July 13, 2021

E3. Consider a network node able to handle traffic at 10 Gbps under normal conditions. The node is subject to attacks, that arrive according to a Poisson process of rate $\lambda = 1/T_0$. For each attack, the node has a prob. $\boxed{1-\alpha \text{ of being infected}}$ whereas with prob. α the attack has no consequence. When a node gets infected, it automatically starts a clean-up process that lasts T_1 and occupies 70 % of it resources, so that during this phase the node only handle 3Gbps. The clean-up process is succesful with prob. β (in which case the node stars working normally), whereas with prob. $\boxed{1-\beta \text{ it fails}}$ and the node needs to be restored manually by a human operator which takes T_2, during which time the node does not handle any traffic. After manually restored, the node starts working normally.

(a) By identifying an appropriate renewal cycle, compute the fraction of the time the node is not handling any traffic and the average traffic per unit time (in Gbps) handled by the node.

λ = arrival rate

$\lambda\alpha$ = rate of infection $= (1-\alpha)\frac{1}{T_0}$

Normal	Cleaning	manual
T_0'	T_1	T_2

$$T_0' = \frac{1}{\lambda\alpha} = \frac{T_0}{(1-\alpha)}$$

Interarrival time between effective attacks

$$P[\text{No traffic}] = \frac{(1-\beta)T_2}{T_0' + T_1 + \beta T_2} = \frac{(1-\beta)T_2}{\left(\frac{T_0}{1-\alpha}\right) + T_1 + (1-\beta)T_2} \text{ ; } P[N\,Tr] \approx 0.01734$$

using minutes

using min.

$$E[\text{traffic}] = \frac{T_0' \, 10\text{Gbps} + T_1 \, 3\text{Gbps} + \beta T_2 \cdot 0\text{ Gbps}}{T_0' + T_1 + T_2(1-\beta)} = 9.692 \text{ Gbps}$$

b) Compute how often (eg. how many times a day on average) a human operator's intervention is needed.

$$E[\text{cycle}] = \frac{E[\text{Time Not working}]}{P[\text{No Traffic}]} = \frac{T_2}{P[\text{No traffic}]} = 10380.62 \text{ with min}$$

$$10380.62 \left(\frac{1}{60 \times 24}\right) = 7.2 \text{ Days}$$

For all the above quantities, find mathematical expressions as a function of the parameters and then compute their numerical values for $T_0 = 20$ min, $T_1 = 20$ min, $T_2 = 3$ hours, $\alpha = 0.98$ $\beta = 0.9$).

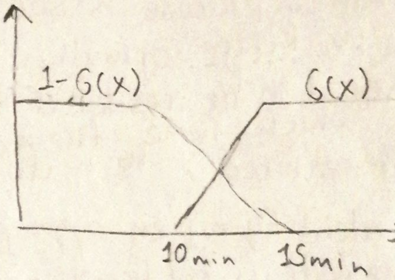
E4. Consider an exhibition where visitors arrive according to a Poisson process with rate $\lambda = 12$ customers per hour. Each visitor spends a time uniformly distributed between 10 and 15 min, and then leaves. The room in which the exhibition is shown is large enough to ensure there is never a need to block customers at entrance due to too many ~~#~~people inside. The exhibition is open from 8 AM. to 6 PM.

(a) Compute the prob. that fewer than 2 visitors arrive during the first fifteen minutes.

System M/G/∞

$\lambda = 12$ per hours. $\lambda = \frac{1}{5}$ per min. $\frac{1}{\lambda} = 5$

$\lambda = 3$ per 15 min



$$P[X(0.25) < 2] = P[X(0.25) = 0] + P[X(0.25) = 1]$$

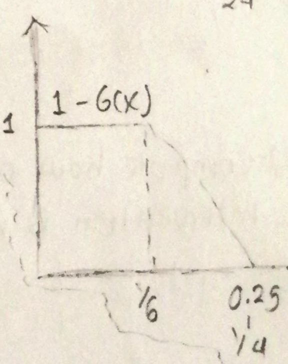
$$= \frac{e^{-3}(3)^0}{0!} + \frac{e^{-3}(3)^1}{1!} = 0.1991$$

b) Compute the probability that at 8:15 AM there is only one visitor in the room, and the prob. that at closing time (6 PM) the room is empty.

1) $$\Lambda = \lambda \int_0^t [1 - G(z)]dz = 12 \int_0^{0.25} [1 - G(z)]dz = 12\left(\frac{5}{24}\right) = \frac{5}{2}$$

$t = 0.25$

$\text{TArea} = \frac{\frac{1}{4} - \frac{1}{6}}{2} = \frac{1}{24}$ $\quad \text{TArea} = \frac{1}{6} + \frac{1}{24} = \frac{5}{24}$



$$P[M(0.25) = 1] = \frac{e^{-\Lambda}\Lambda^1}{1!} = \frac{e^{-5/2}\left(\frac{5}{2}\right)^1}{1} = 0.205$$

1) $\Lambda = \lambda \cdot \frac{5}{24}$ $\Lambda = 12 \cdot \left(\frac{5}{24}\right) = 2.5$

average

$$P[M(0.25) = 1] = \frac{e^{-2.5}\Lambda^1}{1} = 0.205$$

$P[\text{Not empty at 6 PM}] = e^{-\Lambda}$; $\Lambda = \lambda E[\text{service}] = 12(0.2083)$; $e^{-\Lambda} = e^{-2.499} = 0.$

$E[\text{service}] = E(U[10', 15']) = \frac{25'}{2} = 12.5 \text{ min} \rightarrow 0.2083$

60

12.5

1

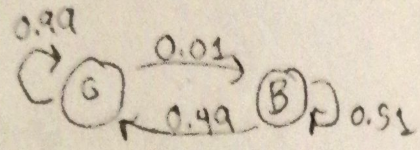
$$P[\text{empty at 6 PM}] = e^{-5/2} = 0.8208$$

June 22, 2021.

E 2. Consider a two-state Markov channel, where the steady-state probability that the channel is in the bad state is 0.02 and the average number of consecutive good slots is 100. The packet error probability is 1 for a bad slot and 0 for a good slot, respectively. The round-trip time is $m=2$, i.e. a packet that is erroneous in slot t is retransmitted in slot $t+2$ (if a retransmission protocol is used).

(a) Compute the throughput that could be obtained if packets were directly transmitted over the channel without using any protocol.

$E(G) = 100$; $\rightarrow$ $P_{01} = \frac{1}{E(G)} = 0.01$ $\quad \pi_B = 0.02$



$$P = \begin{pmatrix} 0.99 & 0.01 \\ 0.49 & 0.51 \end{pmatrix}$$

$$\pi_B = \frac{P_{01}}{P_{01}+P_{10}} = 0.02$$

$$\pi_B P_{01} + P_{10}\pi_B = P_{01}$$

$$P^2 = \begin{pmatrix} 0.985 & 0.015 \\ 0.735 & 0.265 \end{pmatrix}$$

$$P_{10} = \frac{P_{01}}{\pi_B} - \frac{\pi_B P_{01}}{\pi_B} = \frac{0.01}{0.02} - 0.01$$

$$P_{10} = 0.49$$

$$T = \pi_G = 0.98$$

b) Compute the throughput of the Go-Back-N protocol for an error free feedback channel.

$$T = \frac{P_{10}^{(2)}}{P_{10}^{(2)} + (2)P_{01}} = \frac{0.735}{0.735 + (2)(0.01)} = 0.9735$$

c) ~~Compute the throughput of the Go-Back-N protocol for a feedback channel,~~

i. Compute the throughput of the Go-Back-N protocol for a feedback channel subject to iid errors with prob. 0.1.

$$T = \frac{(1-\delta)\,P_{10}(m)}{(1-\delta)\,P_{10}^{(m)} + m\left((1-\delta)P_{01} + \delta P_{01}^{(m)} + \delta P_{10}^{(m)}\right)} = 0.797468$$

=

E4 Consider a node that contains 2 identical and independent servers, each able to stream data at 1 Gbps. Each server is subject to attacks according to a Poisson process with rate $\lambda = 10$ attacks/hour, and each attack is effective with probability 1/9, whereas it has no consequence with probability 8/9. (Hint: only consider the process of effective attacks). As a result of each effective attack, the server will remain inoperational (i.e with zero streaming rates) for an exp. time with average T = 6 min, during which any arriving attack will have no effect, and then will resume normal operations.

a) Compute the fraction of time during which the node doesn't stream any data (both servers are inaoperational), and the average duration of a period of time during which data is streamed.

λ' effective attack $= 10\left(\frac{1}{9}\right) = \frac{10}{9}$ effective attacks per hour.

$T' = \frac{1}{\lambda'} = \frac{9}{10} \frac{h}{\text{attacks}} \cdot 60 = 54 \text{ min/eff. Att.}$

$P[1 \text{ serv. down}] = \frac{T}{\frac{1}{\lambda'} + T} = \frac{6\text{ min}}{54\text{ min} + 6\text{ min}} = 0.1$

$P(2 \text{ serv down}) = P(\text{system down}) = (0.1)^2 = 0.01.$

$E[\text{Node doesn't work}] = \frac{1}{2} T = 3 \text{ min}$

b) by considering an appropriate renewal cycle, compute the average duration of the time interval during which the node is able to stream data without interruption. (there is always at least one server working).

$$E[\text{cycle}] = \frac{T/2}{P[\text{system down}]} = \frac{3\text{ min}}{0.01} = 300$$

c) Compute the average total streaming rate of the node in Gbps

1 server = 1 Gbps (0.9) = 0.9 Gbps

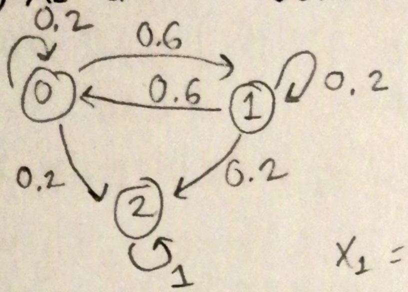
2 server = 1.8 Gbps

2 (0.9) = 1.8

E3 Consider a Markov chain X_n with the following transition matrix (states are numbered from 0 to 2):

$$P = \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0 & 1 \end{pmatrix}$$

a) Draw the transition diagram, and find the probability distribution of X_1, X_2 and X_{1000}, given $X_0 = 0$.



$$P = \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0 & 1 \end{pmatrix} \qquad P^2 = \begin{pmatrix} 0.4 & 0.24 & 0.36 \\ 0.24 & 0.4 & 0.36 \\ 0 & 0 & 1 \end{pmatrix}$$

$X_1 = (0.2, 0.6, 0.2)$ $\quad X_2 = (0.4, 0.24, 0.36)$.

2 is an absorbtion state, then $X_{1000} = (0, 0, 1)$

(b) Let $W_{ij}^{(n)} = E\left[\sum_{k=0}^{n-1} I\{X_k = j\} \mid X_0 = i\right]$ be the average number of visits to state j during the first n time slots, given that the chain starts in state i. Compute $\lim_{n\to\infty} W_{ij}^{(n)}$ for $j = 0, 1, 2$.

As 2 is an absorbing state, then $\lim_{n\to\infty} W_{02}^{(n)} = \infty$.

By simetry in the channel

$V_{00} = 1 + P_{00} V_{00} + P_{01} V_{10} = 1 + 0.2\, V_{00} + 0.6\, V_{10}$; $V_{00} = \frac{20}{7}$; $V_{11} = \frac{20}{7}$

$V_{10} = 0 + P_{10} V_{00} + P_{11} V_{10} = 0 + 0.6\, V_{00} + 0.2\, V_{10}$; $V_{10} = \frac{15}{7}$; $V_{01} = \frac{15}{7}$

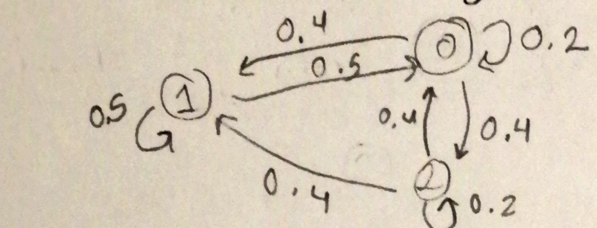
c) Compute the average duration of the transient evolution of the chain, i.e. the time index at which the chain is absorbed.

$V_1 = \frac{20}{7} + \frac{15}{7} = 5$. By simetry, $V_0 = 5$.

E1 Consider a Markov chain X_n with the following transition matrix (States are numbered from 0 to 2):

$$P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$$

a) Draw the transition diagram, and find the probability distribution of X_1, X_2 and X_{500}, given $X_0 = 0$.



$$P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0 \\ 0.4 & 0.4 & 0.2 \end{pmatrix} \qquad P^2 = \begin{pmatrix} 0.4 & 0.44 & 0.16 \\ 0.35 & 0.45 & 0.2 \\ 0.36 & 0.44 & 0.2 \end{pmatrix}$$

$X_1 = (0.2, 0.4, 0.4)$ $\quad X_2 = (0.4, 0.44, 0.16)$

$\pi_0 = 0.2\pi_0 + 0.5\pi_1 + 0.4\pi_2$ $\qquad \pi_0 = 0.3703$

$\pi_2 = 0.4\pi_0 + 0.2\pi_2$ $\qquad \pi_1 = 0.4444$

$\pi_0 + \pi_1 + \pi_2 = 1$ $\qquad \pi_2 = 0.1851$

b) Compute the average first passage times from states 0, 1 and 2 to state 2

$E[T \mid X_0 = 2] = \frac{1}{\pi_2} = 5.4$

$\nu_0 = 1 + \nu_0 P_{00} + \nu_1 P_{01} = 1 + \nu_0\, 0.2 + \nu_1\, 0.4$ $\quad \nu_0 = \frac{9}{2}$ $\nu_1 = \frac{1}{}$

$\nu_1 = 1 + \nu_0 P_{10} + \nu_1 P_{11} = 1 + \nu_0\, 0.5 + \nu_1\, 0.5$ $\quad (\frac{9}{2}, \frac{13}{2}, 5.$

c) Let $W_{ij}^{(n)} = E[\sum_{k=0}^{n-1} I\{X_k = j\} \mid X_0 = i]$ be the average number of visits to state j during the first n time slots, given that the chain starts in state i. Compute $W_{0j}^{(3)}$ and $W_{0j}^{(50000)}$ for $j = 0, 1, 2$.

$$W_{0j}^{(3)} = P_{0j}^{0} + P_{0j}^{(1)} + P_{0j}^{(2)} = \begin{cases} 1 + 0.2 + 0.4 = 1.6 & j=0 \\ 0 + 0.4 + 0.44 = 0.84 & j=1 \\ 0 + 0.4 + 0.16 = 0.56 & j=2 \end{cases}$$

$$W_{0j}^{(50000)} \simeq 5000\pi_j = \begin{bmatrix} 1852 & j=0 \\ 2222 & j=1 \\ 926 & j=2 \end{bmatrix}$$

Average recurrent time $m = \begin{bmatrix} 1/\pi_0 \\ 1/\pi_1 \\ 1/\pi_2 \end{bmatrix}$

June 24, 2022

$\lambda_1 = 1$ and $\lambda_2 = 1$

a) $P[X_1(1)=1 \mid X_1(1)+X_2(1)=2] = P[X_1(1)=1 \mid \underline{X}(1)=2] = \binom{2}{1}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^1\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{2-1}$ $= 1/2$

$P[X_1(1)+X_2(1)=2 \mid X_1(1)=1] = P[\underline{X}(1)=2 \mid X_1(1)=1] = P[X_2(1)=1] = \frac{e^{-\lambda_2}(\lambda_2)^1}{1!} = 0.3678$

b)

$P[X_1(1)=1 \mid X_2(2)+X_2(2)=4] = P[X_1(1)=1 \mid \underline{X}(2)=4]$

$= \binom{4}{1}\left(\frac{1\lambda_1}{(2)\lambda_1+\lambda_2}\right)^1\left(1-\frac{(1)\lambda_1}{(2)\lambda_1+\lambda_2}\right)^{4-1} = 0.4218$

$P[X_1(2)+X_2(2)=4 \mid X_1(1)=1)]$

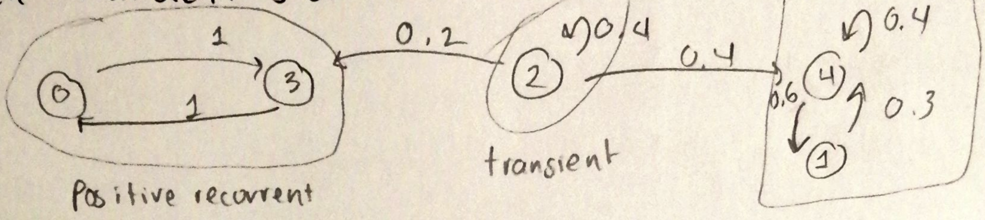
$P(\underline{X}(2)=4 \mid X_1(1)=1] = P[X_1(2)+X_2(2)-X_1(1)=3] =$

$= \frac{(2\lambda_1+2\lambda_2-\lambda_1)^3 e^{-(2\lambda_1+2\lambda_2-\lambda_1)}}{3!} = 0.2240$

E3 Consider a Markov chain with the following transition matrix (states are numbered from 0 to 4)

$$P \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0,7 & 0 & 0 & 0,3 \\ 0 & 0 & 0,4 & 0.2 & 0.4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0.4 \end{pmatrix}$$

a) Draw the transition diagram identify the classes, classify the states and compute the prob. absorption in all recurrent classes starting from each transient state.



positive recurrent aperiodic $d=1$

Positive recurrent aperiodic $d=2$

$P[\text{abs in } \{0,3\} \text{ starting from } \{2\}] = \frac{0.2}{0.4+0.2} = 0{,}33$ $P[\text{abs in } \{4,1\} \text{ from } 3\} = \frac{0,4}{0,4+0,2} = 0.66$

b) Compute $\lim_{n\to\infty} P^n$ and $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k$

$$\begin{cases} \pi_1 = 0,7 + 0,6\pi_4 \\ \pi_1 + \pi_4 = 1 \end{cases}$$

$\pi_1 = 2/3$

$\pi_4 = 1/3$

$$\lim_{n\to\infty} P^n = \begin{bmatrix} X & 0 & 0 & X & 0 \\ 0 & 2/3 & 0 & 0 & 1/3 \\ X & 2/3\ 2/3 & 0 & X & 2/3\ 1/3 \\ X & 0 & 0 & X & 0 \\ 0 & 2/3 & 0 & 0 & 1/3 \end{bmatrix}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{n=0}^{n-1} P^k = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 0 & 1/3 \\ 1/6 & 4/9 & 0 & 1/6 & 2/9 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 0 & 1/3 \end{bmatrix}$$

c) Compute the average recorrence time for all states, and the average first passage time from any state to state 4.

Average recorrence time

$m_1 = \frac{1}{\pi_1} = \frac{3}{2}$ $m_4 = \frac{1}{\pi_4} = 3$ Because they are positive recorrent.

$m_0 = m_3 = 2$ guaranteed to comeback in 2 steps.

$m_2 = +\infty$ for transient, because won't comeback; $m = (2, \frac{3}{2}, +\infty, 2, 3)$

Average first passage time from any to 4.

$V_{04} = V_{2,3} = \infty$ because recorrent from different classes

$V_{24} = +\infty$ because transient.

$V_{44} = \frac{1}{\pi_4} = 3$

$V_{14} = 1 + P_{11} \cdot V_{14}$; $V_{14} = \frac{1}{1-P_{11}} = \frac{1}{0.3} = 3.33.$

14 July 2006

E3 Consider an exhibition where visitors arrive according to a Poisson process with rate $\lambda = 10$ customers per hour. Each visitor spends a time uniformly distributed between 20 and 30 min, and then leaves. The room in which the exhibition is shown is large enough to ensure there is never a need to block customers at the entrance due to too many people inside. The exibition is open 8AM to 6PM.

a) Compute the probability that fewer than 3 visitors arrive during the firs half hour.

$$P[X(0.5) < 3] = P[X(0.5) = 0] + P[X(0.5) = 1] + P[X(0.5) = 2]$$

$$= \frac{e^{-5}(5)^0}{0!} + \frac{e^{-5}(5)^1}{1!} + \frac{e^{-5}(5)^2}{2!} \approx 0.12465$$

b) Compute the prob. that at 8:15 AM there is only one visitor in the room.

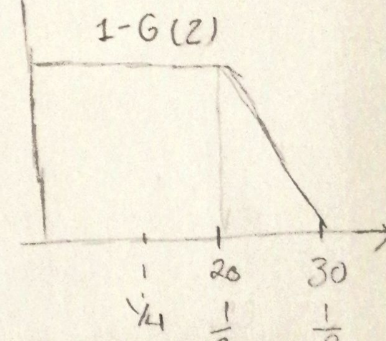
$$\Lambda = \lambda \int_0^t [1 - G(z)]dz = 10 \int_0^{.5} [1 - G(z)]dz = 10\left(\frac{5}{12}\right) = \frac{25}{6}$$

Total

$$\frac{\frac{1}{2} - \frac{1}{3}}{2} = \frac{1}{12}$$

$$\frac{1}{3} + \frac{1}{12} = \frac{5}{12}$$

1-G(z)



1)
$$\Lambda = \lambda \int_0^{1/4} [1 - G(z)]dz = 10\,(0.25) = 2.5$$

$$P[M(0.25) = 1] = \frac{e^{-\Lambda}\Lambda^1}{1!} = e^{-2.5}(2.5) = 0.2052$$

The integral until is asking 15 min

c) Compute the probability that at closing time (6 PM) the room is empty.

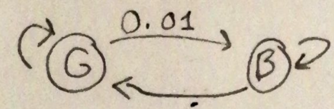
Total integral.

$$P\left[\begin{smallmatrix}\text{empty}\\ \text{at 6PM}\end{smallmatrix}\right] = e^{-\Lambda} = e^{-25/6} = 0.0155$$

— July 13 2021 —

E1 Consider a Go-Back-N protocol over a two-state Markov channel, where the average number of consecutive good slots is 100 and the average number of consecutive bad slots is 100/9. The packet error probability is 1 for a bad slot and 0 for a good slot respectively. The round-trip time is $m = 2$ slots, i.e. packet that is erroneous in slot t will be retransmitted in slot $t+2$.

(a) Compute the throughput that could be obtained if packets were directly transmitted over the channel without using any protocol.



Average good slots: 100.

$E(G) = 100$

↳ Prob. of leaving the good state is $P_{01} = \frac{1}{100} = 0.01$

0 → Good state

1 → Bad state

Average of bad slots: $100/9$

$$C = \begin{array}{cc} P_{00} & P_{01} \end{array}$$
$$C = \begin{pmatrix} 0.99 & 0.01 \\ 0.09 & 0.91 \end{pmatrix}$$

$P_{10} = \frac{9}{100}$

In this case $T: \dfrac{P_{10}}{P_{10} + P_{01}} = \dfrac{0.09}{0.09 + 0.01} = 0.9$

b) Compute the throughput of the Go-Back-N Protocol for an error-free feedback channel.

$$P^2 = \begin{pmatrix} 0.981 & 0.019 \\ 0.171 & 0.829 \end{pmatrix} \qquad C = \begin{bmatrix} P_{00} & P_{01} \\ P_{10}^{(m)} & P_{11}^{(m)} \end{bmatrix} \qquad C^2 = \begin{bmatrix} 0.99 & 0.01 \\ 0.171 & 0.829 \end{bmatrix}$$

$$T = \frac{P_{10}^{(m)}}{P_{10}^{(m)} + mP_{01}} = \frac{0.171}{0.171 + (2)(0.01)} = 0.8952$$

c) Compute the throughput of the Go-Back-N protocol for a feedback channel to iid errors wit prob. $0.1 = \delta$

$$T = \frac{(1-\delta)\, P_{10}^{(m)}}{(1-\delta) P_{10}^{(m)} + m\left((1-\delta)P_{01} + \delta P_{01}^{(m)} + \delta P_{10}^{(m)}\right)}$$

$$= \frac{(1-.1)(.171)}{(1-.1).171 + (2 \times ((.9 \times 0.01) + (.1 \times 0.019) + (.1 \times .171)))}$$

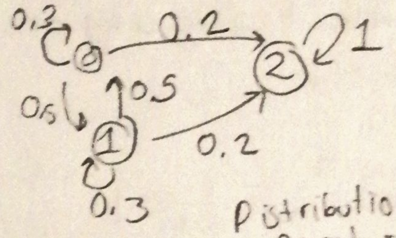
$$= 0.7332$$

E2 Consider a Markov Chain X_n, with the following transition matrix (states are numbered from 0 to 2)

If this was a positive recurrent chain, the limit(2) will be infinit.

$$P = \begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Draw the transition diagram, and find the probability distribution of X_1, X_2, and X_{1000}, given $X_0 = 0$.



$$P^2 = \begin{pmatrix} 0.34 & 0.3 & 0.36 \\ 0.3 & 0.34 & 0.36 \\ 0 & 0 & 1 \end{pmatrix}$$

Distribution of X_1 is the first row of P

$X_1 = (0.3, 0.5, 0.2)$

Distribution of X_2 is the first row of P_2, $X_2 = (0.34, 0.3, 0.36)$

Considering a long term run and 2 is an absorbing state $P_{1000} = (0, 0, 1)$

(b) Let $W_{ij}^{(n)} = E[\sum_{k=0}^{n-1} I\{X_k = j\} \mid X_0 = j]$ be the average number of visits to state j during the firs n time slots, given that the Chain starts in state i. Compute $\lim_{n\to\infty} W_{0j}^{(n)}$ for $j = 0, 1, 2$.

Transient part $\{0, 1\}$ Absorbing $\{2\}$

So we can say $\lim_{n\to\infty} W_{02}^{(n)} = \infty$ because that's a recorrent state.

d_{ij} is 1 if $i = j$ and $= 0$ otherwise. Because then I already have a visit

$V_{ij} = \lim_{n\to\infty} W_{ij}^{(n)}$ $i, j \neq 2$.

$V_{ij} = d_{ij} + P_{i0} V_{0j} + P_{i1} V_{1j}$ $i, j = 0, 1$

if $j = 0$, d_{i0} ... one

$V_{00} = 1 + P_{00} V_{00} + P_{01} V_{10} = 1 + 0.2 V_{00} + 0.6 V_{10}$

$V_{10} = 0 + P_{10} V_{00} + P_{11} V_{10} = 0.6 V_{00} + 0.2 V_{10}$.

$V_{00} = 20/7$

$V_{10} = 15/7$

I can do the same for state 1, V_{11}, V_{01}. or can observed that 1, 0 are symetric so $V_{00} = V_{11}$ $V_{10} = V_{10}$

c) Compute the average duration of the transient evolution of the chain, i.e. the time index at which the chain is absorbed.

$V_i = E[\text{absortion time} \mid X_0 = i] = E\left[\begin{matrix}\text{visits} \\ \text{to 0 or 1}\end{matrix} \mid X_0 = i\right]$

↳ length of transient phase

↳ Time the chain spents in 0 or 1 before 2.

$$V_{00} + V_{10} = \frac{20 + 15}{7}$$

$$\| $$

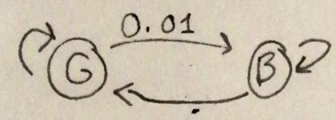
$$5$$

And just because the transient part is symetry, it doesn't depend on the initial state.

- July 13 2021 -

E1 Consider a Go-Back-N protocol over a two-state Markov channel, where the average number of consecutive good slots is 100 and the average number of consecutive bad slots is 100/9. The packet error probability is 1 for a bad slot and 0 for a good slot respectively. The round-trip time is $m = 2$ slots, i.e. packet that is erroneous in slot t will be retransmitted in slot $t+2$.

(a) Compute the throughput that could be obtained if packets were directly transmitted over the channel without using any protocol.



Average good slots : 100.

$E(G) = 100$

↳ Prob. of leaving the good state is $P_{01} = \frac{1}{100} = 0.01$

$0 \rightarrow$ Good state

$1 \rightarrow$ Bad state

Average of bad slots : $100/9$

$$\begin{array}{ccc} & P_{00} & P_{01} \\ C = & \begin{pmatrix} 0.99 & 0.01 \\ 0.09 & 0.91 \end{pmatrix} \end{array}$$

$P_{10} = \frac{9}{100}$

In this case T: $\frac{P_{10}}{P_{10} + P_{01}} = \frac{0.09}{0.09 + 0.01} = 0.9$

b) Compute the throughput of the Go-Back-N Protocol for an error-free feedback channel.

$$P^2 = \begin{pmatrix} 0.981 & 0.019 \\ 0.171 & 0.829 \end{pmatrix} \quad C = \begin{bmatrix} P_{00} & P_{01} \\ P_{10}^{(m)} & P_{11}^{(m)} \end{bmatrix} \quad C^2 = \begin{bmatrix} 0.99 & 0.01 \\ 0.171 & 0.829 \end{bmatrix}$$

$$T = \frac{P_{10}^{(m)}}{P_{10}^{(m)} + mP_{01}} = \frac{0.171}{0.171 + (2)(0.01)} = 0.8952$$

c) Compute the throughput of the Go-Back-N protocol for a feedback channel to iid errors wit prob. $0.1 = \delta$

$$T = \frac{(1-\delta)\ P_{10}^{(m)}}{(1-\delta)P_{10}^{(m)} + m((1-\delta)P_{01} + \delta P_{01}^{(m)} + \delta P_{10}^{(m)})}$$

$$= \frac{(1-.1)(.171)}{(1-.1).171 + (2 \times ((.9 \times 0.01) + (.1 \times 0.019) + (.1 \times .171)))}$$

$= 0.7332$

Matrix Markov chain

Go Back-N

λ, no queue, time o P.

Prob eq. (2) Processes sums.

1 Matrix Markov chain

2 Identical machines with period of time

3 Semimarkov, Network 1 node
sleep, awake, Reception

4 system with poisson request, service period and no queue

5 Network with one node, 2 outgoing links.
Each one with queue, restriction on TX.

6 1 node wit 2 <u>incoming</u> links.
indepent poisson process.
2 independent Poisson process in equations.

7 • 2 independent
• Markov channel with good state and

8 bad state. GO-BACK-N protocol

9 Matrix Markov chain

10 • 1 poison proces of arrival, queue, restriction to TX.
• Poisson process with rate λ,
No queue, exp. service. compute ~~[illegible]~~ people in the system at time t_1, t_2, t_3.

11 Go Back-N Protocol

12 Web server
uniformly distributed
steady condition.

13 CSMA
Poisson process rate.
Delay, throughput.

14 Matrix Markov Chain.

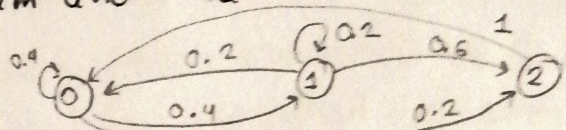
15 Network node normal
alarm condition - capacity limited.

Exam 09/07/2007

E1 . Consider a Markov chain X_n with the following transition matrix (states are numbered from 0 to 2, and initial state $X_0 = 0$.

$$P = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.6 \\ 1 & 0 & 0 \end{pmatrix}$$

a) Draw the transition diagram and find the probability distributions of X_1, X_2 and X_{500}



$P^2 = \Big(0.44 \;\; 0.24 \;\; 0.32$
0.12

$P(X_1) = (0.4, 0.4, 0.2)$
$P(X_2) = (0.44, 0.24, 0.32)$
$P(X_{500}) =$ We can assome the chain is in long run behavior, so we can use steady state distribution. Then compute $\begin{cases} \pi = \pi \cdot P \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$

$$\begin{cases} \pi_0 \cdot (0.4 = P_{00}) + \pi_1 (0.2 = P_{10}) + \pi_2 \cdot (1 = P_{20}) = \pi_0 \\ \pi_0 (0.4 = P_{01}) + \pi_1 (0.2 = P_{11}) + \pi_2 \cdot (0 = P_{21}) = \pi_1 \\ \pi_0 (0.2 = P_{02}) + \pi_1 (0.6 = P_{12}) + \pi_2 \cdot (0 = P_{22}) \pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$\pi_0 = 0.5 \quad \pi_1 = 0.25 \quad \pi_2 = 0.25 \quad \rightarrow P(X_{500}) = (0.5, 0.25, 0.25)$

b) Compute the average first passage times from states 0, 1 and 2 to state 2

$$\nu_0 = 1 + P_{00}\nu_0 + P_{01}\nu_1 = 1 + 0.4\,\nu_0 + 0.4\,\nu_1$$
$$\nu_1 = 1 + P_{10}\nu_0 + P_{11}\nu_1 = 1 + 0.2\,\nu_0 + 0.2\,\nu_1$$

$\nu_2 = \frac{1}{\pi_2} = \frac{1}{1/4} = 4 \qquad \nu_2 = 1 + P_{20} \cdot \nu_0$
$4 = 1 + P_{20} \cdot \nu_0 \rightarrow \nu_0 = 3$

$3 = 1 + 0.4(3) + 0.4\,\nu_1 \qquad \nu_0 = 3 \quad \nu_1 = 2 \quad \nu_2 = 4$

c) Compute $P[X_1 = 1, X_3 = 1 \mid X_2 = 1]$ and $P[X_2 = 1 \mid X_1 = 1, X_3 = 1]$

$$P[X_1 = 1, X_3 = 1 \mid X_2 = 1] = \frac{P[X_1 = 1, X_2 = 1, X_3 = 1 \mid X_0 = 0]}{P[X_2 = 1 \mid X_0 = 0]} = \frac{P_{01} \cdot P_{11} \cdot P_{11}}{P_{01}^{(2)}}$$
$$= \frac{(0.4)(0.2)(0.2)}{0.24} = 0.67$$

$$P[X_2 = 1 \mid X_1 = 1, X_3 = 1] = \frac{P[X_3 = 1, X_2 = 1, X_1 = 1 \mid X_0 = 0]}{P[X_1 = 1, X_3 = 1 \mid X_0 = 0]} = \frac{P_{01} \cdot P_{11} \cdot P_{11}}{P_{01}\;P_{11}^{(2)}}$$
$$= \frac{(0.4)(0.2)(0.2)}{(0.4)(0.12)} = 0.33$$

c) Extra question in other exams

Let $W_{ij}^{(n)} = E[\sum_{k=0}^{n-1} I\{X_k = j\} | X_0 = i]$ be the average num of visits to state j during the first n time slots, given that the chain starts in state i. Compute $W_{0j}^{(3)}$ and $W_{0j}^{(5000)}$ for $j = 0, 1, 2$.

$$W_{ij}^{(n)} = E[\sum_{k=0}^{n-1} \chi\{X_k = j\} | X_0 = i] = \sum_{k=0}^{n-1} P_{ij}^{(k)}$$

In 3:

$$= P_{ij}^{(0)} + P_{ij}^{(1)} + P_{ij}^{(2)} \rightarrow W_{00}^3 = P_{00}^0 + P_{00}^1 + P_{00}^2$$

$$W_{01}^3 = P_{01}^0 + P_{01}^1 + P_{01}^2$$

$$W_{02}^3 = P_{02}^0 + P_{02}^1 + P_{02}^2$$

In $W_{0j}^{(500)}$ we can use steady state distribution multiplied by 5000

$W_{0j}^{(5000)} \simeq 5000\pi_j$

$W_{00}^{500} \simeq 5000\pi_0$

$W_{01}^{500} \simeq 6000\pi_1$

$\pi_{02}^{500} \simeq 5000\pi_2$

E2 Consider a factory with 2 identical machines. Each machine alternates periods of time in which it is working or not working, of exp. duration with mean $1/\alpha = 27$ days (working) and $1/\beta = 1/(9\alpha)$ (not working). $\boxed{\text{Each machine, whose operation is independent of the other, can produce 12 pieces per hour when is working}}$

(a) Compute the fraction of time in which there is no production (i.e. both machines are not working).

| 1/α | 1/β | 1/α | 1/β | 1/α | 1/β |

$$P[\text{Machine not work}] = \frac{1/\beta}{1/\alpha + 1/\beta} = 0.1$$

$$P[\text{No production}] = (0.1)^2 = \underline{0.01}$$

b) Compute the average number of pieces per hour produced by the factory.

$$E[\text{NUM PROD}] = \left(12 \frac{\text{pieces}}{\text{hour}} \cdot (1 - 0.01)\right] \cdot 2 \text{ machines} = 21.6 \frac{\text{pieces}}{\text{hour}}$$

c) Compute the average number of pieces per hour produced by the factory if the number of pieces produced per hour is 12 when only one machine is working and 30 when they are both working

$$E[\text{Produc}] = \underbrace{0.81}_{\text{both}} \cdot 30 + \underbrace{0.18}_{\text{one}} \cdot 12 + \underbrace{0.01}_{\text{no working even 1}} \cdot 0 = 26.46$$

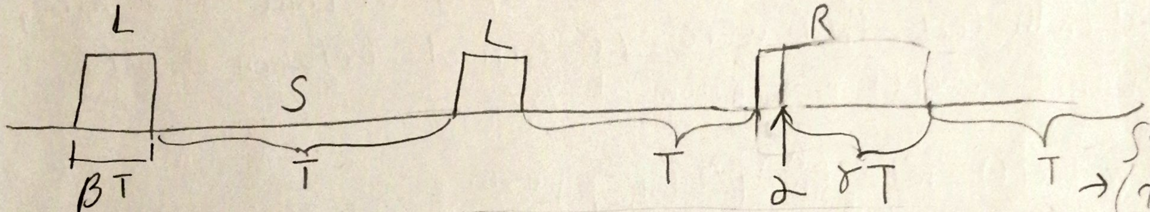
$0.81 = 0.9 \cdot 0.9$, $0.9 = 1 - 0.01$

$0.18 = 1 - 0.01 - 0.81$

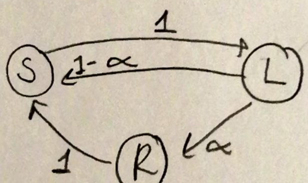
09/07/2007

E3 Consider a network node that works as follows, If there is no traffic, the node alternates between a sleep state for an exponential duration of average T an an awake state for a fixed duration βT. When in the awake state, the node can receive it entirely (even if requires it to remain awake for a total time longer than βT) and goes to sleep immediately after. If instead while the node is awake there is no transmisson, the node goes back to time at which such transmi. starts is uniformly distributed in $[0, \beta T]$ and the average packet transmission time is γT. Develop and solve a semi-Markov model for the node, and in particular:

a) Consider the 3 states sleep(S), listening (L) and receiving (R), determinate the matrix of the trans. prob. of the embeded Markov Chain and draw its transition diagram.



$$\begin{cases} \Pi_L(1+\alpha)+\Pi_S=1 \\ \to \Pi_L(2+\alpha)=1 \end{cases}$$



$$P=\begin{array}{c|ccc} & S & L & R \\ \hline S & 0 & 1 & 0 \\ L & 1-\alpha & 0 & \alpha \\ R & 1 & 0 & 0 \end{array}$$

$$\begin{cases} \Pi_L=\Pi_S \\ \Pi_R=\alpha\Pi_S \\ \Pi_S+\Pi_L+\Pi_R=1 \end{cases}$$

$\Pi_L=\Pi_S=\frac{1}{2+\alpha}$

$\Pi_R=\frac{\alpha}{2+\alpha}$

b) Determine the matrix of the average times associated to each transmission, T, and the average times associated to the visits to each state μ_S, μ_L, μ_R

$$T=\begin{array}{c|ccc} & S & L & R \\ \hline S & - & T & - \\ L & \beta T & - & \beta T/2 \\ R & \gamma T & - & - \end{array}$$ uniforme

$\mu_i=\sum_j P_{ij}T_{ij} \to \mu_S=T$, $\mu_R=\gamma T$

$\mu_L=(1-\alpha)\beta T+\alpha\frac{\beta T}{2}=\left(1-\frac{\alpha}{2}\right)\beta T$

c) Find an expression for the fraction of time the node spends in each of the 3 states and find its nomerical value for $\alpha=0.5$, $\beta=0.1$, $\gamma=0.2$

Fraction of time given by $P_i=\frac{\mu_i\Pi_i}{\sum_j\mu_j\Pi_j}=\frac{(2+\alpha)(\Pi_i\mu_i)}{T(1+\beta-\frac{\alpha\beta}{2}+\alpha\gamma)}$

$\sum_j\mu_j\Pi_j=\underbrace{\left(\frac{1}{2+\alpha}\right)(T)}_{S}+\underbrace{\left(\frac{1}{2+\alpha}\right)\left(1-\frac{\alpha}{2}\right)\beta T}_{L}+\underbrace{\frac{\alpha}{2+\alpha}(\gamma T)}_{R}$

$=\frac{2+\alpha}{T(1+\beta-\frac{\alpha\beta}{2}+\alpha\gamma)}$

$P_S=\frac{T}{2+\alpha}\frac{2+\alpha}{T(1+\beta-\frac{\alpha\beta}{2}+\alpha\gamma)}=0.851$

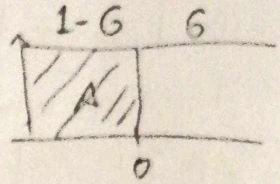
$P_L=\frac{(1-\frac{\alpha}{2})\beta T}{2+\alpha}\frac{2+\alpha}{T(1+\beta-\frac{\alpha\beta}{2}+\alpha\gamma)}=0.064$

$P_R=\frac{\gamma T\alpha}{2+\alpha}\frac{2+\alpha}{T(1+\beta-\frac{\alpha\beta}{2}+\alpha\gamma)}=0.085$

E4 Consider a system which receives service requests according to a Poisson process of rate $\lambda = 20$ requests per hour. Each request remains in the system for a service time equal to 6 minutes, and there is no limit to the number of requests simultaneously in service. Assume that the system started its operation at time $t=0$, 4 (4)

a) Compute the probability that the system is empty at time $t = 30$ min.

$\lambda = 20$ req/h $= \frac{1}{3}$ req/min



The service time Y is deterministic, $G_Y(x) = \begin{cases} 1 & x \geq 6 \\ 0 & x \leq 6 \end{cases}$

If $M(t)$ is the r.v. that counts the users in the system at time t.

$Pr[M(0.5)=0] = \frac{e^{-\Lambda}(\Lambda)^0}{0!}$ where $\Lambda = \lambda \int_0^{0.5} [1 - G(x)]\, dx = \frac{\lambda}{10} = 2$

$\Lambda = \lambda E[Y] = \frac{20}{60} \cdot 6 = 2 \quad \rightarrow Pr[M(0.5)=0] = e^{-2} = 0.135$

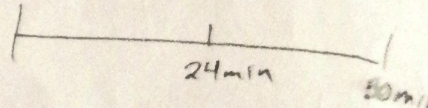
b) Compute the probability that the system is empty at time $t = 30$ min, conditioned on the fact there were 10 arrival between 0 and t.

$M(t)$ conditioned on $X(t)$ is a Binomial r.v.

$Pr[M(t) = m \mid X(t) = n] = \binom{n}{m} p^m (1-p)^{n-m}$

$Pr[M(0.5) = 0 \mid X(0.5) = 10] = \binom{10}{0} (0.2)^0 (0.8)^{10} = 0.10737$

where $p = \frac{1}{t} \int_0^t [1 - G_Y(z)]\, dz = 0.2$

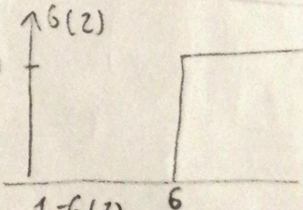


$1 - p = \frac{1}{30} \int_0^{24} 1\, dx = \frac{24}{30}$

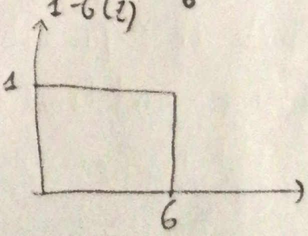
---

a) $P(M(t) = 0] = e^{-\lambda p t} = e^{-\lambda \int_0^t (1-G(z))dz}$

$= e^{-1/3 \cdot 6} = e^{-2} = 0.135$



b) $P[M(t) = 0 \mid X(t) = 10] = \binom{10}{0} p^0 (1-p)^{10} = \left(\frac{4}{5}\right)^{10} = 0.107$



$p = \frac{1}{t} \int^t [1 - G(z)]\, dz = \frac{1}{30}(6) = \frac{1}{5}$

$= \frac{1}{30} \int_0^{30} (1 - G(z))\, dz = \frac{6}{30} = \frac{1}{5}$

24/09/2007

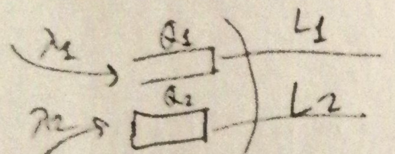
[5] ①

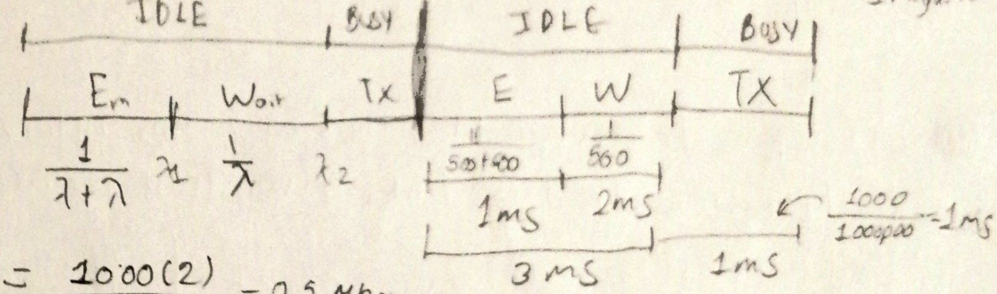
E1*

E2* Consider a network node with 2 outgoing links L1 and L2. The two links have a capacity of 1 Mbps, and are fed by 2 seperate queus Q1 and Q2 which can contain rates $\lambda_1 = \lambda_2 = \lambda = 500$ pak/s, and that flow λ_i is releated only to link L1 (and therefore queue Q1), $i = 1, 2$. Two packets are transmited simultaneously when both queues are full 1 packet. when there is a packet in only one of the queues, the node waits until the other queue also receives a packet, and only then does it send both packets, each on its own link. (Note that a time interval in which both queues are empty is followed by one in whic one queue is empty and the other is full, which is turn followed by another, in which both packets are being transmitted.) Futhermore, assume that when Qi is full (i.e. wheen there is a packet waiting to be transmited) any arrival of flow λ_i is rejected. All packets are 1000 bits long.

a) Compute the througput of the node in terms of total bps transmitted

1 Mega = 10^6





a) $\text{Througput} = \dfrac{\text{bixel TX}}{\text{Tempo cycle}} = \dfrac{1000(2)}{4\ ms} = 0.5\ Mbps$

b) ~~[illegible]~~ Compute the fraction of the total traffic that is rejected.

Use PASTA

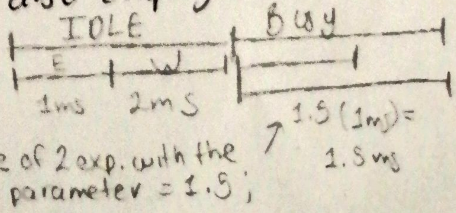
System empty — it depends which type arrive before

$\text{REJECTION PROB} = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot 1 = \frac{1}{2}$

Prob of rejection in that time

Fraction the time of the proces — proportional to the intervals

c) Repeat the previous calculations if the length of the packets, instead of being fixed to 1000 bits has exp. duration with mean 1000 bits. In this case, assume that the 2 queues are considered empty (and therefore the system again can accept incoming packets) only when both packets have completely tx. (I.e the queue that had shorter packet will be able to acept packets only when the other queue is also empty.



$\text{throughput} = \dfrac{2000}{4.5\ ms} = 0.44\ Mbps$

$\text{Prob of rejection} = \left(\frac{1}{4.5}\right) \cdot 0 + \left(\frac{2}{4.5}\right) \cdot \frac{1}{2} + \left(\frac{1.5}{4.5}\right) 1 = \frac{5}{9} = 0.556$

Average of 2 exp. with the same parameter = 1.5

E3. Consider a network node with 2 incoming links, through which pakets are received according to 2 independent Poisson process with rates $\lambda_1 = \lambda_2 = 500$ packets per second.

a) Compute the probability that in a 3-ms interval the node receives 2 packets from the first ~~lik~~ link and one from the second.

$$\text{a) } P[X_1(0{,}00\underset{t}{3}) = 2] = \frac{e^{\overset{(-\lambda)}{-500}\,\overset{t}{(0.003)}}\,(\overset{(\lambda)}{500}\cdot\overset{(t)}{0.003})^{\overset{\leftarrow \#p}{2}}}{2! \leftarrow \#p} = 0.25$$

$P[X_2(0.003) = 1] = 0.335$ $\quad P_T = (0.335)(0.25) = 0.0837$

(b) Compute the probability that in a 3ms interval the node receives 3 packets ~~[illegible]~~ in total.

$$\lambda_{TOT} = \lambda_1 + \lambda_2 = 1000$$

$$P[X_{TOT}(0.003) = 3] = \frac{e^{\overset{-\lambda}{-(1000)}\cdot\overset{t}{(0.003)}}\,(\overset{\lambda}{1000}\cdot\overset{t}{0.003})^{3}}{3!} = 0.224$$

c) Compute the probability that in a 3ms interval the nodes receives 2 packets from the first link, given that it received 3 packets in total

$$P[X_1(0{,}000) = 2 \mid X(0{,}003) = 3] = \binom{3}{2}\left(\frac{2\lambda_1}{3(\lambda_1+\lambda_2)}\right)^2\left(1 - \frac{3\lambda_1}{3(\lambda_1+\lambda_2)}\right)^{3-2}$$