Network Modeling
Solutions for 18/06/2018

Problem 1

$$\Pr[X_1(1)=1|X(2)=3] = \binom{3}{1}\left(\frac{\lambda_1}{2(\lambda_1+\lambda_2)}\right)^1\left(1-\frac{\lambda_1}{2(\lambda_1+\lambda_2)}\right)^{3-1}$$

$$\Pr[X(2)=3|X_1(1)=1] = \Pr[X_1(2)+X_2(2)-X_1(1)=3-1]$$

$$\Pr[X(2)=3|X_1(3)=0] = \Pr[X_2(2)=3] = \frac{e^{-2\lambda_2}(2\lambda_2)^3}{3!}$$

$$\begin{aligned}\Pr[X(2)=3|X_1(3)=1] &\overset{\text{(TPL)}}{=} \Pr[X(2)=3|X_1(2)=0]\Pr[X_1(2)=0|X_1(3)=1]+\\ &+\Pr[X(2)=3|X_1(2)=1]\Pr[X_1(2)=1|X_1(3)=1]\\ &=\Pr[X_1(2)+X_2(2)-X_1(2)=3-0]\binom{1}{0}\left(\frac{2}{3}\right)^0\left(1-\frac{2}{3}\right)^1+\\ &+\Pr[X_1(2)+X_2(2)-X_1(2)=3-1]\binom{1}{1}\left(\frac{2}{3}\right)^1\left(1-\frac{2}{3}\right)^0\end{aligned}$$

Problem 2

Average number of consecutive good slots is 100, so $p_{01}=\frac{1}{100}=0.01$ and so $p_{00}=1-p_{01}=0.99$.

Then we solve $\begin{cases}\pi_G = p_{00}\pi_G + p_{10}\pi_B\\ \pi_G = 1-\pi_B\end{cases} \Rightarrow p_{10} = \frac{\pi_G(1-p_{00})}{\pi_B} = 0.49$. So $p_{11}=1-p_{10}=0.51$.

Embedded MC is $C=\begin{bmatrix}0.99 & 0.01\\ 0.49 & 0.51\end{bmatrix}$.

For the first question we have $\mathcal{T}_1=\frac{p_{10}}{p_{10}+p_{01}}=0.98$.

Protocol matrix is $C=\begin{bmatrix}- & 0.015\\ 0.735 & -\end{bmatrix}$.

For the second question we have $\mathcal{T}_2=\frac{p_{10}^{(2)}}{p_{10}^{(2)}+2p_{01}}\simeq 0.9735$.

For the last question we first compute $\mathcal{T}_3=\frac{(1-\delta)p_{10}^{(2)}}{(1-\delta)p_{10}^{(2)}+2\left((1-\delta)p_{01}+\delta p_{01}^{(2)}+\delta p_{10}^{(2)}\right)}\simeq 0.7974$.

Then average throughput is $\mathcal{T}_{\ni}=\frac{E[\text{perfect}]T_2+E[\text{iid}]T_3}{E[\text{cycle}]}\simeq 0.88545$

Problem 3

$A=\{0,2\}$ is positive recurrent periodic ($d=2$) class.
$\{3\}$ is transient class.
$B=\{1,4\}$ is positive recurrent aperiodic class.
Using first step analysis we have:
$\Pr[\text{absorption in }\{0,2\}|X_0=3]=\pi_3(\{0,2\})=\frac{5}{8}$.
$\Pr[\text{absorption in }\{1,4\}|X_0=3]=\pi_3(\{1,4\})=\frac{3}{8}$.

$$\lim_{n\to\infty} P^n = \begin{bmatrix} X & 0 & X & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ X & 0 & X & 0 & 0 \\ X & \frac{1}{3}\frac{3}{8} & X & 0 & \frac{2}{3}\frac{3}{8} \\ 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix} \qquad \lim_{n\to\infty} \sum_{k=0}^{n-1} P^k = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2}\frac{5}{8} & \frac{1}{3}\frac{3}{8} & \frac{1}{2}\frac{5}{8} & 0 & \frac{2}{3}\frac{3}{8} \\ 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

Then $\left[\overline{\theta}_{04},\, \overline{\theta}_{14},\, \overline{\theta}_{24},\, \overline{\theta}_{34},\, \overline{\theta}_{44}\right] = \left[\infty,\, \frac{5}{3},\, \infty,\, \infty,\, \frac{3}{2}\right]$ and $\mathbf{m} = \left[m_0,\, m_1,\, m_2,\, m_3,\, m_4,\right] = \left[2,\, 3,\, 2,\, \infty,\, \frac{3}{2}\right]$.

Problem 4

$\Pr[\text{both OFF}] = \left(\dfrac{\frac{1}{\beta}}{\frac{1}{\alpha} + \frac{1}{\beta}}\right)^2 = \left(\dfrac{1}{20}\right)^2 = 0.0025.$

$\Pr[\text{both ON}] = \left(\dfrac{\frac{1}{\alpha}}{\frac{1}{\alpha} + \frac{1}{\beta}}\right)^2 = \left(\dfrac{19}{20}\right)^2 = 0.9025.$

$E[\text{both OFF}] = E[\min\{\xi(\beta), \xi(\beta)\}] = \dfrac{1}{2\beta} = 500$ seconds.

Then we can compute average cycle duration: $E[\text{cycle}] = \dfrac{E[\text{both OFF}]}{\Pr[\text{both OFF}]} = 200000$ seconds.

The probability that at least one processor is working is $\Pr[\geq 1 \text{ ON}] = 1 - \Pr[\text{both OFF}] = 0.9975$.

So $E[\geq 1 \text{ ON}] = \Pr[\geq 1 \text{ ON}]E[\text{cycle}] = 199500$ seconds.

Finally $T = \Pr[\text{both ON}] \cdot 1\,\text{Gbps} + (1 - \Pr[\text{both ON}] - \Pr[\text{both OFF}]) \cdot 0.3\,\text{Gbps} = 0.931$ Gbps.