Network Modeling
Solutions for 05/09/2007

Problem 1

Let processing time be $T_{proc} = 0.02$ s. Let packet length be $L \sim \mathcal{U}(8 \cdot 10^6, 16 \cdot 10^6)$ bits. Let server capacity be $C = 100 \cdot 10^6$ bits.
Time required for a transmission is $T_{TX} = T_{proc} + \dfrac{L}{C} \sim \mathcal{U}(0.1, 0.18)$, random variable with mean $E[T_{TX}] = 0.14$ s.
Problem asks the first t for which the system can be considered in steady-state condition. Clearly t is the first instant from which the function $1 - G(z)$ becomes zero: by inspection $t = 0.18$ s.
In this condition $\Lambda = \lambda E[T_{TX}] = 0.14\lambda$. Then $\Pr[M(t) = k] = \dfrac{e^{-\Lambda}(\Lambda)^k}{k!}$.
For the second point recall that $M(t)$ conditioned on $X(t)$ is a Binomial random variable:

$$\Pr[M(T) = m | X(T) = N] = \binom{N}{m} p^m (1-p)^{N-m}$$

where

$$p = \frac{1}{T} \int_0^T [1 - G_{T_{TX}}(z)] dz$$

$\Pr[M(0.1) = 0 | X(0.1) = 2] = \dbinom{2}{0} 1^0 (0)^{2-0} = 0.$
$\Pr[M(1) = 0 | X(1) = 20] = \dbinom{20}{0} (0.14)^0 (0.86)^{20-0} = 0.0489.$

Problem 2

$\{0, 1\}$ is positive recurrent periodic class. This class presents a "ping pong" behavior.
$\{3\}$ is transient class.
$\{2, 4\}$is positive recurrent periodic class.
It is useful to compute the probability in being absorbed in each recurrent class given we start in the transient:
$\Pr[\text{absorption in 0,1}] = u_0 = p_{33}u_0 + p_{30} \Rightarrow u_0 = \dfrac{p_{30}}{1 - p_{33}} = \dfrac{5}{8}$.
$\Pr[\text{absorption in 0,1}] = u_4 = p_{33}u_0 + p_{34} \Rightarrow u_0 = \dfrac{p_{34}}{1 - p_{33}} = \dfrac{3}{8}$.

$$\lim_{n\to\infty} P^n = \begin{bmatrix} X & X & 0 & 0 & 0 \\ X & X & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ X & X & \frac{1}{2}\frac{3}{8} & 0 & \frac{1}{2}\frac{3}{8} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} P^k = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2}\frac{5}{8} & \frac{1}{2}\frac{5}{8} & \frac{1}{2}\frac{3}{8} & 0 & \frac{1}{2}\frac{3}{8} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Average recurrence times are given by $\mathbf{m} = \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ \infty \\ 2 \end{bmatrix}$.

Problem 3

Let $\Pr[H] = p$ be the probability that returned side is H. Using first step analysis we have

$$\begin{cases} u_0 = \Pr[X_T = TT | X_0 = 0] = p u_H + (1-p) u_T \\ u_H = (1-p) u_T \\ u_T = 1 - p + p u_H \end{cases} \Rightarrow \begin{cases} u_0 = p\dfrac{(1-p)^2}{1-p+p^2} + (1-p)\dfrac{1-p}{1-p+p^2} \\ u_H = \dfrac{(1-p)^2}{1-p+p^2} \\ u_T = \dfrac{1-p}{1-p+p^2} \end{cases}$$

For $p = \frac{1}{2}$ we have $u_0 = \frac{1}{2}$.
For the second point we need to solve:
$\begin{cases} u_H = (1-p) u_T \\ u_T = 1 - p + p u_H \end{cases}$ which are the last two equations of previous system.
For $p = \frac{1}{2}$ we have $u_H = \frac{1}{3}$.
Last point is just computation with $p = \frac{1}{4}$.

Problem 4

Transition matrix is $P = \begin{bmatrix} 0.98 & 0.02 \\ 0.1 & 0.9 \end{bmatrix}$. Protocol matrix is $C = \begin{bmatrix} p_{00} & p_{01} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix}$.

First question is straightforward: $\mathcal{T} = \dfrac{p_{10}^{(m)}}{p_{10}^{(m)} + m p_{01}}$, where $m = 2$.
We do not need to compute all C^2 but only $p_{10}^{(2)} = p_{10}p_{00} + p_{11}p_{10} = 0.188$. Then $T = 0.8245$.
For the second point we can model the system as an alternating process: in the first phase we have a Markov behavior already discussed, while in the second one we have the forward IID errors behavior.
For the latter we have $\mathcal{T}_{iid} = \dfrac{1-\epsilon}{1-\epsilon+m\epsilon} = 0.98$.
Then $\mathcal{T} = \mathcal{T}_{Markov}\dfrac{E\left[\text{Markov behavior}\right]}{E\left[\text{cycle duration}\right]} + \mathcal{T}_{iid}\dfrac{E\left[\text{IID behavior}\right]}{E\left[\text{cycle duration}\right]} = \mathcal{T}_{Markov}\dfrac{1}{3} + \mathcal{T}_{iid}\dfrac{2}{3} = 0.9281.$