6,7,8

martedì 25 giugno 2024 12:37

6) T3 Prove that for a Poisson process $X(t)$ the statistics of $X(s)$ conditioned on $X(t)$, $s < t$, is binomial, and provide the espression of $P[X(s) = k | X(t) = n]$.

$$P[X(s) = k \mid X(t) = n] = \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

Proof:

Since $X(t) = n$, the n events are i.i.d. $\sim U[0,t]$, the prob. that each falls in $[0,s]$ is $\frac{s}{t}$ therefore $X(s)$ is Binomial with parameters $(n, \frac{s}{t})$

7) T2 Prove that if states i and j of a Markov chain communicate and i is recurrent, then j is also recurrent.

Corollary: If $j \leftrightarrow i$ and i is recurrent then j is recurrent.

Proof: $j \leftrightarrow i \Longrightarrow \exists n, m\colon P_{ij}^{(n)} P_{ji}^{(m)} > 0.$

Let $k > 0$

$$\sum_{k=0}^{\infty} P_{jj}^{(k)} \geq \sum_{k=0}^{\infty} P_{jj}^{(m+k+n)} \geq \sum_{k=0}^{\infty} P_{ji}^{(m)} P_{ii}^{(k)} P_{ij}^{(n)}$$

$$= P_{ji}^{(m)} \left(\sum_{k=0}^{\infty} P_{ii}^{(k)} \right) P_{ij}^{(n)} = \infty$$

Proof: $i \leftrightarrow j \Rightarrow \exists n, m : P_{ij}^{(n)}, P_{ji}^{(m)} > 0$

Let $k > 0$

$$P_{jj}^{(n+m+k)} \geq P_{ji}^{(m)} \cdot P_{ii}^{(k)} P_{ij}^{(n)}$$

$$\Rightarrow \underbrace{\sum_{k}^{\infty} P_{jj}^{(k)}}_{*} \geq \sum_{k}^{\infty} P_{jj}^{(n+m+k)} \geq \sum_{k}^{\infty} P_{ji}^{(m)} \cdot P_{ii}^{(k)} P_{ij}^{(n)} =$$

$$= P_{ji}^{(m)} P_{ij}^{(n)} \underbrace{\sum_{k}^{\infty} P_{ii}^{(k)}}$$

$\Rightarrow$ If diverges, $*$ diverges

$\Rightarrow$ if i is recurrent, j is recurrent

8) T1 For a Poisson process of rate λ, prove that the interarrival times are iid exponential with mean $1/\lambda$.

Let S_n = time between $(n-1)$st and nth event

① $P[S_0 > t] = P[\text{no arr. in } [0,t)] = e^{-\lambda t}$

$\Rightarrow S_0 \sim \exp(\lambda)$ mean $= \frac{1}{\lambda}$

② $P[S_1 > t \mid S_0 = s] = P[\text{no arrivals in } (s, s+t] \mid \cancel{S_0 = s}]$

$= e^{-\lambda t}$ $\Rightarrow S_1 \sim \exp(\lambda)$ mean $= \frac{1}{\lambda}$

$\downarrow$

indep. and stationary increments

and independent of S_0

③ $P[S_n > t \mid S_i = s_i, i = 0, \dots, n-1] =$

$= P[\text{no arr. in } (S_0 + \dots + S_{n-1}, S_0 + \dots + S_{n-1} + t] \mid S_i \cancel{= s_i, i = 0, \dots, n-1}] =$

$= e^{-\lambda t}$ $\Rightarrow S_n \sim \exp(\lambda)$ with mean $\frac{1}{\lambda}$

$\downarrow$

ind. and stat increments

and indep. of $S_0, \dots, S_{n-1}$

■