Network Modeling
Solutions for 17/06/2008

Problem 1

The embedded Markov chain is given by $P = \begin{bmatrix} 0 & 1 & 0 \\ \beta & 0 & 1-\beta \\ 1 & 0 & 0 \end{bmatrix}$.

Time matrix is given by $T = \begin{bmatrix} 0 & T & 0 \\ \frac{\alpha_1 T}{2} & 0 & \frac{\alpha_2 T}{2} \\ \delta T & 0 & 0 \end{bmatrix}$.

Let now N denote the fully working state, M the semi-working state and G the faulty state.

Mean vector is given by $\mu = \begin{bmatrix} \mu_N \\ \mu_M \\ \mu_G \end{bmatrix} = \begin{bmatrix} T \\ \beta\frac{\alpha_1 T}{2} + (1-\beta)\frac{\alpha_2 T}{2} \\ \delta T \end{bmatrix}$.

The fraction of time spent in each state is given by $P_i = \dfrac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}$.

$$P_N = \frac{T}{\mu_N + \mu_M + (1-\beta)\mu_G} = \frac{T}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + (1-\beta)\delta T}.$$

$$P_M = \frac{\dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2}}{\mu_N + \mu_M + (1-\beta)\mu_G} = \frac{\dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2}}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + (1-\beta)\delta T}.$$

$$P_G = \frac{(1-\beta)\delta T}{\mu_N + \mu_M + (1-\beta)\mu_G} = \frac{(1-\beta)\delta T}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + (1-\beta)\delta T}.$$

Limiting distribution is computed solving the usual system $\pi = \pi P$: $\begin{cases} \pi_N = \dfrac{1}{3-\beta} \\ \pi_M = \dfrac{1}{3-\beta} \\ \pi_G = \dfrac{1-\beta}{3-\beta} \end{cases}$.

Average throughput is is given by $\mathcal{T} = 100 P_N + 50 P_M$.
Using renewal theory we have to identify a suitable renewal cycle: NMNM...NMNMG.
Let S be the number of consecutive $N \mapsto M$ cycles: this is a geometric random variable.
Clearly $\mathrm{P}[S \geq k] = \beta^k$, hence $E[S] = \sum_{k=1}^{\infty} \beta^k = \dfrac{\beta}{1-\beta}$.
Then $E[\text{cycle}] = E[S]T\left(1 + \dfrac{\alpha_1}{2}\right) + T\left(1 + \dfrac{\alpha_2}{2}\right) + \delta T$. Then:

$$P_N = \frac{T + TE[S]}{E[\text{cycle}]} = \frac{T}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + \delta T(1-\beta)}.$$

$$P_M = \frac{E[S]\dfrac{\alpha_1}{2}T + \dfrac{\alpha_2}{2}T}{E[\text{cycle}]} = \frac{\dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2}}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + \delta T(1-\beta)}.$$

$$P_G = \frac{\delta T}{E[\text{cycle}]} = \frac{\delta T(1-\beta)}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + \delta T(1-\beta)}.$$

Obviously these results are the same as the previous ones.

Reward metric is $\mathbf{R} = \begin{bmatrix} 100 \cdot T(E[S]+1) \\ 50 \cdot \left(E[S]\frac{\alpha_1 T}{2} + \frac{\alpha_2 T}{2}\right) \\ 0 \cdot \delta T \end{bmatrix}$. Note that time metric is still μ.

Last point is now straightforward: $\mathcal{T} = \dfrac{100T(E[S]+1) + 50\dfrac{E[S]\alpha_1 T + \alpha_2 T}{2}}{E[\text{cycle}]}$.

Problem 2

Recall the usual Binomial formula: $P[X_1(s) = k | X(t) = n] = \binom{n}{k}\left(\frac{\lambda_1 s}{(\lambda_1+\lambda_2)t}\right)^k \left(1 - \frac{\lambda_1 s}{(\lambda_1+\lambda_2)t}\right)^{n-k}$

$P[X_1(3) = 2 | X(3) = 3] = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{3-2}$

$P[X_1(2) = 2 | X(2) = 2] = \binom{2}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{2-2}$

$P[X_1(1) = 2 | X(2) = 3] = \binom{3}{2}\left(\frac{1}{4}\right)^2\left(1-\frac{1}{4}\right)^{3-2}$

$P[X(2) = 3 | X_1(1) = 2] = P[X_1(2) + X_2(2) - X_1(1) = 1] = \dfrac{(2\lambda_1 + 2\lambda_2 - \lambda_1)e^{-(2\lambda_1+2\lambda_2-\lambda_1)}}{1!}$

For the last point we can see that this is a $M/G/\infty$ queue.

Service time is deterministic: $Y = \dfrac{L}{C} = 0.001$ s.

Notice that in this case $G(z)$ is step function before 1 ms and 1 after 1 ms.

We consider the arrivals as a Poisson process of intensity $\lambda_{TOT} = \lambda_1 + \lambda_2 = 2\lambda = 1000$ packets/s.

Let $M(t)$ be a random variable counting packets in the system at time t. Let $\Lambda = \lambda_{TOT}\int_0^t [1-G(z)]dz$.

$\Lambda_1 = \lambda_{TOT}\int_0^{0.0005}[1-G(z)]dz = 0.0005\lambda_{TOT} = 0.5$.

$\Pr[M(0.0005) = 2] = \dfrac{e^{-\Lambda_1}(\Lambda_1)^2}{2!} = 0.0758$.

For the second point, we realize that we are in steady-state condition: $\Lambda_2 = \lambda_{TOT} Y = 1$.

$\Pr[M(0.003) = 2] = \dfrac{e^{-\Lambda_2}(\Lambda_2)^2}{2!} = 0.1839$.

Problem 3

$\{0, 2\}$ is positive recurrent periodic ($d = 2$) class.

$\{3\}$ is transient class.

$\{1, 4\}$ is positive recurrent aperiodic class.

It is useful to compute the probability of being absorbed in the two classes given we start in the transient state:

P[absorption in 0,2|start in 3] $= \dfrac{0.4}{0.4+0.4} = \dfrac{1}{2}$.

P[absorption in 1,3|start in 3] $= \dfrac{0.4}{0.4+0.4} = \dfrac{1}{2}$.

$$\lim_{n\to\infty} P^n = \begin{bmatrix} X & 0 & X & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ X & 0 & X & 0 & 0 \\ X & \frac{1}{2}\frac{1}{2} & X & 0 & \frac{1}{2}\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \text{ and } \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{\infty} P^k = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2}\frac{1}{2} & \frac{1}{2}\frac{1}{2} & \frac{1}{2}\frac{1}{2} & 0 & \frac{1}{2}\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

For the first passage times, notice that $\hat{\theta}_{04}$ and $\hat{\theta}_{24}$ are both ∞. By basic limit theorem we also have

$\hat{\theta}_{44} = m_4 = \dfrac{1}{\pi_4} = 2$. Then $\hat{\theta}_{34} = 1 + p_{31}\hat{\theta}_{14} + p_{33}\hat{\theta}_{34} = \infty$. Finally $\hat{\theta}_{14} = \dfrac{1}{1-p_{11}} = \dfrac{10}{7}$.

Problem 4

Transition matrix is $P = \begin{bmatrix} 0.99 & 0.01 \\ 0.1 & 0.9 \end{bmatrix}$. The protocol matrix is $C = \begin{bmatrix} p_{00} & p_{01} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix}$.

Reward and time vectors are $\mathbf{R} = \begin{bmatrix} R_G \\ R_B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\quad \mathbf{T} = \begin{bmatrix} T_G \\ T_B \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Throughput is then $\mathcal{T} = \dfrac{\sum_i \pi_i R_i}{\sum_i \pi_i T_i} = \dfrac{\pi_G}{\pi_G + 2\pi_B} = \dfrac{p_{10}^{(2)}}{p_{10}^{(2)} + 2p_{01}} \simeq 0.9043$.

For the last question we have $\mathcal{T} = \dfrac{(1-\delta)p_{10}^{(2)}}{(1-\delta)p_{10}^{(2)} + 2\left((1-\delta)p_{01} + \delta p_{01}^{(2)} + \delta p_{10}^{(2)}\right)}$, where $\delta = 0.02$.