Network Modeling
Solutions for 14/07/2008

Problem 1

$$\Pr[X(0.5) \geq 2|X(1) = 5] = 1 - \Pr[X(0.5) = 0|X(1) = 5] - \Pr[X(0.5) = 1|X(1) = 5]$$
$$= 1 - \binom{5}{0}\left(\frac{0.5\lambda}{\lambda}\right)^0\left(1 - \frac{0.5\lambda}{\lambda}\right)^5 - \binom{5}{1}\left(\frac{0.5\lambda}{\lambda}\right)^1\left(1 - \frac{0.5\lambda}{\lambda}\right)^4 = 0.8125$$

For the second point, the average time required for a packet to be processed and then transmitted is $T_{TOT} = E[\mathcal{U}(10, 30)] + \dfrac{E[L]}{C} = 100$ ms. It must be:

$$\sum_{i=0}^{N} \Pr[i \text{ requests in 100 ms}] > 1 - 0.001 = 0.999$$

We can do this by inspection: $N = 4$ yields solution, so correct capacity is $C_r = 4 \cdot C = 400$ Mbps.

Problem 2

let TT be the state corresponding to two consecutive tails.
For the first question we need to solve the following systems:

$$\begin{cases} u_0 = \Pr[X_T = TT|X_0 = 0] = \frac{1}{2}u_T + \frac{1}{2}u_H \\ u_H = \frac{1}{2}u_T \\ u_T = \frac{1}{2} + \frac{1}{2}u_H \end{cases} \Rightarrow u_0 = \frac{1}{2}$$

$$\begin{cases} v_0 = E[X_T = TT|X_0 = 0] = \frac{1}{2}v_T + \frac{1}{2}v_H \\ v_H = \frac{1}{2}v_T \\ u_T = 1 + \frac{1}{2}v_H \end{cases} \Rightarrow v_0 = 3$$

Problem 3

$$\Pr[\text{no disposal}] = \left(\frac{\frac{1}{\beta}}{\frac{1}{\alpha} + \frac{1}{\beta}}\right)^2 = \left(\frac{\alpha}{\alpha + \beta}\right)^2 = \left(\frac{1}{20}\right)^2 = 0.0025.$$

$$E[\text{no disposal}] = \min\{\xi(\beta), \xi(\beta)\} = \frac{1}{38\alpha} = \frac{1}{2} \text{ days.}$$

$$E[\text{cycle}] = \frac{E[\text{no disposal}]}{\Pr[\text{no disposal}]} = 200 \text{ days.}$$

$$E[\text{1 p. working}] = \Pr[\text{1 p. working}] \cdot E[\text{cycle}] = \left(1 - \left(1 - \frac{1}{20}\right)^2 - \left(\frac{1}{20}\right)^2\right)(200) = 19 \text{ days.}$$

Let now $X \sim \xi(\alpha)$ and $Y \sim \xi(\beta)$.

We want to compute the probability that X happens before Y: $\Pr[X < Y]$. We can do it conditioning and averaging:

$$\begin{aligned}
\Pr[X < Y] &= \int_0^\infty \Pr[X < y | Y = y] dF_Y(y) \\
&= \int_0^\infty \left(1 - e^{-\alpha y}\right) \beta e^{-\beta y} dy \\
&= 19\alpha \int_0^\infty e^{-19\alpha y} dy - 19\alpha \int_0^\infty e^{-(\alpha+\beta)y} dy = \cdots = 0.05
\end{aligned}$$

Average disposed traffic is

$$\begin{aligned}
\mathcal{T} &= 2.5 \Pr[2 \text{ p. working}] + \Pr[1 \text{ p. working}] \\
&= 2.5 \left(1 - \frac{1}{20}\right)^2 + \left(1 - \left(1 - \frac{1}{20}\right)^2 - \left(\frac{1}{20}\right)^2\right) = 2.35215 \text{ Gbps.}
\end{aligned}$$

Problem 4

Probability distribution of X_1 given $X_0 = 0$ is first row of P. Probability distribution of X_1 given $X_0 = 0$ is first row of P^2. For X_{500} we can use the approximation of the steady state distribution: transition matrix is doubly stochastic, so $\pi = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$.
For the average first passage times we need to solve the usual system:

$$\begin{cases} \hat{\theta}_{02} = 1 + p_{00}\hat{\theta}_{02} + p_{01}\hat{\theta}_{12} \\ \hat{\theta}_{12} = 1 + p_{10}\hat{\theta}_{02} + p_{11}\hat{\theta}_{12} \end{cases} \Rightarrow \begin{cases} \hat{\theta}_{02} = \frac{5}{2} \\ \hat{\theta}_{12} = \frac{5}{2} \end{cases}. \text{ Then } \hat{\theta}_{22} = m_2 = \pi_2^{-1} = 3.$$

Average recurrent times are given by $\mathbf{m} = \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \pi_0^{-1} \\ \pi_1^{-1} \\ \pi_2^{-1} \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$.

Last point requires to compute $W_{ij}^{(n)} = E\left[\sum_{k=0}^{n-1} \chi\{X_k = j\} | X_0 = i\right] = \sum_{k=0}^{n-1} p_{ij}^{(k)}$.

In vector form we can write $W_{0j}^{(3)} = \begin{bmatrix} p_{00}^{(0)} + p_{00}^{(1)} + p_{00}^{(2)} \\ p_{01}^{(0)} + p_{01}^{(1)} + p_{01}^{(2)} \\ p_{02}^{(0)} + p_{02}^{(1)} + p_{02}^{(2)} \end{bmatrix}$.