of time spent empty is $P_{empty} = \frac{E[empty]}{E[empty] + E[busy]} = 0.536$

b) Compute the average delay (i.e. the average time a packet spend in the queue).

If a packet finds the queue non empty, the transmission is immediate. Otherwise, it has to wait $\frac{1 - e^{-2\lambda}}{\lambda}$ on average. By the law of total prob. we have:

$$E[delay] = E[delay \mid empty] P_{empty} + E[delay \mid busy] P_{busy}$$
$$= (0.864)(0.536) = 0.463$$

E3 Consider a frequency division transmission system in which the number of channels is so large that the probability they are all occupied is negligible. Such system receives connection requests according to a Poisson process with rate $\lambda = 100$ calls per hour, and the duration of each call is exp. with mean 6 minutes. Let $X(t)$ be the number of occupied channels at time $t$.

a) Compute the average of $X(t)$ at $t = 6, 10$ minutes and for $t = \infty$.

$\lambda = 100 \frac{calls}{h} \frac{1h}{60min} = \frac{10}{6} \frac{call}{min}$ $\quad \mu = \frac{1}{6}$

This scheme is $M/G/\infty$ queue.

Let $X(t)$ be the number of occupied channel at time $t$. $X(t) \sim P(\Lambda)$

$$E[X(t)] = \Lambda, \text{ where } \Lambda = \lambda \int_0^t [1 - G(z)] dz = \lambda \int_0^t e^{-\mu t} = \frac{\lambda}{\mu}(1 - e^{-\mu t}) = \begin{cases} 10(1 - e^{-1}) & t = 6 \\ 10(1 - e^{-\frac{10}{6}}) & t = 10 \\ 10 & t = \infty \end{cases}$$

b) Compute $P[X(t) = 10]$ for $t = 6$ and $t = \infty$

Very important: using the values from the previous point

$$P[X(t) = 10] = \frac{\Lambda^{10} e^{-\Lambda}}{10!} = \begin{cases} 0.05 & t = 6 \\ 0.102 & t = 10 \\ 0.125 & t = \infty \end{cases}$$

12 / 12 / 2006



E 3 Consider two independant Poisson process $X_1(t)$ and $X_2(t)$ where $X_i(t)$ is the number of arrivals for process $i$ during $[0, t]$. The average number of arrivals per unit of time of the 2 process is $\lambda_1 = 0.5$ and $\lambda_2 = 1$ respectively.

a) Compute $P[X_1(3) = 1 \mid X_1(3) + X_2(3) = 3]$ and $P[X_1(3) + X_2(3) = 3 \mid X_1(3) = 1]$ with $X(t)$ as the sum of the 2 processes

$$P[X_1(3) = 1 \mid X(3) = 3] = \binom{3}{1}\left(\frac{3\lambda_1}{3(\lambda_1+\lambda_2)}\right)^1\left(1 - \frac{3\lambda_1}{3(\lambda_1+\lambda_2)}\right)^{3-1} = \frac{4}{9}$$

$$P[X(3) = 3 \mid X_1(3) = 1] = P[X_2(3) = 2) = \frac{e^{-3\lambda_2}(3\lambda_2)^2}{2!} = 0.224$$

b) Compute $P[X_1(2) = 1 \mid X_1(3) = 3]$ and $P[X_1(3) = 3 \mid X_1(2) = 1]$

$$P[X_1(2) = 1 \mid X_1(3) = 3] = \binom{3}{1}\left(\frac{2\lambda_1}{3\lambda_1}\right)\left(1 - \frac{2\lambda_1}{3\lambda_1}\right)^{3-1} = \frac{2}{9}$$

$$P[X_1(3) = 3 \mid X_1(2) = 1] = \frac{e^{-(3\lambda_1 - 2\lambda_1)}(3\lambda_1 - 2\lambda_1)^2}{2!} = 0.0758$$

"c")

$$P[X_1(2) = 1 \mid X(3) = 3] = \binom{3}{1}\left(\frac{2}{9}\right)\left(\frac{7}{9}\right)^2 = \frac{98}{243}$$

$$P[X(3) = 3 \mid X_1(2) = 1] = P(X_1(1) + X_2(3) = 2] \; (X_1(3) - X_1(2)).$$

E4 Considera two state Markov channel with transition prob. 0.98 (from the good state to itself) and 0.1 (from the bad state to the good state). The packet error probability is 1 for a bad slot and 0 for a good slot respectly

a) Compute the throoghput (average number of successes per slot) of a protocol that transmits packets directly on the channel, with no retransmissions.

$$\text{Transition matrix } P = \begin{array}{c|cc} & G & B \\ G & 0.98 & 0.02 \\ B & 0.1 & 0.9 \end{array} \quad R = \begin{bmatrix} R_G \\ R_B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad T = \begin{bmatrix} T_G \\ T_B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathcal{T} = \frac{\sum_i \pi_i R_i}{\sum_i \pi_i T_i} = \frac{\pi_G}{\pi_G + \pi_B} \overset{\text{In this case;}}{} \frac{P_{10}}{P_{10} + P_{01}} = \frac{0.1}{0.1 + 0.02} = 0.833$$

b) Compute the througput of a Go-Back-N protocol if the round-trip time is $m = 2$ slots (i.e. a packet that is erroneous in slot $t$ is retransmited in slot $t+2$), in the presence of an error-free feedback channel

$$P^2 = \begin{pmatrix} 0.9624 & 0.376 \\ 0.188 & 0.812 \end{pmatrix} \quad \text{Protocol matrix } C = \begin{bmatrix} P_{00} & P_{01} \\ P_{10}^{(m)} & P_{11}^{(m)} \end{bmatrix} \; m=2 \rightarrow C = \begin{bmatrix} 0.98 & 0.02 \\ 0.188 & 0.812 \end{bmatrix}$$

$$T = \begin{bmatrix} T_G \\ T_B \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix} \quad \mathcal{T} = \frac{P_{10}^{(m)}}{P_{10}^{(m)} + m P_{01}} = \frac{0.188}{0.188 + (2)(0.02)} = 0.8245$$

(C) Same as in the previous point, with the difference that now the feedback [8]

channel is subject to iid errors with prob. 0.1.

$$\tau = \frac{(1-\delta) P_{10}^{(m)}}{(1-\delta) P_{10}^{(m)} + m((1-\delta) P_{01} + \delta P_{01}^{(m)} + \delta P_{10}^{(m)})},$$

"(b)" 05/09/2007

Consider now a system where a channel behavior as in point (a) (Markow model for the forward channel and error-free feedback channel) alternates with a channel behavior in whic the forward channel is subject to iid errors with prob. $\varepsilon = 0.01$. In particular the channel follows the Markov model for a geometric number of slots with mean 2 000 000 slots, then again the Markov model and so on. Compute the overal average througput of the Go-Back-N protocol in this case.

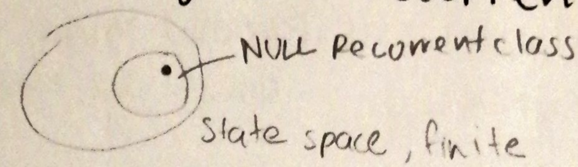
we can model the system as an alternating process: in the first phase we have a Markov behaviour, while in the second one we have the forward iid errors behaviour.

$$\tau_{iid} = \frac{1-\varepsilon}{1-\varepsilon+m\varepsilon} = 0.98$$

$$\tau = \tau_{MARKOV} \frac{E[\text{Markov behaviour}]}{E[\text{cycle duration}]} + \tau_{iid} \frac{E[\text{IID behavior}]}{E[\text{cycle duration}]}$$

$$= \underbrace{\tau_{MARKOV}}_{\text{Previous}} \frac{1}{3} + \underbrace{\tau_{iid}}_{\text{New}} \frac{2}{3} = \frac{1}{3} 0.825 + \frac{2}{3} 0.98 = 0.928$$

T3 Prove that a Markov chain with a finite number of states can't have any null recurrent state.




1: this class is MC
2- this class is finite

Suposo there is one no recurrent state, and then we find the class where this orecurrent state belongs, All thestates in that class will be no recurrent, (because transity is class property).

Note: a recurrent class is Markov chain by itself, because if start ina recurrent class, I never leave, Not the same with transient class that will be left at some point

Contradiction we find a Markov chain with finite states with no positive recurrent states an this violates the following result which says: In a finite Markov chain we must have a positive recurrent state, at least one.

Proof by contradiction: No positive recurrent state.

then $\sum_{j=1}^{N} P_{ij}^{(n)} = 1 \quad \forall i, \forall n.$

In the limits to infinity

$1 = \lim_{n\to\infty} \sum_{j=1}^{N} P_{ij}^{(n)}$ since, the number ofstates is finite, then we can write

$1 = \sum_{j=1}^{N} \lim_{n\to\infty} P_{ij}^{(n)}$ given the assumption there arenot positive recurrent states, all these limits are $=0$. Because the limit is positive only for recurrent states

$1 \doteq 0$ Contradiction

12/12/2006

T1 State and prove the elementary renewal theorem

Pag. 107 ROSS NOT SEEN IN CLASS (THE PROOF) WAS SAID AS OPTIONAL

Ross 3.8 theorem 2nd edition

$$\lim_{t \to \infty} \frac{m(t)}{t} \to \frac{1}{\mu}$$

You need to prove this that is another question in the exam, so you only said is alredy proved. Otherwise, write it at the end.

Proof

suppose $\mu < \infty$

$$S_{N(t)+1} \geq t$$

by the corollary $E[S_{N(t)+1}] = \mu(1 + M(t))$

then

$$\mu(m(t) + 1) \geq t$$

On the limit

$$\lim_{t \to \infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

- 05/09/2007

T3 Prove that for a Markovchain the n-step transition probabilities, $P_{ij}^{(n)}$ satisfy the relantionship

$$P_{ij}^{(n)} = \sum_{m} P_{im}^{(k)} P_{mj}^{(n-k)}, \quad k = 0, 1, \dots n$$

$r \leq n$

For a generic $k$

$$P_{ij}^{(n)} = P[X_n = j \mid X_0 = i] = \sum_{m}^{\infty} P[X_n = j, X_k = m \mid X_0 = i]$$

$$= \sum_{m} P[X_n = j \mid X_k = m, X_0 = i] \, P[X_k = m \mid X_0 = i]$$

$$= \sum_{m} P[X_n = j \mid X_k = m] \, P[X_k = m \mid X_0 = i]$$

$$= \sum_{m} P_{im}^{(k)} P_{mj}^{(n-k)}$$

22/06/2021

T3. For a renewal process state precisely (with proof) the value of:
Ross book 2nd Edition Pag 102 Proposition 3.3.1 or class 18

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad w.p. 1.$$

Proof since $S_{N(t)} \leq t \leq S_{N(t)+1}$, we see that $\frac{S_{N(t)}}{N(t)} \leq \frac{1}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$

However since $\frac{S_{N(t)}}{N(t)}$ is the average of the first $N(t)$ interarrivals time, it follows by the strong law of large numbers that $\frac{S_{N(t)}}{N(t)} \to \mu$ as $N(t) \to \infty$

where $N(t) \to \infty$ when $t \to \infty$ then:

$$\lim_{t \to \infty} \frac{S_{N(t)}}{N(t)} \to \mu$$

Futhermore

$$\frac{S_{N(t)+1}}{N(t)} = \left[\frac{S_{N(t)+1}}{N(t)+1}\right]\left[\frac{N(t)+1}{N(t)}\right]$$

We have then

$$\frac{S_{N(t)+1}}{N(t)} \to \mu \text{ as } t \to \infty$$

Since $\frac{t}{N(t)}$ is between 2 numbers, each of which converges to $\mu$ as $t \to \infty$

then $\lim_{t \to \infty} \frac{t}{N(t)} = \mu \to \lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu}$

T1 If $i \leftrightarrow j$ and if $i$ is recorrent, then $j$ is recorrent. (transity, recorrency is a class property)
class 9 or corollary 3.1 (Samuel karlin 3rd edition page 242)

Since $i \leftrightarrow j$, there exits $m, n \geq 1$ Such that $P_{ij}^{(n)} > 0$ and $P_{ji}^{(m)} > 0$

Let $K > 0$

$$\sum_{K=0}^{\infty} P_{JJ}^{(m+n+k)} \geq \sum_{K=0}^{\infty} P_{Ji}^{(m)} P_{ii}^{(K)} \cdot P_{iJ}^{(n)}$$

Both must to be explained.

$$= P_{Ji}^{(m)} P_{ij}^{(n)} \sum_{V=0}^{\infty} P_{ii}^{(K)}$$

also needs proof.

Hence $i$ is recorrent $\sum_{K=0}^{\infty} P_{ii}^{(K)} = +\infty$.

that implays $\sum_{K=0}^{\infty} P_{JJ}^{(K)} = +\infty$ ; this implays that $J$ is recorrent

T2 For a Poisson process $X(t)$ of rate $\lambda$, state and derive the expression of $P[X(u)=k \mid X(t)=n]$ for tho 2 cases

(i) $0 < u < t, \; 0 \le k \le n$ Samuel Karlin 3rd Edition page 294, class 14

$$P\{X(u)=k \mid X(t)=n\} = \frac{P\{X(u)=k, \; X(t)=n\}}{P\{X(t)=n\}}$$

$$= \frac{P\{X(u)=k, \; X(t)-X(u)=n-k\}}{P\{X(t)=n\}}$$

$$= \frac{\{e^{-\lambda u}(\lambda u)^k/k!\}\{e^{-\lambda(t-u)}[\lambda(t-u)]^{n-k}/(n-k)!\}}{e^{-\lambda t}(\lambda t)^n/n!}$$

$$= \frac{n!}{k!(n-k)!} \frac{u^k(t-u)^{n-k}}{t^n}$$

ii) $0 < t < u, \; 0 \le n \le k$

$$P[X(u)=k \mid X(t)=n] =$$

$$\frac{P[X(u)=k, X(t)=n]}{P[X(t)=n]} = \frac{P[X(t)=n, \; X(u)-X(t)=k-n]}{[X(t)=n]}$$

By independent interarrivals

$$\frac{\cancel{P[X(t)=n]}\, P[X(s)-X(t)=k-n]}{\cancel{P[X(t)=n]}} = e^{-\lambda(u-t)} \frac{(\lambda(u-t))^{k-n}}{(k-n)!}$$

— 09/07/2007

T3 Prove that for a Poisson process $X(t)$ the statistics of $X(s)$ conditioned on $X(t)$, $s < t$, is binomial and provide the expression of $P[X(s) = k \mid X(t) = n]$

$$P[X(s) = k \mid X(t) = n] = \binom{n}{k} \left(\frac{s}{t}\right)^{k} \left(1 - \frac{s}{t}\right)^{n-k}$$

Between 0 and $t$ we have $n$ arrivals, whic are iid according to a uniform random variable $U[0,t]$. The probabibility that each falls in $[0,s]$ is $\frac{s}{t}$. Therefore $X(s)$ can be seen as a binomial random variable with parameters $n$, $\frac{s}{t}$.

[redacted] 13/07/21

**T1 For a Poisson process of rate $\lambda$, prove that the interarrival times are iid with mean $1/\lambda$** Ross book Interarrival and waiting distributions

If $X_n$ is denote as the time from $(n-1)$st to the nth event, the sequence $\{X_n, n=1,2,\dots\}$ is the sequence of intearrival times.

Note that

$$P\{X_1 > t\} = P\{N(t)=0\} = e^{-\lambda t}$$

$X_1$ has an exp. distribution with mean $1/\lambda$.

$$P\{X_2 > t\} = E[P\{X_2 > t \mid X_1\}] \quad \dots \quad (*)$$

But

$$P\{X_2 > t \mid X_1 = s\} = P\{0 \text{ events in } (s, s+t] \mid X_1 = s\}$$
$$= P\{0 \text{ events in } (s, s+t]\}$$
$$= e^{-\lambda t}$$

The previous equations followed from independent and stationary increments.
From (*) we conclude $X_2$ is an exp. r.v. with mean $1/\lambda$ and $X_2$ is independent of $X_1$

**T2 state precisely and formally prove the result that establishes that in a Markov chain the period is a class property.** (Periodicity is a class property).

class 8, Ross Book pag. 66 2nd edition

if $i \leftrightarrow j$, then $d(i) = d(j)$.

Assume $i \leftrightarrow j$ communicates, then exits $\underbrace{m, n, \text{such that } P_{ij}^{(m)} > 0 \quad P_{ji}^{(n)} > 0}$ Very important

Then $P_{jj}^{(m+n)} \geq P_{ji}^{(n)} \cdot P_{ij}^{(m)} > 0$ strictly positive

state why inequality

Let $s \in \{n \geq 1 : P_{ii}^{n} > 0\}$

$$P_{jj}^{(m+s+n)} \geq P_{ji}^{(n)} P_{ii}^{(s)} P_{ji}^{(m)} > 0$$

$m+n,\ m+s+n \in \{k \geq 1 : P_{jj}^{k} > 0\}$

$d(j)$ divids $m+n$ and $m+s+n$

$d(j)$ divids $m+s+n-(m+n) = s$

$d(j)$ divids $d(i)$

Similar way can be proved that $\underbrace{d(i) \text{ divides } d(j) \quad \text{then } d(i) = d(j)}$

recomended to prove it

Carefull, in theory professors can be deadly precise. You could lose points just for not explaining why you use an inequality.

14 / 07 / 2006

T1. Similar as 13/07/21 T1

T2. Prove that in a Markov chain the period is a class property.
Same as T2 13/07/21

T3. Prove that for a renewal process $E[S_{N(t)+1}] = E[X](M(t)+1)$

we know the renewal equation $A(t) = a(t) + \int_0^t A(t-x)\,dF(x)$ if $a(t)$ is bounded, the solotion of that equation is

$$A(t) = a(t) + \int_0^t a(t-x)\,dM(x)$$

Let's suppose $A(t) = E[S_{N(t)+1}]$

We can derive $E[S_{N(t)+1} \mid X_1 = x] = \begin{cases} x & \text{if } x > t \\ x + A(t-x) & \text{if } x \le t \end{cases}$

$$A(t) = E[S_{N(t)+1}] = \int_0^\infty E[S_{N(t)+1} \mid X_1 = x]\,dF(x)$$

$$= \int_t^\infty x\,dF(x) + \int_0^t x + A(t-x)\,dF(x) =$$

$$\int_0^\infty x\,dF(x) + \int_0^t A(t-x)\,dF(x) = E[X] + \int_0^t A(t-x)\,dF(x)$$

If $E[X]$ is bounded $E[S_{N(t)+1}] = A(t) = E[X] + \int_0^t E[X]\,dM(x)$

$$= E[X]\left(1 + \int_0^t dM(x)\right)$$

$$= E[X](1 + M(t)$$