Stochastic Processes – Cybersecurity – AY 2023/2024
written test – July 1, 2024 – part A (90 minutes)

E1 Consider a two-state Markov channel, where the average probability of a bad channel is 0.1 and the average number of consecutive good slots is 20. The packet error probability is 1 for a bad slot and 0 for a good slot, respectively. The round-trip time is $m = 2$ slots, i.e., a packet that is erroneous in slot t is retransmitted in slot $t+2$ (if a retransmission protocol is used).

(a) Compute the throughput that could be obtained if packets were directly transmitted over the channel without using any protocol

(b) compute the throughput of a Go-Back-N protocol that transmits packets over the Markov channel described above, in the presence of an error-free feedback channel

(c) compute the throughput of a Go-Back-N protocol that transmits packets over the Markov channel described above, with a feedback channel subject to iid errors with probability $\delta = 0.1$.

E2 Consider a network node able to handle traffic at 20 Gbps under normal conditions. The node is subject to attacks, that arrive according to a Poisson process of rate $\lambda = 1/T_1$. For each attack, the node has a probability β of being infected, whereas with probability $1-\beta$ the attack has no consequence. When a node gets infected, it automatically starts a clean-up process that lasts T_2 and occupies 70% of its resources, so that during this phase the node can only handle 6 Gbps. The clean-up process is successful with probability $1-\alpha$ (in which case the node starts working normally), whereas with probability α it fails and the node needs to be restored manually by a human operator, which takes T_3, during which time the node does not handle any traffic. After being manually restored, the node starts working normally.

(a) By identifying an appropriate renewal cycle, compute the fraction of the time the node is not handling any traffic and the average traffic per unit time (in Gbps) handled by the node,

(b) Compute how often (e.g., how many times a day on average) a human operator's intervention is needed.

(For all the above quantities, find mathematical expressions as a function of the parameters, and then compute their numerical values for $T_1 = 10$ minutes, $T_2 = 30$ minutes, $T_3 = 3$ hours, $\alpha = 0.1$, $\beta = 0.01$.)

E3 Consider a Markov chain with the following transition matrix (states are numbered from 0 to 4):

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.8 & 0 \\ 0 & 0 & 0.2 & 0.1 & 0.7 \end{pmatrix}$$

(a) Draw the transition diagram, identify the classes, classify the states, and compute the probabilities of absorption in all recurrent classes starting from each transient state

(b) compute $\lim_{n\to\infty} P^n$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P^k$

(c) compute the average recurrence time for all states, and the average first passage time from any state to state 0.

E4 Consider two independent Poisson processes $X_1(t)$ e $X_2(t)$, where $X_i(t)$ is the number of arrivals for process i during $[0,t]$. The average number of arrivals per unit time of the two processes is $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively.

(a) Compute $P[X_1(2) = 1 | X_1(3) = 2]$ and $P[X_1(3) = 2 | X_1(2) = 1]$

(b) Compute $P[X_2(1) = 1 | X_1(2) + X_2(2) = 3]$ and $P[X_1(2) + X_2(2) = 3 | X_2(1) = 1]$

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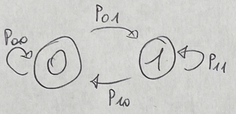
E 1



0: Good state
1: Bad state

$$\pi = (\pi_0, \pi_1) \qquad \pi_1 = 0.1 = \frac{P_{01}}{P_{01} + P_{10}(m)}$$

$$\Rightarrow \pi_0 = 1 - \pi_1 = 0.9$$

$$\mathbb{E}[\#\text{ cons. good slot}] = 20 = \frac{1}{P_{01}}$$

$$\Rightarrow P_{01} = \frac{1}{20} = 0.05 \Rightarrow P_{00} = 1 - P_{01} = 0.95$$

$$\Rightarrow P_{10} = \frac{P_{01} - \pi_1 P_{01}}{\pi_1} = 0.45 \Rightarrow P_{11} = 1 - P_{10} = 0.55$$

$$C = \begin{pmatrix} 0.95 & 0.05 \\ 0.45 & 0.55 \end{pmatrix} \qquad \underset{m=2}{C^m} = \begin{pmatrix} 0.925 & 0.075 \\ 0.675 & 0.325 \end{pmatrix}$$

a) No protocol means every packet is a success

$$\Rightarrow \eta = \pi_0 = 0.9$$

b) Error free feedback channel

$$\eta = \frac{P_{10}(m)}{P_{10}(m) + m \cdot P_{01}} = \frac{0.675}{0.675 + 2 \cdot 0.05} \simeq 0.871$$

c) Feedback iid errors $\delta = 0.1$

$$\eta = \frac{(1-\delta)\, P_{10}(m)}{(1-\delta+m\delta)\, P_{10}(m) + m\left[(1-\delta) P_{01} + P_{01}(m)\delta\right]} = 0.7168$$

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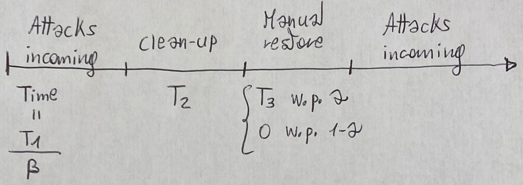
E2



a)
① $$P[\text{no traffic}] = \frac{T_3\, a}{\frac{T_1}{\beta} + T_2 + T_3\, a}$$

② $$\mathbb{E}[\text{traffic}] = \frac{\frac{T_1}{\beta}\cdot 20 \text{ Gbps} + T_2 \cdot 6 \text{ Gbps} + T_3 \cdot a \cdot 0}{\frac{T_1}{\beta} + T_2 + T_3\, a}$$

b)
$$P[\text{no traffic}] = \frac{\mathbb{E}[\text{time no traffic}]}{\mathbb{E}[\text{cycle}]}$$

$\Rightarrow$ ③ $$\mathbb{E}[\text{cycle}] = \frac{\mathbb{E}[\text{time no traffic}]}{\mathbb{P}[\text{no traffic}]} = \frac{T_3}{P[\text{no traffic}]}$$

$T_1 = 10$ min
$T_2 = 30$ min
$T_3 = 180$ min
$a = 0.1$
$\beta = 0.01$

① = 0.0171755

② = 19.256 Gbps

③ = 10480 min = 7.278 days

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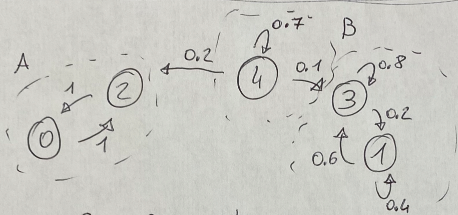
E3 1

a)



Class $A = \{0, 2\}$, positive recurrent
period $= 2$

Class $B = \{4\}$ transient

Class $C = \{1, 3\}$ positive recurrent aperiodic

$$\Pi_4(A) = \frac{0.2}{0.2+0.1} = \frac{2}{3} \qquad \Pi_4(C) = \frac{0.1}{0.2+0.1} = \frac{1}{3}$$

prob. of absorption in A starting from 4

$$\frac{P_{42}}{P_{42}+P_{43}} \qquad \frac{P_{43}}{P_{43}+P_{42}}$$

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b)

$$\begin{cases} \Pi_1 = \Pi_1 P_{11} + \Pi_3 P_{31} \\ \Pi_3 = \Pi_1 P_{13} + \Pi_3 P_{33} \\ \Pi_1 + \Pi_3 = 1 \end{cases}$$

$$\begin{array}{c|c} & \begin{array}{cc} 1 & 3 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$$

$$\Pi_1 (1 - 0.4) = \Pi_3 \, 0.2 \qquad \Pi_1 = \frac{0.2}{0.6} \Pi_3$$

$$\frac{1}{3}\Pi_3 + \Pi_3 = 1 \Rightarrow \Pi_3 = \frac{3}{4} \Rightarrow \Pi_1 = \frac{1}{4}$$

$$\Pi_{41} = \Pi_4(C) \cdot \Pi_1 = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$$\Pi_{43} = \Pi_4(C) \cdot \Pi_3 = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

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E3 | 2

$$\lim_{n\to\infty} P^n = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & X_1 & 0 & X_2 & 0 & 0 \\ 1 & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 2 & X_3 & 0 & X_4 & 0 & 0 \\ 3 & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 4 & X_5 & \frac{1}{12} & X_6 & \frac{1}{4} & 0 \end{array}$$

$$\begin{cases} \pi_0 = 0 \cdot \pi_0 + 1 \cdot \pi_2 \\ \pi_2 = 1 \cdot \pi_0 + 0 \cdot \pi_2 \\ \pi_0 + \pi_2 = 1 \Rightarrow \pi_0 = \pi_2 = \frac{1}{2} \end{cases}$$

$$\begin{array}{c|cc} & 0 & 2 \\ \hline 0 & 0 & 1 \\ 2 & 1 & 0 \end{array}$$

$\Rightarrow \pi_{40} = \pi_4(A) \cdot \pi_0 = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$

$\pi_{42} = \pi_4(A)\, \pi_2 = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$

$\Rightarrow X_1 = X_2 = X_3 = X_4 = \pi_0 = \pi_2 = \frac{1}{2}$

$X_5 = X_6 = \pi_{40} = \pi_{42} = \frac{1}{3}$ ↘ Adding these to the matrix gives $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k$

Check row 4 $\quad \frac{1}{12} + \frac{4}{12} + \frac{4}{12} + \frac{3}{12} = 1$

c) Avg. recurrence times $m_i = \frac{1}{\pi_i}$

$m_4 = \infty$ (transient)

$m_0 = 2 \quad m_2 = 2 \quad m_3 = \frac{4}{3} \quad m_1 = 4$

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E3 | 3 First pass. times

$$E[\theta_{30}] = E[\theta_{10}] = \infty$$

$$E[\theta_{20}] = 1 + \overset{0}{\overset{\shortparallel}{P_{21}}} E[\theta_{10}] + \overset{0}{\overset{\shortparallel}{P_{22}}} E[\theta_{20}] + $$

$$+ \overset{0}{\overset{\shortparallel}{P_{23}}} E[\theta_{30}] + \overset{0}{\overset{\shortparallel}{P_{24}}} E[\theta_{40}] = 1$$

$$E[\theta_{00}] = 1 + \overset{0}{\overset{\shortparallel}{P_{01}}} E[\theta_{10}] + \overset{1}{\overset{\shortparallel}{P_{02}}} E[\theta_{20}] +$$

$$+ \overset{0}{\overset{\shortparallel}{P_{03}}} E[\theta_{30}] + \overset{0}{\overset{\shortparallel}{P_{04}}} E[\theta_{40}] =$$

$$= E[\theta_{20}] = 2 = 1 + 1$$

For transient state it doesn't apply ?

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E4

a)

Binomial

$$P[X_1(2)=1 \mid X_1(3)=2] =$$

$$= \binom{2}{1}\left(\frac{2}{3}\right)^1\left(1-\frac{2}{3}\right)^{2-1} = \frac{2!}{1!\,1!}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{4}{9}$$

$$P[X_1(3)=2 \mid X_1(2)=1] =$$

$$= P[X_1(3)-X_1(2)=1] = \frac{e^{-\lambda_1\cdot(3-2)}\cdot(\lambda_1)^1}{1!} =$$

$$= e^{-1} \simeq 0.3679$$

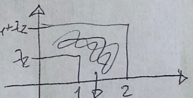
b) $P[X_2(1)=1 \mid X_1(2)+X_2(2)=3] =$

$$= \binom{3}{1}\left(\frac{\lambda_2\cdot 1}{(\lambda_1+\lambda_2)\cdot 2}\right)^1\left(\frac{\lambda_2(2-1)+\lambda_1\cdot 2}{(\lambda_1+\lambda_2)\cdot 2}\right)^2 =$$

?

$$= \frac{3!}{2!} = \frac{4}{9}$$

$\frac{3!}{2!} = 3$



$$P[X_1(2)+X_2(2)=3 \mid X_2(1)=1] =$$

$$= P[X_1(2)+X_2(2)-X_2(1)=2] =$$

$$= \frac{e^{-\lambda t}(\lambda t)^k}{k!} = \frac{e^{-4}\,4^2}{2!} \simeq 0.1465$$

$\lambda t = (\lambda_1+\lambda_2)\cdot 2 - \lambda_2\cdot 1 = 6-2=4$

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$\boxed{T1}$

$$i \leftrightarrow j \Rightarrow \exists\, m,n \geq 0 \text{ s.t. } \begin{array}{l} P_{ij}^{(m)} > 0 \\ P_{ji}^{(n)} > 0 \end{array}$$

Let $K > 0$

NO why?

$$P_{jj}^{(K)} \geq P_{jj}^{(n+m+K)} \geq P_{ji}^{(n)} P_{ii}^{(K)} P_{ij}^{(m)} > 0$$

$$\Rightarrow \underbrace{\sum_{K=0}^{+\infty} P_{jj}^{(K)}}_{1} \geq \sum_{K=0}^{+\infty} P_{jj}^{(n+m+K)} \geq \sum_{K=0}^{+\infty} P_{ji}^{(n)} P_{ii}^{(K)} P_{ij}^{(m)} > 0$$

$$\sum_{K=0}^{+\infty} P_{ji}^{(n)} P_{ii}^{(K)} P_{ij}^{(m)} = P_{ji}^{(n)} P_{ij}^{(m)} \underbrace{\sum_{K=0}^{+\infty} P_{ii}^{(K)}}_{2}$$

By the recurrence criterion, if

2 diverges $\Leftrightarrow$ i is recurrent

So, if i is recurrent (hypothesis) $\Rightarrow$ 2 diverges

If 2 diverges $\Rightarrow$ 1 diverges (following chain of inequality)

If 1 diverges $\Leftrightarrow$ j is recurrent

proving the claim

Also recurrence criterion

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T2

Let S_n = time between $(n-1)st$ and nth event

- $P[S_0 > t] = P[\text{no arrivals in } [0,t]] = e^{-\lambda t}$

$\Rightarrow S_0 \sim exp(\lambda)$, $\mathbb{E}[S_0] = \frac{1}{\lambda}$

- $P[S_1 > t \mid S_0 = s] = P[\text{no arrivals in } (s, s+t] \mid S_0 = s] =$

why? $= e^{-\lambda t}$ $\Rightarrow S_1 \sim exp(\lambda)$, $\mathbb{E}[S_1] = \frac{1}{\lambda}$

independent and stationary increments

and S_1 independent from S_0

- $P[S_n > t \mid S_i = s_i \ \forall i = 0, \dots, n-1] =$

$= P[\text{no arrivals in } (s_0+s_1+\dots+s_{n-1}, s_0+s_1+\dots+s_{n-1}+t] \mid S_i = s_i \ \forall i=0,\dots,n-1] =$

$= e^{-\lambda t}$ $\Rightarrow S_n \sim exp(\lambda)$, $\mathbb{E}[S_n] = \frac{1}{\lambda}$

independent and stationary increments

and S_n independent from $S_0, S_1, S_2, \dots, S_{n-1}$

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T 3

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \text{with probability } 1$$

Proof:

$$S_{N(t)} \leq t < S_{N(t)+1}$$

$$\Rightarrow \frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

① $\lim_{t \to \infty} \frac{S_{N(t)}}{N(t)} = \lim_{n \to \infty} \frac{S_n}{n} = \mu$ with prob. 1

Since $N(t) \to \infty$ as $t \to \infty$

Law of large numbers

② $\lim_{t \to \infty} \frac{S_{N(t)+1}}{N(t)} = \lim_{t \to \infty} \underbrace{\frac{S_{N(t)+1}}{N(t)+1}}_{\mu} \cdot \frac{N(t)+1}{N(t)} = \mu \cdot 1$ w. p. 1

$\frac{N(t)+1}{N(t)} \to 1$ as $t \to \infty$

$$\Rightarrow \quad \overset{①}{\mu} \leq \lim_{t \to \infty} \frac{t}{N(t)} \leq \overset{②}{\mu}$$

$$\Rightarrow \lim_{t \to \infty} \frac{t}{N(t)} = \mu \quad \text{w. p. } 1$$

$$\Rightarrow \lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \text{w. p. } 1$$