9,11,12

martedì 25 giugno 2024 17:00

9) T3 Consider a random walk over the non-negative integers with the following transition probabilities: $P_{01} = 1$, $P_{i,i+1} = p, P_{i,i-1} = q, i > 0$, with $p + q = 1$. Study its behavior, and in particular characterize its recurrence or transiency and derive the steady-state distribution.

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "1-p"] \arrow[r, bend left, "p"] & 2 \arrow[l, bend left, "1-p"] \arrow[r, bend left, "p"] & \cdots \arrow[l, bend left, "1-p"] \end{tikzcd}$$

$$P = \begin{array}{c c} & \begin{matrix} 0 & 1 & 2 & \cdots\cdots \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & 0 & \\ 0 & 1-p & 0 & p & 0 & \\ \vdots & & & & & \ddots \end{pmatrix} \end{array}$$

To derive steady state distrib. we need to solve

$$x_i = \sum_{j=0}^{+\infty} x_j P_{ji} = p\, x_{i-1} + q\, x_{i+1}$$

where $\sum_{k=0}^{\infty} x_k = 1$. (2)

Firstly

$$x_0 = q\, x_1 \qquad x_1 = \frac{x_0}{q}$$

$$x_1 = x_0 + q\, x_2 \qquad x_2 = \frac{x_1 - x_0}{q} = \frac{x_0 - q x_0}{q^2} =$$

$$= \frac{(1-q)\, x_0}{q^2} = \frac{p\, x_0}{q^2}$$

$\Rightarrow$ Generally, $x_i = \frac{1}{p}\left(\frac{p}{q}\right)^i x_0$

So using (2)

$$1 = \sum_k x_k = \frac{1}{p} x_0 \sum_k \left(\frac{p}{q}\right)^k \Rightarrow x_0 = \frac{p}{\sum_k \left(\frac{p}{q}\right)^k} \quad (3)$$

$\Rightarrow$ From (3) we can conclude that

- If $p < q$, sum converges so chain is positive recurrent
- If $p \geq q$, sum diverges so chain is transient

11) T2 For a Poisson process $X(t)$ of rate λ, state and derive the expression of $P[X(u) = k | X(t) = n]$ for the two cases (i) $0 < u < t, 0 \leq k \leq n$ and (ii) $0 < t < u, 0 \leq n \leq k$.

Binomial theorem

i) $0 < u < t, \quad 0 \leq k \leq n$

$$= \frac{P[X(u) = k,\ X(t) = n]}{P[X(t) = n]} =$$

$$= \frac{P[X(u) = k,\ X(t) - X(u) = n-k]}{P[X(t) = n]} =$$

$$= \frac{\left(\cancel{e^{-\lambda u}} \frac{(\lambda u)^k}{k!}\right)\left(\cancel{e^{-\lambda(t-u)}} \frac{(\lambda(t-u))^{n-k}}{(n-k)!}\right)}{\cancel{e^{-\lambda t}} \frac{(\lambda t)^n}{n!}} =$$

$$= \binom{n}{k} \frac{u^k}{\underbrace{t^n}_{t^k\, t^{n-k}}} (t-u)^{n-k} = \binom{n}{k} \left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k}$$

ii) $0 < t < u \quad 0 \leq n \leq k$

$$= \frac{P[X(u) = k,\ X(t) = n]}{P[X(t) = n]} =$$

$$= \frac{P[\overbrace{X(t) = n}^{a},\ \overbrace{X(u) - X(t) = k-n}^{b}]}{P[X(t) = n]} =$$

disjoint intervals $\Rightarrow$ independent

$$= \frac{\cancel{P[a]}\ P[b]}{\cancel{P[X(t) = n]}} =$$

$$= P[X(u-t) = k-n] =$$

stationary increments

$$= e^{-\lambda(u-t)} \frac{(\lambda(u-t))^{k-n}}{(k-n)!}$$

12) T3 For a renewal process, state precisely (also providing a formal proof) what is the value of

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$$

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \text{w.p. 1}$$

Proof:

$$S_{N(t)} \leq t < S_{N(t)+1}$$

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

$$\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} = \lim_{n\to\infty} \frac{S_n}{n} = \mu \quad \text{w.p. 1}$$

LLN

$N(t) \to \infty$ as $t \to \infty$

$$\lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} = \mu \cdot 1 \quad \text{w.p. 1}$$

$\frac{S_{N(t)+1}}{N(t)+1} \to \mu$, $\frac{N(t)+1}{N(t)} \to 1$, LLN

$$\Rightarrow \mu \leq \lim_{t\to\infty} \frac{t}{N(t)} \leq \mu \quad \text{w.p. 1}$$