Network Modeling
Solutions for 12/12/2006

Problem 1

States $\{0, 4\}$ belong to an absorbing class of period $d = 2$.
State $\{2\}$ is transient.
States $\{1, 3, 5\}$ belong to a positive recurrent aperiodic class ($d = 1$).
Since absorbing class has a "ping-pong" behavior, the limiting distribution is not defined for the whole chain.
Obviously $\pi_2 = 0$ since state is transient.
We still can compute the π_i for the recurrent class by solving the usual system $\pi = \pi P$.
Notice that the submatrix induced by class $\{1, 3, 5\}$ is doubly stochastic, meaning that each row and column sums to one. In this case it is immediate that $\pi_1 = \pi_3 = \pi_5 = \frac{1}{3}$.
We have $\mathrm{P}\left[\text{absorption in } \{0, 4\} | \text{start in } 2\right] = \dfrac{0.2}{0.2 + 0.3} = \dfrac{2}{5}$ and
$\mathrm{P}\left[\text{absorption in } \{1, 3, 5\} | \text{start in } 2\right] = \dfrac{0.3}{0.2 + 0.3} = \dfrac{3}{5}$.

We obtain $$\lim_{n\to\infty} P^n = \begin{bmatrix} X & 0 & 0 & 0 & X & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ X & \frac{3}{5}\frac{1}{3} & 0 & \frac{3}{5}\frac{1}{3} & X & \frac{3}{5}\frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ X & 0 & 0 & 0 & X & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

Instead what always exists is $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} P^k = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{5}\frac{1}{2} & \frac{3}{5}\frac{1}{3} & 0 & \frac{3}{5}\frac{1}{3} & \frac{2}{5}\frac{1}{2} & \frac{3}{5}\frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

Remember that this quantity is the average time spent in a state: it is the probability of being absorbed in that class times the average time spent in that specific state.
For the last point we use Bayes rule:
$$\mathrm{P}\left[X_4 = 5, X_2 = 3 | X_3 = 1, X_1 = 3\right] = \frac{\mathrm{P}\left[X_4 = 5, X_3 = 1, X_2 = 3 | X_1 = 3\right]}{\mathrm{P}\left[X_3 = 1 | X_1 = 3\right]} = \frac{p_{33}p_{31}p_{15}}{p_{31}^{(2)}}$$

Problem 2

This system can be modeled as a semi-Markov process, so we need to specify a transition matrix P and a time matrix T.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ p_{AF} & 0 & p_{AG} \\ 1 & 0 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} - & 99T & - \\ T & - & \Omega \\ 20T & - & - \end{bmatrix}$$

$p_{AF} = \mathrm{P}\left[\xi\left(\frac{1}{2T}\right) > T\right] = e^{-\dfrac{T}{2T}} = e^{-\dfrac{1}{2}} = 0.6065.$
$p_{AG} = 1 - p_{AF} = 0.3935.$

$F(t) = \mathrm{P}\left[\xi \leq t | \xi < T\right] = \dfrac{\mathrm{P}\left[\xi \leq t, \xi < T\right]}{\mathrm{P}\left[\xi < T\right]} = \begin{cases} \dfrac{\mathrm{P}\left[\xi \leq t\right]}{\mathrm{P}\left[\xi < T\right]} & t < T \\ 1 & t \geq T \end{cases}$

$\Omega = E\left[\xi | \xi < T\right] = \int\limits_0^\infty \left(1 - F(t)\right) dt = \int\limits_0^T \left(1 - \dfrac{1 - e^{-\frac{t}{2T}}}{1 - e^{-\frac{T}{2T}}}\right) dt = \int\limits_0^\infty \dfrac{1 - e^{-\frac{1}{2}} - 1 + e^{-\frac{t}{2T}}}{1 - e^{-\frac{1}{2}}} dt = \dfrac{2T - 3Te^{-\frac{1}{2}}}{1 - e^{-\frac{1}{2}}}$

Metric for the time is $\mathbf{T} = \begin{bmatrix} 99T \\ Te^{-\frac{1}{2}} + 2T - 3Te^{-\frac{1}{2}} \\ 20T \end{bmatrix} = \begin{bmatrix} 99T \\ 2T\left(1 - e^{-\frac{1}{2}}\right) \\ 20T \end{bmatrix} = \begin{bmatrix} \mu_F \\ \mu_A \\ \mu_G \end{bmatrix}.$

Stationary distribution is given by $\begin{cases} \pi_A = \pi_F \\ \pi_G = \left(1 - e^{-\frac{1}{2}}\right)\pi_A \\ \pi_A + \pi_F + \pi_G = 1 \end{cases} \Rightarrow \begin{cases} \pi_F = \dfrac{1}{3 - e^{-\frac{1}{2}}} \\ \pi_A = \dfrac{1}{3 - e^{-\frac{1}{2}}} \\ \pi_G + \dfrac{1 - e^{-\frac{1}{2}}}{3 - e^{-\frac{1}{2}}} \end{cases}$

Then the fraction of time the system spends in state i is $P_i = \dfrac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}$.

For the non-working state it is $P_G = \dfrac{\left(1 - e^{-\frac{1}{2}}\right) 20T}{99T + 2T\left(1 - e^{-\frac{1}{2}}\right) + \left(1 - e^{-\frac{1}{2}}\right) 20T} = 0.073.$

The average time between two subsequent substitutions is $E\left[\text{cycle time}\right] = \dfrac{E\left[G\right]}{P_G}$.

Now the reward vector is $\mathbf{R} = \begin{bmatrix} 99T \\ 2T\left(1 - e^{-\frac{1}{2}}\right)\frac{1}{4} \\ 0 \end{bmatrix}.$

Throughput is finally computed as $\mathcal{T} = \dfrac{\sum_i R_i \pi_i}{\sum_i T_i \pi_i}$.

Problem 3

Let $X(t)$ be the sum of the two Poisson processes.

$\mathrm{P}\left[X_1(3) = 1 | X(3) = 3\right] = \binom{3}{1} \left(\frac{3\lambda_1}{3(\lambda_1+\lambda_2)}\right)^1 \left(1 - \frac{3\lambda_1}{3(\lambda_1+\lambda_2)}\right)^{3-1}.$

$\mathrm{P}\left[X(3) = 3 | X_1(3) = 1\right] = \mathrm{P}\left[X_2(3) = 2\right] = \frac{e^{-3\lambda_2}(3\lambda_2)^2}{2!}.$

$\mathrm{P}\left[X_1(2) = 1 | X_1(3) = 3\right] = \frac{\mathrm{P}\left[X_1(2) = 1, X_2(3) = 3\right]}{\mathrm{P}\left[X_1(3) = 3\right]} = \mathrm{P}\left[X_1(3) = 3 | X_1(2) = 1\right] \frac{\mathrm{P}\left[X_1(2) = 1\right]}{\mathrm{P}\left[X_1(3) = 3\right]}$

$\mathrm{P}\left[X_1(3) = 3 | X_1(2) = 1\right] = \mathrm{P}\left[X_1(3) - X_1(2) = 2\right] = \frac{e^{3\lambda_1 - 2\lambda_1}(3\lambda_1 - 2\lambda_1)^2}{2!}.$

Problem 4

Transition matrix is $P = \begin{bmatrix} 0.98 & 0.02 \\ 0.1 & 0.9 \end{bmatrix}.$

For the first question, we introduce reward and time vector: $\mathbf{R} = \begin{bmatrix} R_G \\ R_B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} T_G \\ T_B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

Then we simply compute $\mathcal{T} = \frac{\sum_i \pi_i R_i}{\sum_i \pi_i T_i} = \frac{\pi_G}{\pi_G + \pi_B} = 0.8333.$

For the second question we need to introduce the usual protocol matrix $C = \begin{bmatrix} p_{00} & p_{01} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix}$ and the new time vector $\mathbf{T} = \begin{bmatrix} T_G \\ T_B \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}.$

Then computation is straightforward: $\mathcal{T} = \frac{p_{10}^{(m)}}{p_{10}^{(m)} + m p_{01}}$, where $m = 2$.

We do not need to compute all P^2 but only $p_{10}^{(2)} = p_{10}p_{00} + p_{11}p_{10} = 0.188$. Then $T = 0.8245$.

For the last question we have $\mathcal{T} = \frac{(1-\delta)p_{10}^{(m)}}{(1-\delta)p_{10}^{(m)} + m\left((1-\delta)p_{01} + \delta p_{01}^{(m)} + \delta p_{10}^{(m)}\right)}$, where $\delta = 0.1$.