Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2007/2008
prova scritta – 14 luglio 2008 – parte A (90 minuti)

E1 Si consideri un server web a cui arrivano richieste di download secondo un processo di Poisson di intensita' $\lambda = 5$ richieste al secondo. Ognuna di esse, dopo un tempo di elaborazione uniformemente distribuito fra 10 e 30 ms, da' luogo al trasferimento di un file la cui dimensione e' esponenziale di media 1 MByte. Una richiesta si dice "attiva" da quando arriva fino a quando il trasferimento del file corrispondente e' terminato. Si supponga che la capacita' del server in termini di numero di richieste simultanee che puo' elaborare sia infinita, e che si voglia trasferire ciascun file a 100 Mbit/s, indipendentemente dal numero di file che vengono contemporaneamente trasferiti. Si supponga che il server venga acceso al tempo $t = 0$, e sia $X(t)$ il numero di richieste attive nel sistema al tempo t.

(a) Si calcoli la probabilita' che, dato che sono arrivate 5 richieste nell'intervallo da 0 a 1 s, almeno due di queste siano arrivate entro $t = 0.5$ s.

(b) Si dimensioni la capacita' del link di uscita dal server, in modo che la probabilita' che il numero di file da trasferire ecceda tale capacita' sia minore di 0.001

E2 Una moneta e' lanciata finche' non si verificano due teste (TT) o due croci (CC) in sequenza.

(a) Si calcoli la probabilita' che il gioco termini con la sequenza CC, e la durata media del gioco.

(b) Come la domanda precedente, nel caso in cui il gioco finisca quando si verificano due lanci *diversi* in sequenza (cioe' CT o TC).

E3 Si consideri uno switch in cui vi sono due processori uguali e indipendenti. Ogni processore alterna periodi di funzionamento e di guasto di durata esponenziale con media $1/\alpha$ (funzionamento) e $1/\beta = 1/(19\alpha) = 1$ giorno (guasto). Il traffico totale smaltito dallo switch e' pari a 2.5 Gbps se entrambi i processori sono attivi, 1 Gbps se ne funziona uno solo, e zero altrimenti.

(a) Si calcoli la frazione del tempo in cui lo switch non smaltisce traffico

(b) si calcoli la durata media di un intervallo di tempo durante il quale lo switch non smaltisce traffico

(c) si calcoli la durata media di un intervallo di tempo in cui c'e' un solo processore funzionante, e la probabilita' che in questo caso esso si guasti prima che l'altro torni in funzione

(d) si calcoli il traffico medio smaltito dallo switch

E4 Si consideri una catena di Markov X_n con la seguente matrice di transizione (gli stati sono ordinati da 0 a 2)

$$P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$$

(a) Si disegni il diagramma di transizione, e si calcoli la distribuzione di probabilita' di X_1, X_2 e X_{500}, dato che $X_0 = 0$.

(b) Si calcoli il tempo medio di primo passaggio dagli stati 0, 1 e 2 verso lo stato 2, e il tempo medio di ricorrenza di tutti gli stati.

(c) Sia $W_{ij}^{(n)} = E\left[\sum_{k=0}^{n-1} I\{X_k = j\} \middle| X_0 = i\right]$ il numero medio di visite allo stato j a partire dallo stato i durante i primi n istanti dell'evoluzione della catena. Si calcolino $W_{0j}^{(3)}$ e $W_{0j}^{(5000)}$ per $j = 0, 1, 2$.

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2007/2008
prova scritta – 14 luglio 2008 – parte B (60 minuti)

T1 Si enunci e si dimostri il teorema elementare del rinnovamento.

T2 Si dimostri che in una catena di Markov il periodo e' una proprieta' di classe.

T3 Si dimostri che per un processo di Poisson $X(t)$ la statistica di $X(s)$ condizionata a $X(t), s < t$, e' binomiale, e si fornisca l'espressione di $P[X(s) = k|X(t) = n]$.

Network Modeling
Solutions for 14/07/2008

Problem 1

$$\Pr[X(0.5) \geq 2|X(1)=5] = 1 - \Pr[X(0.5)=0|X(1)=5] - \Pr[X(0.5)=1|X(1)=5]$$
$$= 1 - \binom{5}{0}\left(\frac{0.5\lambda}{\lambda}\right)^0\left(1-\frac{0.5\lambda}{\lambda}\right)^5 - \binom{5}{1}\left(\frac{0.5\lambda}{\lambda}\right)^1\left(1-\frac{0.5\lambda}{\lambda}\right)^4 = 0.8125$$

For the second point, the average time required for a packet to be processed and then transmitted is $T_{TOT} = E[\mathcal{U}(10,30)] + \dfrac{E[L]}{C} = 100$ ms. It must be:

$$\sum_{i=0}^{N} \Pr[i \text{ requests in } 100 \text{ ms}] > 1 - 0.001 = 0.999$$

We can do this by inspection: $N = 4$ yields solution, so correct capacity is $C_r = 4 \cdot C = 400$ Mbps.

Problem 2

let TT be the state corresponding to two consecutive tails.
For the first question we need to solve the following systems:

$$\begin{cases} u_0 = \Pr[X_T = TT|X_0 = 0] = \frac{1}{2}u_T + \frac{1}{2}u_H \\ u_H = \frac{1}{2}u_T \\ u_T = \frac{1}{2} + \frac{1}{2}u_H \end{cases} \Rightarrow u_0 = \frac{1}{2}$$

$$\begin{cases} v_0 = E[X_T = TT|X_0 = 0] = \frac{1}{2}v_T + \frac{1}{2}v_H \\ v_H = \frac{1}{2}v_T \\ u_T = 1 + \frac{1}{2}v_H \end{cases} \Rightarrow v_0 = 3$$

Problem 3

$\Pr[\text{no disposal}] = \left(\dfrac{\frac{1}{\beta}}{\frac{1}{\alpha}+\frac{1}{\beta}}\right)^2 = \left(\dfrac{\alpha}{\alpha+\beta}\right)^2 = \left(\dfrac{1}{20}\right)^2 = 0.0025.$

$E[\text{no disposal}] = \min\{\xi(\beta), \xi(\beta)\} = \dfrac{1}{38\alpha} = \dfrac{1}{2}$ days.

$E[\text{cycle}] = \dfrac{E[\text{no disposal}]}{\Pr[\text{no disposal}]} = 200$ days.

$E[1 \text{ p. working}] = \Pr[1 \text{ p. working}] \cdot E[\text{cycle}] = \left(1 - \left(1-\dfrac{1}{20}\right)^2 - \left(\dfrac{1}{20}\right)^2\right)(200) = 19$ days.

Let now $X \sim \xi(\alpha)$ and $Y \sim \xi(\beta)$.

We want to compute the probability that X happens before Y: $\Pr[X < Y]$. We can do it conditioning and averaging:

$$\begin{aligned}
\Pr[X < Y] &= \int_0^\infty \Pr[X < y | Y = y] dF_Y(y) \\
&= \int_0^\infty \left(1 - e^{-\alpha y}\right) \beta e^{-\beta y} dy \\
&= 19\alpha \int_0^\infty e^{-19\alpha y} dy - 19\alpha \int_0^\infty e^{-(\alpha+\beta)y} dy = \cdots = 0.05
\end{aligned}$$

Average disposed traffic is

$$\begin{aligned}
\mathcal{T} &= 2.5 \Pr[2 \text{ p. working}] + \Pr[1 \text{ p. working}] \\
&= 2.5\left(1 - \frac{1}{20}\right)^2 + \left(1 - \left(1 - \frac{1}{20}\right)^2 - \left(\frac{1}{20}\right)^2\right) = 2.35215 \text{ Gbps.}
\end{aligned}$$

Problem 4

Probability distribution of X_1 given $X_0 = 0$ is first row of P. Probability distribution of X_1 given $X_0 = 0$ is first row of P^2. For X_{500} we can use the approximation of the steady state distribution: transition matrix is doubly stochastic, so $\pi = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$.
For the average first passage times we need to solve the usual system:
$\begin{cases} \hat{\theta}_{02} = 1 + p_{00}\hat{\theta}_{02} + p_{01}\hat{\theta}_{12} \\ \hat{\theta}_{12} = 1 + p_{10}\hat{\theta}_{02} + p_{11}\hat{\theta}_{12} \end{cases} \Rightarrow \begin{cases} \hat{\theta}_{02} = \frac{5}{2} \\ \hat{\theta}_{12} = \frac{5}{2} \end{cases}$. Then $\hat{\theta}_{22} = m_2 = \pi_2^{-1} = 3$.

Average recurrent times are given by $\mathbf{m} = \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \pi_0^{-1} \\ \pi_1^{-1} \\ \pi_2^{-1} \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$.

Last point requires to compute $W_{ij}^{(n)} = E\left[\sum_{k=0}^{n-1} \chi\{X_k = j\} | X_0 = i\right] = \sum_{k=0}^{n-1} p_{ij}^{(k)}$.

In vector form we can write $W_{0j}^{(3)} = \begin{bmatrix} p_{00}^{(0)} + p_{00}^{(1)} + p_{00}^{(2)} \\ p_{01}^{(0)} + p_{01}^{(1)} + p_{01}^{(2)} \\ p_{02}^{(0)} + p_{02}^{(1)} + p_{02}^{(2)} \end{bmatrix}$.

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2007/2008
prova scritta – 17 giugno 2008 – parte A (90 minuti)

E1 Si consideri un nodo di rete che in condizioni di funzionamento normale N smaltisce un traffico pari a 100 Mbps. Dopo un tempo di funzionamento casuale con distribuzione esponenziale di media T, il nodo entra in uno stato di malfunzionamento M in cui la sua capacita' di smaltire traffico si riduce a 50 Mbps. Quando il nodo entra nello stato M, con probabilita' β torna a funzionare normalmente dopo un tempo distribuito uniformemente in $[0, \alpha_1 T]$, mentre con probabilita' $1 - \beta$ smette di funzionare dopo un tempo distribuito uniformemente in $[0, \alpha_2 T]$ e deve essere sostituito, operazione che richiede un tempo deterministico pari a δT.

(a) Si costruisca un modello semi-Markoviano per il sistema. In particolare, indicando gli stati possibili con N, M e G (dove G e' lo stato durante il quale il nodo non funziona), si scrivano la matrice di transizione della catena di Markov inclusa e la matrice dei tempi medi associati alle transizioni.

(b) Usando il modello semi-Markoviano sviluppato al punto precedente, si calcoli in maniera parametrica la frazione del tempo che il nodo passa nei tre stati, e si calcoli il throughput medio smaltito dal nodo. Si calcolino inoltre i valori numerici di tali quantita' per $\beta = 0.9, \alpha_1 = 0.1, \alpha_2 = 0.2$ e $\delta = 0.1$.

(c) Si calcolino le quantita' richieste al punto precedente usando la teoria dei processi di rinnovamento, individuando un ciclo di rinnovamento opportuno.

E2 Si consideri un nodo di rete con due link di ingresso, dai quali arrivano pacchetti secondo due processi di Poisson indipendenti $X_1(t)$ e $X_2(t)$ di intensita' $\lambda_1 = \lambda_2 = \lambda = 500$ pacchetti al secondo, dove un pacchetto e' composto da 1000 bit. Sia inoltre $X(t) = X_1(t) + X_2(t)$.

(a) Si calcolino $P[X_1(3) = 2|X(3) = 3]$ e $P[X_1(2) = 2|X(2) = 3]$.

(b) Si calcolino $P[X_1(1) = 2|X(2) = 3]$ e $P[X(2) = 3|X_1(1) = 2]$.

(c) Si supponga che il link di uscita dal nodo sia un sistema di trasmissione costituito da un gran numero di canali paralleli, ognuno caratterizzato da un valore di bit rate pari a 1 Mbps. Supponendo che il sistema sia vuoto al tempo $t = 0$ e di poter trascurare l'eventualita' che un pacchetto che arriva non trovi un canale libero, determinare la probabilita' che vi siano due pacchetti in trasmissione agli istanti $t_1 = 0.5$ ms e $t_2 = 3$ ms.

E3 Si consideri una catena di Markov con la seguente matrice di transizione (gli stati sono numerati da 0 a 4):

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0.7 \\ 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0.2 & 0.4 \\ 0 & 0.7 & 0 & 0 & 0.3 \end{pmatrix}$$

(a) si disegni il diagramma di transizione della catena, se ne classifichino gli stati e si individuino le classi

(b) si calcolino $\lim_{n\to\infty} P^n$ e $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k$

(c) si calcoli la media del tempo di primo passaggio da tutti gli stati allo stato 4

E4 Si consideri un canale Markoviano a due stati con probabilità di transizione 0.99 (dallo stato buono a se stesso) e 0.1 (dallo stato cattivo allo stato buono). La probabilità che un pacchetto sia errato è 1 nello stato cattivo e 0 nello stato buono.

(a) Si calcoli il throughput del protocollo Go-Back-N se il tempo di round-trip è pari a due slot (cioè se una trasmissione nello slot t è errata questa verrà ritrasmessa nello slot $t + 2$), e il canale di ritorno è senza errori.

(b) Come al punto precedente se il canale di ritorno e' affetto da errori indipendenti con probabilita' 0.02.

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2007/2008
prova scritta – 17 giugno 2008 – parte B (60 minuti)

T1 Si dimostri che in una catena di Markov con un numero finito di stati non possono esserci stati ricorrenti nulli.

T2 Dimostrare che per un processo di rinnovamento $E[S_{N(t)+1}] = E[X](M(t) + 1)$.

T3 Dimostrare che per una catena di Markov le probabilità di transizione a n passi, $P_{ij}^{(n)}$ soddisfano la relazione

$$P_{ij}^{(n)} = \sum_m P_{im}^{(k)} P_{mj}^{(n-k)}, k = 0, 1, \ldots, n$$

Network Modeling
Solutions for 17/06/2008

Problem 1

The embedded Markov chain is given by $P = \begin{bmatrix} 0 & 1 & 0 \\ \beta & 0 & 1-\beta \\ 1 & 0 & 0 \end{bmatrix}$.

Time matrix is given by $T = \begin{bmatrix} 0 & T & 0 \\ \frac{\alpha_1 T}{2} & 0 & \frac{\alpha_2 T}{2} \\ \delta T & 0 & 0 \end{bmatrix}$.

Let now N denote the fully working state, M the semi-working state and G the faulty state.

Mean vector is given by $\mu = \begin{bmatrix} \mu_N \\ \mu_M \\ \mu_G \end{bmatrix} = \begin{bmatrix} T \\ \beta\frac{\alpha_1 T}{2} + (1-\beta)\frac{\alpha_2 T}{2} \\ \delta T \end{bmatrix}$.

The fraction of time spent in each state is given by $P_i = \dfrac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}$.

$$P_N = \frac{T}{\mu_N + \mu_M + (1-\beta)\mu_G} = \frac{T}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + (1-\beta)\delta T}.$$

$$P_M = \frac{\dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2}}{\mu_N + \mu_M + (1-\beta)\mu_G} = \frac{\dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2}}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + (1-\beta)\delta T}.$$

$$P_G = \frac{(1-\beta)\delta T}{\mu_N + \mu_M + (1-\beta)\mu_G} = \frac{(1-\beta)\delta T}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + (1-\beta)\delta T}.$$

Limiting distribution is computed solving the usual system $\pi = \pi P$: $\begin{cases} \pi_N = \dfrac{1}{3-\beta} \\ \pi_M = \dfrac{1}{3-\beta} \\ \pi_G = \dfrac{1-\beta}{3-\beta} \end{cases}$.

Average throughput is is given by $\mathcal{T} = 100 P_N + 50 P_M$.
Using renewal theory we have to identify a suitable renewal cycle: NMNM...NMNMG.
Let S be the number of consecutive $N \mapsto M$ cycles: this is a geometric random variable.
Clearly $\mathrm{P}[S \geq k] = \beta^k$, hence $E[S] = \sum_{k=1}^{\infty} \beta^k = \dfrac{\beta}{1-\beta}$.
Then $E[\text{cycle}] = E[S]T\left(1 + \dfrac{\alpha_1}{2}\right) + T\left(1 + \dfrac{\alpha_2}{2}\right) + \delta T$. Then:

$$P_N = \frac{T + TE[S]}{E[\text{cycle}]} = \frac{T}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + \delta T(1-\beta)}.$$

$$P_M = \frac{E[S]\dfrac{\alpha_1}{2}T + \dfrac{\alpha_2}{2}T}{E[\text{cycle}]} = \frac{\dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2}}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + \delta T(1-\beta)}.$$

$$P_G = \frac{\delta T}{E[\text{cycle}]} = \frac{\delta T(1-\beta)}{T + \dfrac{\beta\alpha_1 T + (1-\beta)\alpha_2 T}{2} + \delta T(1-\beta)}.$$

Obviously these results are the same as the previous ones.

Reward metric is $\mathbf{R} = \begin{bmatrix} 100 \cdot T(E[S]+1) \\ 50 \cdot \left(E[S]\frac{\alpha_1 T}{2} + \frac{\alpha_2 T}{2}\right) \\ 0 \cdot \delta T \end{bmatrix}$. Note that time metric is still μ.

Last point is now straightforward: $\mathcal{T} = \dfrac{100T(E[S]+1) + 50\dfrac{E[S]\alpha_1 T + \alpha_2 T}{2}}{E[\text{cycle}]}$.

Problem 2

Recall the usual Binomial formula: $P[X_1(s) = k | X(t) = n] = \binom{n}{k}\left(\frac{\lambda_1 s}{(\lambda_1+\lambda_2)t}\right)^k \left(1 - \frac{\lambda_1 s}{(\lambda_1+\lambda_2)t}\right)^{n-k}$

$P[X_1(3) = 2 | X(3) = 3] = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{3-2}$

$P[X_1(2) = 2 | X(2) = 2] = \binom{2}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{2-2}$

$P[X_1(1) = 2 | X(2) = 3] = \binom{3}{2}\left(\frac{1}{4}\right)^2\left(1-\frac{1}{4}\right)^{3-2}$

$P[X(2) = 3 | X_1(1) = 2] = P[X_1(2) + X_2(2) - X_1(1) = 1] = \dfrac{(2\lambda_1 + 2\lambda_2 - \lambda_1)e^{-(2\lambda_1+2\lambda_2-\lambda_1)}}{1!}$

For the last point we can see that this is a $M/G/\infty$ queue.

Service time is deterministic: $Y = \dfrac{L}{C} = 0.001$ s.

Notice that in this case $G(z)$ is step function before 1 ms and 1 after 1 ms.

We consider the arrivals as a Poisson process of intensity $\lambda_{TOT} = \lambda_1 + \lambda_2 = 2\lambda = 1000$ packets/s.

Let $M(t)$ be a random variable counting packets in the system at time t. Let $\Lambda = \lambda_{TOT}\int_0^t [1-G(z)]dz$.

$\Lambda_1 = \lambda_{TOT}\int_0^{0.0005}[1-G(z)]dz = 0.0005\lambda_{TOT} = 0.5$.

$\Pr[M(0.0005) = 2] = \dfrac{e^{-\Lambda_1}(\Lambda_1)^2}{2!} = 0.0758$.

For the second point, we realize that we are in steady-state condition: $\Lambda_2 = \lambda_{TOT} Y = 1$.

$\Pr[M(0.003) = 2] = \dfrac{e^{-\Lambda_2}(\Lambda_2)^2}{2!} = 0.1839$.

Problem 3

$\{0, 2\}$ is positive recurrent periodic ($d = 2$) class.

$\{3\}$ is transient class.

$\{1, 4\}$ is positive recurrent aperiodic class.

It is useful to compute the probability of being absorbed in the two classes given we start in the transient state:

P[absorption in 0,2|start in 3] $= \dfrac{0.4}{0.4+0.4} = \dfrac{1}{2}$.

P[absorption in 1,3|start in 3] $= \dfrac{0.4}{0.4+0.4} = \dfrac{1}{2}$.

$$\lim_{n\to\infty} P^n = \begin{bmatrix} X & 0 & X & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ X & 0 & X & 0 & 0 \\ X & \frac{1}{2}\frac{1}{2} & X & 0 & \frac{1}{2}\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \text{ and } \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{\infty} P^k = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2}\frac{1}{2} & \frac{1}{2}\frac{1}{2} & \frac{1}{2}\frac{1}{2} & 0 & \frac{1}{2}\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

For the first passage times, notice that $\hat{\theta}_{04}$ and $\hat{\theta}_{24}$ are both ∞. By basic limit theorem we also have

$\hat{\theta}_{44} = m_4 = \dfrac{1}{\pi_4} = 2$. Then $\hat{\theta}_{34} = 1 + p_{31}\hat{\theta}_{14} + p_{33}\hat{\theta}_{34} = \infty$. Finally $\hat{\theta}_{14} = \dfrac{1}{1 - p_{11}} = \dfrac{10}{7}$.

Problem 4

Transition matrix is $P = \begin{bmatrix} 0.99 & 0.01 \\ 0.1 & 0.9 \end{bmatrix}$. The protocol matrix is $C = \begin{bmatrix} p_{00} & p_{01} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix}$.

Reward and time vectors are $\mathbf{R} = \begin{bmatrix} R_G \\ R_B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_G \\ T_B \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Throughput is then $\mathcal{T} = \dfrac{\sum_i \pi_i R_i}{\sum_i \pi_i T_i} = \dfrac{\pi_G}{\pi_G + 2\pi_B} = \dfrac{p_{10}^{(2)}}{p_{10}^{(2)} + 2p_{01}} \simeq 0.9043$.

For the last question we have $\mathcal{T} = \dfrac{(1-\delta)p_{10}^{(2)}}{(1-\delta)p_{10}^{(2)} + 2\left((1-\delta)p_{01} + \delta p_{01}^{(2)} + \delta p_{10}^{(2)}\right)}$, where $\delta = 0.02$.

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2006/2007
prova scritta – 05 settembre 2007 – parte A

E1 Consider a web server which receives download requests according to a Poisson process with rate $\lambda = 20$ requests per second. Each requests, after a fixed processing time of 20 ms, triggers the transfer of a file with size uniformly distributed between 1 and 2 MBytes. A request is said to be "active" from when it arrives to when the corresponding file transfer is completed. Assume that the server capacity, in terms of how many simultaneous requests it can handle, is infinite, and that the transfer data rate for each file is 100 Mbit/s, regardless of the number of files that are being transferred at any given time.

(a) Assuming that the server is switched on at time $t = 0$, when does the statistics of the number of active requests in the system reach its steady-state condition? In such condition, express $P[k$ active requests$]$

(b) Given that in an interval of duration T the system received N requests, find the probability that at the end of such interval there are no active requests in the two cases (b1) $T = 0.1$ s, $N = 2$ and (b2) $T = 1$ s, $N = 20$.

E2 Consider a Markov chain with the following transition matrix (states are numbered from 0 to 4):

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 \\ 0.5 & 0 & 0 & 0.2 & 0.3 \\ 0 & 0 & 0.6 & 0 & 0.4 \end{pmatrix}$$

(a) Classify the states and identify the classes

(b) Compute the probabilities of absorption in all recurrent classes starting from each transient state

(c) Compute $\lim_{n\to\infty} P^n$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k$

(d) Compute the average recurrence times for all states

E3 A fair coin is flipped until two consecutive heads (HH) or two consecutive tails (TT) are observed.

(a) Compute the probability that the game ends with sequence TT

(b) Compute the probability that the game ends with sequence TT given that the first flip is H

(c) Respond to the two questions above if the coin is unfair, where the probability that a flip returns H is $p = 1/4$

E4 Consider a two-state Markov channel with transition probabilities 0.98 (from the good state to itself) and 0.1 (from the bad state to the good state). The packet error probability is 1 for a bad slot and 0 for a good slot, respectively.

(a) Compute the throughput of a Go-Back-N protocol if the round-trip time is $m = 2$ slots (i.e., a packet that is erroneous in slot t is retransmitted in slot $t + 2$), in the presence of an error-free feedback channel

(b) Consider now a system where a channel behavior as in point (a) (Markov model for the forward channel and error-free feedback channel) alternates with a channel behavior in which the forward channel is subject to iid errors with probability $\varepsilon = 0.01$. In particular, the channel follows the Markov model for a geometric number of slots with mean 1000000 slots, then follows the iid model for a geometric number of slots with mean 2000000 slots, then again the Markov model and so on. Compute the overall average throughput of the Go-Back-N protocol in this case.

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2006/2007
prova scritta – 05 settembre 2007 – parte B

T1 Prove that a Markov chain with a finite number of states cannot have any null recurrent state.

T2 Prove that for a renewal process $E[S_{N(t)+1}] = E[X](M(t) + 1)$.

T3 Prove that for a Markov chain the n-step transition probabilities, $P_{ij}^{(n)}$, satisfy the relationship

$$P_{ij}^{(n)} = \sum_m P_{im}^{(k)} P_{mj}^{(n-k)}, k = 0, 1, \ldots, n$$

Network Modeling
Solutions for 05/09/2007

Problem 1

Let processing time be $T_{proc} = 0.02$ s. Let packet length be $L \sim \mathcal{U}(8 \cdot 10^6, 16 \cdot 10^6)$ bits. Let server capacity be $C = 100 \cdot 10^6$ bits.
Time required for a transmission is $T_{TX} = T_{proc} + \dfrac{L}{C} \sim \mathcal{U}(0.1, 0.18)$, random variable with mean $E[T_{TX}] = 0.14$ s.
Problem asks the first t for which the system can be considered in steady-state condition. Clearly t is the first instant from which the function $1 - G(z)$ becomes zero: by inspection $t = 0.18$ s.
In this condition $\Lambda = \lambda E[T_{TX}] = 0.14\lambda$. Then $\Pr[M(t) = k] = \dfrac{e^{-\Lambda}(\Lambda)^k}{k!}$.
For the second point recall that $M(t)$ conditioned on $X(t)$ is a Binomial random variable:

$$\Pr[M(T) = m | X(T) = N] = \binom{N}{m} p^m (1-p)^{N-m}$$

where

$$p = \frac{1}{T} \int_0^T [1 - G_{T_{TX}}(z)] dz$$

$\Pr[M(0.1) = 0 | X(0.1) = 2] = \dbinom{2}{0} 1^0 (0)^{2-0} = 0.$
$\Pr[M(1) = 0 | X(1) = 20] = \dbinom{20}{0} (0.14)^0 (0.86)^{20-0} = 0.0489.$

Problem 2

$\{0, 1\}$ is positive recurrent periodic class. This class presents a “ping pong” behavior.
$\{3\}$ is transient class.
$\{2, 4\}$is positive recurrent periodic class.
It is useful to compute the probability in being absorbed in each recurrent class given we start in the transient:
Pr[absorption in 0,1] $= u_0 = p_{33}u_0 + p_{30} \Rightarrow u_0 = \dfrac{p_{30}}{1 - p_{33}} = \dfrac{5}{8}$.
Pr[absorption in 0,1] $= u_4 = p_{33}u_0 + p_{34} \Rightarrow u_0 = \dfrac{p_{34}}{1 - p_{33}} = \dfrac{3}{8}$.

$$\lim_{n\to\infty} P^n = \begin{bmatrix} X & X & 0 & 0 & 0 \\ X & X & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ X & X & \frac{1}{2}\frac{3}{8} & 0 & \frac{1}{2}\frac{3}{8} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} P^k = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2}\frac{5}{8} & \frac{1}{2}\frac{5}{8} & \frac{1}{2}\frac{3}{8} & 0 & \frac{1}{2}\frac{3}{8} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Average recurrence times are given by $\mathbf{m} = \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ \infty \\ 2 \end{bmatrix}$.

Problem 3

Let $\Pr[H] = p$ be the probability that returned side is H. Using first step analysis we have

$$\begin{cases} u_0 = \Pr[X_T = TT | X_0 = 0] = pu_H + (1-p)u_T \\ u_H = (1-p)u_T \\ u_T = 1 - p + pu_H \end{cases} \Rightarrow \begin{cases} u_0 = p\dfrac{(1-p)^2}{1-p+p^2} + (1-p)\dfrac{1-p}{1-p+p^2} \\ u_H = \dfrac{(1-p)^2}{1-p+p^2} \\ u_T = \dfrac{1-p}{1-p+p^2} \end{cases}$$

For $p = \frac{1}{2}$ we have $u_0 = \frac{1}{2}$.
For the second point we need to solve:
$\begin{cases} u_H = (1-p)u_T \\ u_T = 1 - p + pu_H \end{cases}$ which are the last two equations of previous system.
For $p = \frac{1}{2}$ we have $u_H = \frac{1}{3}$.
Last point is just computation with $p = \frac{1}{4}$.

Problem 4

Transition matrix is $P = \begin{bmatrix} 0.98 & 0.02 \\ 0.1 & 0.9 \end{bmatrix}$. Protocol matrix is $C = \begin{bmatrix} p_{00} & p_{01} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix}$.
First question is straightforward: $\mathcal{T} = \dfrac{p_{10}^{(m)}}{p_{10}^{(m)} + mp_{01}}$, where $m = 2$.
We do not need to compute all C^2 but only $p_{10}^{(2)} = p_{10}p_{00} + p_{11}p_{10} = 0.188$. Then $T = 0.8245$.
For the second point we can model the system as an alternating process: in the first phase we have a Markov behavior already discussed, while in the second one we have the forward IID errors behavior.
For the latter we have $\mathcal{T}_{iid} = \dfrac{1-\epsilon}{1-\epsilon+m\epsilon} = 0.98$.
Then $\mathcal{T} = \mathcal{T}_{Markov}\dfrac{E\,[\text{Markov behavior}]}{E\,[\text{cycle duration}]} + \mathcal{T}_{iid}\dfrac{E\,[\text{IID behavior}]}{E\,[\text{cycle duration}]} = \mathcal{T}_{Markov}\dfrac{1}{3} + \mathcal{T}_{iid}\dfrac{2}{3} = 0.9281.$

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2006/2007
prova scritta – 09 luglio 2007 – parte A

E1 Consider a Markov chain X_n with the following transition matrix (states are numbered from 0 to 2), and initial state $X_0 = 0$:

$$P = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.6 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Draw the transition diagram, and find the probability distributions of X_1, X_2 and X_{500}

(b) Compute the average first passage times from states 0, 1 and 2 to state 2.

(c) Compute $P[X_1 = 1, X_3 = 1|X_2 = 1]$ and $P[X_2 = 1|X_1 = 1, X_3 = 1]$.

E2 Consider a factory with two identical machines. Each machine alternates periods of time in which it is working or not working, of exponential duration with mean $1/\alpha = 27$ days (working) and $1/\beta = 1/(9\alpha)$ (not working). Each machine, whose operation is independent of the other, can produce 12 pieces per hour when it is working.

(a) Compute the fraction of time in which there is no production (i.e., both machines are not working).

(b) Compute the average number of pieces per hour produced by the factory.

(c) Compute the average number of pieces per hour produced by the factory if the number of pieces produced per hour is 12 when only one machine is working, and 30 when they are both working.

E3 Consider a network node that works as follows. If there is no traffic, the node alternates between a sleep state for an exponential duration of average T and an awake state for a fixed duration βT. When in the awake state, the node can receive, whereas in the sleep state it cannot. If while the node is awake a packet is transmitted, the node receives it entirely (even if this requires it to remain awake for a total time longer than βT), and goes to sleep immediately after. If instead while the node is awake there is no transmission, the node goes back to sleep after being awake for βT. The probability that a packet is transmitted while the node is awake is α, the time at which such transmission starts is uniformly distributed in $[0, \beta T]$, and the average packet transmission time is γT. Develop and solve a semi-Markov model for the node, and in particular:

(a) Consider the three states sleep (S), listening (L) and receiving (R), determine the matrix of the transition probabilities of the embedded Markov chain, and draw its transition diagram.

(b) Determine the matrix of the average times associated to each transition, $\mathbf{T}$, and the average times associated to the visits to each state, μ_S, μ_L, μ_R.

(c) Find an expression for the fraction of time the node spends in each of the three states, and find its numerical value for $\alpha = 0.5, \beta = 0.1, \gamma = 0.2$.

E4 Consider a system which receives service requests according to a Poisson process of rate $\lambda = 20$ requests per hour. Each request remains in the system for a service time equal to 6 minutes, and there is no limit to the number of requests simultaneously in service. Assume that the system started its operation at time $t = 0$.

(a) Compute the probability that the system is empty at time $t = 30$ minutes

(b) Compute the probability that the system is empty at time $t = 30$ minutes, conditioned on the fact that there were 10 arrivals between 0 and t.

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2006/2007
prova scritta – 09 luglio 2007 – parte B

T1 State and prove the elementary renewal theorem.

T2 Prove that in a Markov chain the period is a class property.

T3 Prove that for a Poisson process $X(t)$ the statistics of $X(s)$ conditioned on $X(t), s < t$, is binomial, and provide the espression of $P[X(s) = k|X(t) = n]$.

Network Modeling
Solutions for 09/07/2007

Problem 1

Distribution of X_1 is the first row of P.
Distribution of X_2 is the first row of P^2.
Distribution of X_{500} is the first row of P^{500}. Obviously it is not required to compute the $500-th$ power of P: we can assume the chain is in long run behavior, so we can use the steady-state distribution. $\pi = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right]$.
For the average first passage times we need to solve the usual system:
$\begin{cases} \hat{\theta}_{02} = 1 + p_{00}\hat{\theta}_{02} + p_{01}\hat{\theta}_{12} \\ \hat{\theta}_{12} = 1 + p_{10}\hat{\theta}_{02} + p_{11}\hat{\theta}_{12} \end{cases} \Rightarrow \begin{cases} \hat{\theta}_{02} = 3 \\ \hat{\theta}_{12} = 2 \end{cases}$. Then last average first passage time is $\bar{\theta}_{22} = \dfrac{1}{\pi_2} = 4$.

$$\Pr[X_1 = 1, X_3 = 1 | X_2 = 1] = \frac{\Pr[X_1 = 1, X_2 = 1, X_3 = 1 | X_0 = 0]}{\Pr[X_2 = 1 | X_0 = 0]} = \frac{p_{01}p_{11}^2}{p_{01}^{(2)}}$$

$$\Pr[X_2 = 1 | X_1 = 1, X_3 = 1] = \frac{\Pr[X_2 = 1, X_3 = 1 | X_1 = 1]}{\Pr[X_3 = 1 | X_1 = 1]} = \frac{p_{11}^2}{p_{11}^{(2)}}$$

Problem 2

$$\Pr[\text{no disposal}] = \left(\frac{\frac{1}{\beta}}{\frac{1}{\alpha} + \frac{1}{\beta}}\right)^2 = \left(\frac{\alpha}{\alpha + \beta}\right)^2 = \left(\frac{1}{10}\right)^2 = 0.01.$$

Average pieces produced per hour are:

$$\begin{aligned} T &= 2 \cdot 12 \cdot \Pr[\text{2 p. working}] + 12 \cdot \Pr[\text{1 p. working}] \\ &= 24 \cdot \left(1 - \frac{1}{10}\right)^2 + 12 \cdot \left(1 - \left(1 - \frac{1}{10}\right)^2 - \left(\frac{1}{10}\right)^2\right) = 21.6 \text{ pieces per h.} \end{aligned}$$

Last point is straightforward:

$$\begin{aligned} T &= 30 \cdot \Pr[\text{2 p. working}] + 12 \cdot \Pr[\text{1 p. working}] \\ &= 30 \cdot \left(1 - \frac{1}{10}\right)^2 + 12 \cdot \left(1 - \left(1 - \frac{1}{10}\right)^2 - \left(\frac{1}{10}\right)^2\right) = 26.46 \text{ pieces per h.} \end{aligned}$$

Problem 3

Embedded Markov chain is $P = \begin{bmatrix} 0 & 1 & 0 \\ 1-\alpha & 0 & \alpha \\ 1 & 0 & 0 \end{bmatrix}$. Average times matrix is $\mathbf{T} = \begin{bmatrix} - & T & - \\ \beta T & - & \frac{\beta T}{2} \\ \gamma T & - & - \end{bmatrix}$.

Average times associated to the visits are given by $\mu = \begin{bmatrix} \mu_S \\ \mu_L \\ \mu_R \end{bmatrix} = \begin{bmatrix} T \\ (1-\alpha)\beta T + \alpha\frac{\beta T}{2} \\ \gamma T \end{bmatrix}$.

Fraction of time spent in each state is given by $P_i = \dfrac{\mu_i \pi_i}{\sum_j \mu_j \pi_j}$.

We now solve $\begin{cases} \pi_L = \pi_S \\ \pi_R = \alpha \pi_L \\ \pi_S + \pi_L + \pi_R = 1 \end{cases} \Rightarrow \begin{cases} \pi_L = \frac{1}{2+\alpha} \\ \pi_S = \frac{1}{2+\alpha} \\ \pi_R = \frac{\alpha}{2+\alpha} \end{cases}$.

Finally $P_i = \dfrac{(2+\alpha)(\pi_i \mu_i)}{T\left(1+\beta-\frac{\alpha\beta}{2}+\alpha\gamma\right)}$.

Problem 4

Notice that service time Y is deterministic, hence $G_Y(x) = \begin{cases} 1 & x \geq 6 \\ 0 & x < 6 \end{cases}$.

Let $M(t)$ be the random variable counting users in the system at time t.

$\Pr[M(0.5) = 0] = \dfrac{e^{-\Lambda}(\Lambda)^0}{0!}$, where $\Lambda = \lambda \int\limits_0^{0.5} [1 - G_Y(z)]dz = \dfrac{\lambda}{10} = 2$.

Now recall that $M(t)$ conditioned on $X(t)$ is a Binomial random variable:

$$\Pr[M(t) = m | X(t) = n] = \binom{n}{m} p^m (1-p)^{n-m}$$

where

$$p = \frac{1}{t} \int\limits_0^t [1 - G_Y(z)]dz$$

Finally $\Pr[M(0.5) = 0 | X(0.5) = 10] \binom{10}{0} (0.2)^0 (0.8)^{10}$

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2005/2006
prova scritta – 12 dicembre 2006– parte A (90 minuti)

E1 Consider a Markov chain with the following transition matrix (states are numbered from 0 to 5):

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0.1 & 0 & 0.7 & 0 & 0.2 \\ 0 & 0.3 & 0.5 & 0 & 0.2 & 0 \\ 0 & 0.7 & 0 & 0.2 & 0 & 0.1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.1 & 0 & 0.7 \end{pmatrix}$$

(a) Draw the transition diagram, classify the states, and identify the classes.

(b) Compute $\lim_{n\to\infty} P^n$.

(c) Compute $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k$.

(d) Compute $P[X_4 = 5, X_2 = 3 | X_3 = 1, X_1 = 3]$.

E2 Consider a network node that under normal conditions can handle a traffic equal to 1 Gbps. This node works normally for an exponential time with mean $99T$, and then enters an alarm state, during which its capacity is reduced to 250 Mbps. After being for a time T in the alarm state, the node is instantaneously repaired, and starts again to work normally.

(a) Compute the fraction of time the node spends in the alarm state, and the average traffic handled (suppose that the queues are always full, i.e., there are always packets to transmit).

(b) Suppose now that once entering the alarm state the node completely stops working after an exponential time of mean $2T$, unless it is repaired earlier (as before, the repair requires exactly a time T from when the node enters the alarm state). If the node stops working, it needs to be replaced, and this takes a time $20T$, during which the node cannot handle any traffic (note that this replacement is different from the simple repair considered in the previous case). Compute: (i) the average time between two subsequent substitutions, (ii) the fraction of time in which the node is not working, and (iii) the average system throughput.

E3 Consider two independent Poisson processes $X_1(t)$ and $X_2(t)$, where $X_i(t)$ is the number of arrivals for process i during $[0, t]$. The average number of arrivals per unit time of the two processes is $\lambda_1 = 0.5$ and $\lambda_2 = 1$, respectively.

(a) Compute $P[X_1(3) = 1 | X_1(3) + X_2(3) = 3]$ and $P[X_1(3) + X_2(3) = 3 | X_1(3) = 1]$.

(b) Compute $P[X_1(2) = 1 | X_1(3) = 3]$ e $P[X_1(3) = 3 | X_1(2) = 1]$.

E4 Consider a two-state Markov channel with transition probabilities 0.98 (from the good state to itself) and 0.1 (from the bad state to the good state). The packet error probability is 1 for a bad slot and 0 for a good slot, respectively.

(a) Compute the throughput (average number of successes per slot) of a protocol that transmits packets directly on the channel, with no retransmissions.

(b) Compute the throughput of a Go-Back-N protocol if the round-trip time is $m = 2$ slots (i.e., a packet that is erroneous in slot t is retransmitted in slot $t+2$), in the presence of an error-free feedback channel.

(c) Same as in the previous point, with the difference that now the feedback channel is subject to iid errors with probability 0.1.

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2005/2006
prova scritta – 12 dicembre 2006– parte B (60 minuti)

T1 State and prove the elementary renewal theorem.

T2 Prove that if states i and j of a Markov chain communicate and i is recurrent, then j is also recurrent.

T3 Prove that a Markov chain with a finite number of states cannot have any null recurrent state.

Network Modeling
Solutions for 12/12/2006

Problem 1

States $\{0, 4\}$ belong to an absorbing class of period $d = 2$.
State $\{2\}$ is transient.
States $\{1, 3, 5\}$ belong to a positive recurrent aperiodic class ($d = 1$).
Since absorbing class has a "ping-pong" behavior, the limiting distribution is not defined for the whole chain.
Obviously $\pi_2 = 0$ since state is transient.
We still can compute the π_i for the recurrent class by solving the usual system $\pi = \pi P$.
Notice that the submatrix induced by class $\{1, 3, 5\}$ is doubly stochastic, meaning that each row and column sums to one. In this case it is immediate that $\pi_1 = \pi_3 = \pi_5 = \frac{1}{3}$.
We have $\mathrm{P}\left[\text{absorption in } \{0,4\} | \text{start in } 2\right] = \dfrac{0.2}{0.2 + 0.3} = \dfrac{2}{5}$ and
$\mathrm{P}\left[\text{absorption in } \{1,3,5\} | \text{start in } 2\right] = \dfrac{0.3}{0.2 + 0.3} = \dfrac{3}{5}$.

We obtain $\lim\limits_{n\to\infty} P^n = \begin{bmatrix} X & 0 & 0 & 0 & X & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ X & \frac{3}{5}\frac{1}{3} & 0 & \frac{3}{5}\frac{1}{3} & X & \frac{3}{5}\frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ X & 0 & 0 & 0 & X & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$

Instead what always exists is $\lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=1}^{\infty} P^k = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{5}\frac{1}{2} & \frac{3}{5}\frac{1}{3} & 0 & \frac{3}{5}\frac{1}{3} & \frac{2}{5}\frac{1}{2} & \frac{3}{5}\frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$

Remember that this quantity is the average time spent in a state: it is the probability of being absorbed in that class times the average time spent in that specific state.
For the last point we use Bayes rule:
$\mathrm{P}\left[X_4 = 5, X_2 = 3 | X_3 = 1, X_1 = 3\right] = \dfrac{\mathrm{P}\left[X_4 = 5, X_3 = 1, X_2 = 3 | X_1 = 3\right]}{\mathrm{P}\left[X_3 = 1 | X_1 = 3\right]} = \dfrac{p_{33}p_{31}p_{15}}{p_{31}^{(2)}}$

Problem 2

This system can be modeled as a semi-Markov process, so we need to specify a transition matrix P and a time matrix T.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ p_{AF} & 0 & p_{AG} \\ 1 & 0 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} - & 99T & - \\ T & - & \Omega \\ 20T & - & - \end{bmatrix}$$

$p_{AF} = \mathrm{P}\left[\xi\left(\frac{1}{2T}\right) > T\right] = e^{-\dfrac{T}{2T}} = e^{-\dfrac{1}{2}} = 0.6065.$

$p_{AG} = 1 - p_{AF} = 0.3935.$

$F(t) = \mathrm{P}\left[\xi \leq t | \xi < T\right] = \dfrac{\mathrm{P}\left[\xi \leq t, \xi < T\right]}{\mathrm{P}\left[\xi < T\right]} = \begin{cases} \dfrac{\mathrm{P}\left[\xi \leq t\right]}{\mathrm{P}\left[\xi < T\right]} & t < T \\ 1 & t \geq T \end{cases}$

$\Omega = E\left[\xi | \xi < T\right] = \int\limits_0^\infty (1 - F(t))\, dt = \int\limits_0^T \left(1 - \dfrac{1 - e^{-\frac{t}{2T}}}{1 - e^{-\frac{T}{2T}}}\right) dt = \int\limits_0^\infty \dfrac{1 - e^{-\frac{1}{2}} - 1 + e^{-\frac{t}{2T}}}{1 - e^{-\frac{1}{2}}} dt = \dfrac{2T - 3Te^{-\frac{1}{2}}}{1 - e^{-\frac{1}{2}}}$

Metric for the time is $\mathbf{T} = \begin{bmatrix} 99T \\ Te^{-\frac{1}{2}} + 2T - 3Te^{-\frac{1}{2}} \\ 20T \end{bmatrix} = \begin{bmatrix} 99T \\ 2T\left(1 - e^{-\frac{1}{2}}\right) \\ 20T \end{bmatrix} = \begin{bmatrix} \mu_F \\ \mu_A \\ \mu_G \end{bmatrix}.$

Stationary distribution is given by $\begin{cases} \pi_A = \pi_F \\ \pi_G = \left(1 - e^{-\frac{1}{2}}\right)\pi_A \\ \pi_A + \pi_F + \pi_G = 1 \end{cases} \Rightarrow \begin{cases} \pi_F = \dfrac{1}{3 - e^{-\frac{1}{2}}} \\ \pi_A = \dfrac{1}{3 - e^{-\frac{1}{2}}} \\ \pi_G + \dfrac{1 - e^{-\frac{1}{2}}}{3 - e^{-\frac{1}{2}}} \end{cases}$

Then the fraction of time the system spends in state i is $P_i = \dfrac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}$.

For the non-working state it is $P_G = \dfrac{\left(1 - e^{-\frac{1}{2}}\right) 20T}{99T + 2T\left(1 - e^{-\frac{1}{2}}\right) + \left(1 - e^{-\frac{1}{2}}\right) 20T} = 0.073.$

The average time between two subsequent substitutions is $E\left[\text{cycle time}\right] = \dfrac{E\left[G\right]}{P_G}$.

Now the reward vector is $\mathbf{R} = \begin{bmatrix} 99T \\ 2T\left(1 - e^{-\frac{1}{2}}\right)\frac{1}{4} \\ 0 \end{bmatrix}$.

Throughput is finally computed as $\mathcal{T} = \dfrac{\sum_i R_i \pi_i}{\sum_i T_i \pi_i}$.

Problem 3

Let $X(t)$ be the sum of the two Poisson processes.

$\mathrm{P}\left[X_1(3)=1|X(3)=3\right]=\binom{3}{1}\left(\frac{3\lambda_1}{3(\lambda_1+\lambda_2)}\right)^1\left(1-\frac{3\lambda_1}{3(\lambda_1+\lambda_2)}\right)^{3-1}.$

$\mathrm{P}\left[X(3)=3|X_1(3)=1\right]=\mathrm{P}\left[X_2(3)=2\right]=\frac{e^{-3\lambda_2}(3\lambda_2)^2}{2!}.$

$\mathrm{P}\left[X_1(2)=1|X_1(3)=3\right]=\frac{\mathrm{P}\left[X_1(2)=1,X_2(3)=3\right]}{\mathrm{P}\left[X_1(3)=3\right]}=\mathrm{P}\left[X_1(3)=3|X_1(2)=1\right]\frac{\mathrm{P}\left[X_1(2)=1\right]}{\mathrm{P}\left[X_1(3)=3\right]}$

$\mathrm{P}\left[X_1(3)=3|X_1(2)=1\right]=\mathrm{P}\left[X_1(3)-X_1(2)=2\right]=\frac{e^{3\lambda_1-2\lambda_1}(3\lambda_1-2\lambda_1)^2}{2!}.$

Problem 4

Transition matrix is $P=\begin{bmatrix}0.98 & 0.02\\ 0.1 & 0.9\end{bmatrix}.$

For the first question, we introduce reward and time vector: $\mathbf{R}=\begin{bmatrix}R_G\\ R_B\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}\qquad \mathbf{T}=\begin{bmatrix}T_G\\ T_B\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}.$

Then we simply compute $\mathcal{T}=\frac{\sum_i \pi_i R_i}{\sum_i \pi_i T_i}=\frac{\pi_G}{\pi_G+\pi_B}=0.8333.$

For the second question we need to introduce the usual protocol matrix $C=\begin{bmatrix}p_{00} & p_{01}\\ p_{10}^{(m)} & p_{11}^{(m)}\end{bmatrix}$ and the new time vector $\mathbf{T}=\begin{bmatrix}T_G\\ T_B\end{bmatrix}=\begin{bmatrix}1\\ m\end{bmatrix}.$

Then computation is straightforward: $\mathcal{T}=\frac{p_{10}^{(m)}}{p_{10}^{(m)}+mp_{01}}$, where $m=2$.

We do not need to compute all P^2 but only $p_{10}^{(2)}=p_{10}p_{00}+p_{11}p_{10}=0.188$. Then $T=0.8245$.

For the last question we have $\mathcal{T}=\frac{(1-\delta)p_{10}^{(m)}}{(1-\delta)p_{10}^{(m)}+m\left((1-\delta)p_{01}+\delta p_{01}^{(m)}+\delta p_{10}^{(m)}\right)}$, where $\delta=0.1$.

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2005/2006
prova scritta – 14 luglio 2006– parte A (90 minuti)

E1 Consider a Markov chain X_n with the following transition matrix (states are numbered from 0 to 2):

$$P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$$

(a) Draw the transition diagram, and find the probability distribution of X_1, X_2 and X_{500}, given $X_0 = 0$

(b) Compute the average first passage times from state 0 to states 0, 1 and 2.

(c) Let $W_{ij}^{(n)} = E\left[\sum_{k=0}^{n-1} I\{X_k = j\} \middle| X_0 = i\right]$ be the average number of visits to state j during the first n time slots, given that the chain starts in state i. Compute $W_{0j}^{(3)}$ and $W_{0j}^{(5000)}$ for $j = 0, 1, 2$.

E2 Consider a link with capacity 1 Mbps, shared among many users who collectively produce packets according to a Poisson process with rate $\lambda = 500$ packets per second. All packets are of the same length, equal to 1000 bits. The access protocol is an ideal CSMA, where a packet generated when the channel is idle gets immediate access, whereas a packet that finds the channel busy is rescheduled after an exponential time of average $100/\lambda$. (If this new access attempt again finds a busy channel the protocol keeps trying after random times until success.) Assume that the total traffic (new packets plus all retransmissions) can be approximated as Poisson with rate λ.

(a) Compute the throughput (average traffic handled) on the link.

(b) Compute the average access delay, from when a packet is generated to when it finally gets access to the channel.

(c) If a transmission corresponds to a gain of 1 unit and each failed access attempt (i.e., a packet finding the channel busy) corresponds to a cost of 0.2 units, compute the total gain of the system in units per second.

E3 Consider an exhibition where visitors arrive according to a Poisson process with rate $\lambda = 10$ customers per hour. Each visitor spends a time uniformly distributed between 20 and 30 minutes, and then leaves. The room in which the exhibition is shown is large enough to ensure there is never a need to block customers at the entrance due to too many people inside. The exhibition is open from 8 AM to 6 PM.

(a) Compute the probability that fewer than three visitors arrive during the first half hour.

(b) Compute the probability that at 8:15 AM there is only one visitor in the room.

(c) Compute the probability that at closing time (6 PM) the room is empty.

E4 Consider a two-state Markov channel with transition probabilities 0.98 (from the good state to itself) and 0.1 (from the bad state to the good state). The packet error probability is 1 for a bad slot and 0 for a good slot, respectively.

(a) Compute the throughput of a Go-Back-N protocol if the round-trip time is $m = 2$ slots (i.e., a packet that is erroneous in slot t is retransmitted in slot $t + 2$), in the presence of an error-free feedback channel

(b) Consider now a system where a channel behavior as in point (a) (Markov model for the forward channel and error-free feedback channel) alternates with a channel behavior in which the forward channel is subject to iid errors with probability $\varepsilon = 0.01$. In particular, the channel follows the Markov model for a geometric number of slots with mean 1000000 slots, then follows the iid model for a geometric number of slots with mean 2000000 slots, then again the Markov model and so on. Compute the overall average throughput of the Go-Back-N protocol in this case.

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2005/2006
prova scritta – 14 luglio 2006– parte B (60 minuti)

T1 For a Poisson process of rate λ, prove that the interarrival times are iid exponential with mean $1/\lambda$.

T2 Prove that in a Markov chain the period is a class property.

T3 Prove that for a renewal process $E[S_{N(t)+1}] = E[X](M(t) + 1)$.

Network Modeling
Solutions for 14/07/2006

Problem 1

Distribution of X_1 is the first row of P.
Distribution of X_2 is the first row of P^2.
Distribution of X_{500} is the first row of P^{500}. Obviously it is not required to compute the $500-th$ power of P: we can assume the chain is in long run behavior, so we can use the steady-state distribution.
$\pi = \left[\frac{10}{27}, \frac{4}{9}, \frac{5}{27}\right]$.

Average first passage time from 0 to 0 is $\bar{\theta}_{00} = \frac{1}{\pi_0} = \frac{27}{10}$.

For the other states we need to solve the usual systems:
$\begin{cases} \hat{\theta}_{01} = 1 + p_{00}\hat{\theta}_{01} + p_{02}\hat{\theta}_{21} \\ \hat{\theta}_{21} = 1 + p_{20}\hat{\theta}_{01} + p_{22}\hat{\theta}_{21} \end{cases}$ and $\begin{cases} \hat{\theta}_{02} = 1 + p_{00}\hat{\theta}_{02} + p_{01}\hat{\theta}_{12} \\ \hat{\theta}_{12} = 1 + p_{10}\hat{\theta}_{02} + p_{11}\hat{\theta}_{22} \end{cases}$

Last point requires to compute $W_{ij}^{(n)} = E\left[\sum_{k=0}^{n-1} \chi\{X_k = j\} | X_0 = i\right] = \sum_{k=0}^{n-1} p_{ij}^{(k)}$.

In vector form we can write $W_{0j}^{(3)} = p_{0j}^{(0)} + p_{0j}^{(1)} + p_{0j}^{(2)} = \begin{bmatrix} 1.6 \\ 0.84 \\ 0.56 \end{bmatrix}$.

For $W_{0j}^{(5000)}$ we can just use the steady state distribution multiplied by 5000:
$W_{0j}^{(5000)} \simeq 5000\pi_j = \begin{bmatrix} 1852 \\ 2222 \\ 926 \end{bmatrix}$.

Problem 2

This system can be modeled as an alternating process, where the renewal instant is the first arrival since the link is empty.
A transmission requires exactly $E[\text{tx time}] = \frac{|\text{packet}|}{L} = 1$ ms.
Average waiting time is $E[\text{tx time}] = \frac{100}{\lambda} = 200$ ms.
Fraction of time the link is empty is distributed as an exponential of mean $\frac{1}{\lambda} = 2$ ms.
Now we can compute the average cycle duration: $E[\text{cycle time}] = E[\text{tx time}] + E[\text{tempty time}] = 3$ ms.
Throughput is then $T = L\frac{E[\text{tx time}]}{E[\text{cycle time}]} = 0.333$ Mbps.
Let β be the probability of finding the system in the busy state: $\beta = \frac{E[\text{tx time}]}{E[\text{cycle time}]} = \frac{1}{3}$.
Let N be the number of consecutive failed attempts before a successful transmission: clearly N is a geometric random variable.
We have $\mathrm{P}[N \geq k] = \beta^k$ and so $E[N] = \sum_{k=1}^{\infty} \beta^k = \frac{\beta}{1-\beta} = \frac{1}{2}$.
Then the average delay is $E[\text{delay}] = E[N]\, E[\text{tx time}] = \frac{1}{2}\frac{100}{\lambda} = 100$ ms.
For the last point: $\frac{E[\text{gain in a cycle}]}{E[\text{cycle time}]} = \frac{1 - 0.2E[\text{arrivals in 1 ms}]}{E[\text{cycle time}]} = \frac{1 - 0.2\lambda 10^{-3}}{E[\text{cycle time}]} = 300$.

Problem 3

Clearly this system can be modeled as an $M/G/\infty$ queue.
Arrivals are Poisson of parameter $\lambda = 10$ customers per hour.
Service time Y is uniformly distributed between 20 and 30 minutes. Converting in hours, it is between 0.33 and 0.5 hours.
Let $M(t)$ be the number of users in the whole system at time t.
For first question, we have

$$\begin{aligned}
\mathrm{P}\left[X(0.5) < 3\right] &= \mathrm{P}\left[X(0.5) = 0\right] + \mathrm{P}\left[X(0.5) = 1\right] + \mathrm{P}\left[X(0.5) = 2\right] \\
&= \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^0}{0!} + \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^1}{1!} + \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^2}{2!} = e^{-5}\left(1 + 5 + \frac{25}{2}\right) \simeq 0.1246.
\end{aligned}$$

Second point is straightforward, since $\mathrm{P}\left[M(0.25) = 1\right] = \dfrac{e^{-\lambda pt}\left(\lambda pt\right)^1}{1!}$ where now $\lambda pt = \int\limits_0^t \left[1 - G(z)\right] dz$.

Computed in $t = 0.25$, $\lambda pt = \lambda\,(0.25) = 2.5$. Finally $\mathrm{P}\left[M(0.25) = 1\right] = \dfrac{e^{-2.5}\,(2.5)^1}{1!} \simeq 0.2052$.

For the last point we can simply use the approximation for $t \to \infty$: $\mathrm{P}\left[M(10) = 0\right] = \dfrac{e^{-\Lambda}\,(\Lambda)^0}{0!}$ where $\Lambda = \lambda E\left[Y\right]$.

Problem 4

Transition matrix is $P = \begin{bmatrix} 0.98 & 0.02 \\ 0.1 & 0.9 \end{bmatrix}$. Protocol matrix is $C = \begin{bmatrix} p_{00} & p_{01} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix}$.

First question is straightforward: $\mathcal{T} = \dfrac{p_{10}^{(m)}}{p_{10}^{(m)} + m p_{01}}$, where $m = 2$.
We do not need to compute all C^2 but only $p_{10}^{(2)} = p_{10}p_{00} + p_{11}p_{10} = 0.188$. Then $T = 0.8245$.
For the second point we can model the system as an alternating process: in the first phase we have a Markov behavior already discussed, while in the second one we have the forward IID errors behavior.
For the latter we have $\mathcal{T}_{iid} = \dfrac{1 - \epsilon}{1 - \epsilon + m\epsilon} = 0.98$.
Then $\mathcal{T} = \mathcal{T}_{Markov}\dfrac{E\left[\text{Markov behavior}\right]}{E\left[\text{cycle duration}\right]} + \mathcal{T}_{iid}\dfrac{E\left[\text{IID behavior}\right]}{E\left[\text{cycle duration}\right]} = \mathcal{T}_{Markov}\dfrac{1}{3} + \mathcal{T}_{iid}\dfrac{2}{3} = 0.9281.$

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2004/2005
prova scritta – 22 settembre 2005– parte A

E1 Consider a Markov chain X_n with states 1, 2 and 3, $X(0) = 3$, and transition matrix

$$P = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.2 & 0.6 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Compute the steady-state probabilities and the average recurrence times of all states.

(b) Compute mean and variance of the first passage time from state 3 to state 1.

(c) Compute mean and variance of the first passage time from state 1 to state 3.

(d) Compute $P[X(1) = 1, X(3) = 1|X(2) = 2]$ and $P[X(2) = 2|X(1) = 1, X(3) = 1]$.

E2 Consider a queue where packets arrive according to a Poisson process with rate $\lambda = 1$ packet per second. All packets in the queue are transmitted when either of the following events occurs: (i) there are two packets in the queue, or (ii) there is one packet in the queue and its waiting time reaches two seconds. Transmission is instantaneous, i.e., the queue empties every time there is a packet arrival when one packet is already in the queue, or when the only packet in the queue has been there for enough time.

(a) Compute the fraction of time in which the queue is empty.

(b) Compute the average packet delay (i.e., the average time a packet spends in the queue).

E3 Consider a frequency division transmission system in which the number of channels is so large that the probability they are all occupied is negligible. Such system receives connection requests according to a Poisson process with rate $\lambda = 100$ calls per hour, and the duration of each call is exponential with mean 6 minutes. Let $X(t)$ be the number of occupied channels at time t.

(a) Compute the average of $X(t)$ at $t = 6, 10$ minutes and for $t = \infty$.

(b) Compute $P[X(t) = 10]$ for $t = 6$ and $t = \infty$

(c) Repeat the previous calculations assuming that the call duration is uniformly distributed in $[2, 10]$ (minutes)

E4 Consider a Go-Back-N protocol over a two-state Markov channel with transition probabilities 0.99 (from the good state to itself) and 0.1 (from the bad state to the good state). The packet error probability is 1 for a bad slot and 0 for a good slot, respectively. The round-trip time is $m = 2$ slots, i.e., a packet that is erroneous in slot t will be retransmitted in slot $t + 2$.

(a) Compute the throughput of the protocol for an error-free feedback channel

(b) Compute the throughput of the protocol for a feedback channel subject to iid errors with probability 0.1.

Corso di Modelli e Analisi delle Prestazioni nelle Reti – AA 2004/2005
prova scritta – 22 settembre 2005– parte B

T1 State and prove the elementary renewal theorem.

T2 Prove that in a Markov chain the period is a class property.

T3 Consider a random walk over the non-negative integers with the following transition probabilities: $P_{01} = 1$, $P_{i,i+1} = p, P_{i,i-1} = q, i > 0$, with $p + q = 1$. Study its behavior, and in particular characterize its recurrence or transiency and derive the steady-state distribution.

Network Modeling
Solutions for 22/09/2005

Problem 1

For the first question we need to compute π and $\theta_{11}, \theta_{22}, \theta_{33}$.

First we need to solve the usual system $\pi = \pi P = \begin{cases} \pi_2 = 0.3\pi_1 + 0.2\pi_2 \\ \pi_3 = 0.2\pi_1 + 0.6\pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 0 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{5}{9} \\ \pi_2 = \frac{5}{24} \\ \pi_3 = \frac{17}{72} \end{cases}$.

Then by the Basic Limit Theorem we have $\begin{bmatrix} \overline{\theta}_{11} = m_1 = \frac{1}{\pi_1} \\ \overline{\theta}_{22} = m_2 = \frac{1}{\pi_2} \\ \overline{\theta}_{33} = m_3 = \frac{1}{\pi_3} \end{bmatrix}$.

For the second point we need to solve $\overline{\theta}_{ij} = 1 + \sum\limits_{k \neq j} p_{ik}\overline{\theta}_{kj}$ for two values.

$\begin{cases} \overline{\theta}_{31} = 1 + p_{32}\overline{\theta}_{21} + p_{33}\overline{\theta}_{31} = 1 \\ \overline{\theta}_{12} = 1 + p_{22}\overline{\theta}_{21} + p_{23}\overline{\theta}_{31} = \frac{p_{23}}{1-p_{22}} = \frac{3}{4} \end{cases}$ $\begin{cases} \overline{\theta}_{13} = 1 + p_{11}\overline{\theta}_{13} + p_{12}\overline{\theta}_{23} = 5 \\ \overline{\theta}_{23} = \frac{3}{2} \end{cases}$

For the variance we need to compute the second moments according to $\overline{\theta^2}_{ij} = 2\overline{\theta}_{ij} - 1 + \sum\limits_{k \neq j} p_{ik}\left(1 + \overline{\theta^2}_{kj}\right)$.

Then variance is $var(\overline{\theta}_{ij}) = \overline{\theta^2}_{ij} - (\overline{\theta}_{ij})^2$.

Notice that $var(\overline{\theta}_{31}) = 0$ since this step is deterministic.

$$\mathrm{P}\left[X(1) = 1, X(3) = 1 | X(2) = 2\right] = \frac{\mathrm{P}\left[X(1) = 1, X(2) = 2, X(3) = 1 | X(0) = 3\right]}{\mathrm{P}\left[X(2) = 2 | X(0) = 3\right]} = \frac{p_{31}p_{12}p_{21}}{p_{32}^{(2)}}$$

$$\mathrm{P}\left[X(2) = 2 | X(1) = 1, X(3) = 1\right] = \frac{\mathrm{P}\left[X(1) = 1, X(2) = 2, X(3) = 1 | X(0) = 3\right]}{\mathrm{P}\left[X(1) = 1, X(3) = 1 | X(0) = 3\right]} = \frac{p_{31}p_{12}p_{21}}{p_{31}p_{11}^{(2)}}$$

Problem 2

The distribution of the first arrival is exponential, hence $E[\text{empty}] = \dfrac{1}{\lambda}$.

After the first arrival, we wait until another arrival or up to 2 seconds, then we send. The distribution is a truncated exponential: $E[\text{busy}] = \int_0^2 e^{-\lambda t} dt = \dfrac{1 - e^{-2\lambda}}{\lambda} = 1 - e^{-2} \simeq 0.864$.

Fraction of time spent empty is $P_{empty} = \dfrac{E[\text{empty}]}{E[\text{empty}] + E[\text{busy}]} = 0.536$.

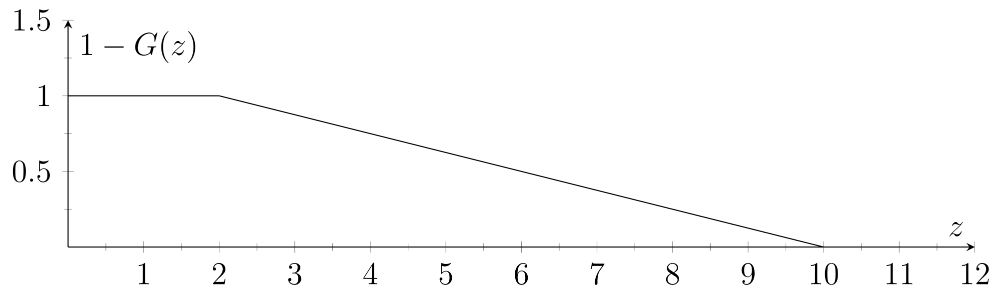
If a packet finds the queue non empty, then transmission is immediate. Otherwise it has to wait $1 - e^{-2}$ on average. By law of total probability we have:

$E[\text{delay}] = E[\text{delay}|\text{empty}]P_{empty} + E[\text{delay}|\text{busy}]P_{busy} = 0.864 \cdot 0.536 = 0.463$.

Problem 3

This scheme is clearly a $M/G/\infty$ queue.

Let $X(t)$ be the number of occupied channels at time t. $X(t) \sim \mathcal{P}(\Lambda)$.



So $E[X(t)] = \Lambda$, where $\Lambda = \lambda \int\limits_0^t [1 - G(z)]dz = \lambda \int\limits_0^t e^{-\mu t} = \frac{\lambda}{\mu}(1 - e^{-\mu t}) = \begin{cases} 10(1 - e^{-1}) & t = 6 \\ 10(1 - e^{-\frac{10}{6}}) & t = 10 \\ 10 & t = \infty \end{cases}$.

For the second point $\mathrm{P}[X(t) = 10] = \dfrac{\Lambda^1 0 e^{-\Lambda}}{10!} = \begin{cases} 0.05 & t = 6 \\ 0.125 & t = \infty \end{cases}$.

For the last point we have to compute Λ again, now for an uniform duration of Y between 2 and 10 minutes:

$\Lambda = \lambda \int\limits_0^t [1 - G(z)]dz = \begin{cases} 6\frac{100}{60} & t = 10, \infty \\ \frac{100}{60}(6 - 1) & t = 6 \end{cases}$.

These results come from the graphical interpretation of $1 - G(z)$. It is 1 between 0 and 2, then it linearly goes to 0 in 10, then it remains 0. The three integrals can be easily computed by analyzing the area under the function.

Problem 4

Transition matrix is $P = \begin{bmatrix} 0.99 & 0.01 \\ 0.1 & 0.9 \end{bmatrix}$. The protocol matrix is $C = \begin{bmatrix} p_{00} & p_{01} \\ p_{10}^{(m)} & p_{11}^{(m)} \end{bmatrix}$.

Reward and time vectors are $\mathbf{R} = \begin{bmatrix} R_G \\ R_B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\quad \mathbf{T} = \begin{bmatrix} T_G \\ T_B \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Throughput is then $\mathcal{T} = \dfrac{\sum_i \pi_i R_i}{\sum_i \pi_i T_i} = \dfrac{\pi_G}{\pi_G + 2\pi_B} = \dfrac{p_{10}^{(2)}}{p_{10}^{(2)} + 2p_{01}}$.

For the last question we have $\mathcal{T} = \dfrac{(1-\delta)p_{10}^{(2)}}{(1-\delta)p_{10}^{(2)} + 2\left((1-\delta)p_{01} + \delta p_{01}^{(2)} + \delta p_{10}^{(2)}\right)}$, where $\delta = 0.1$.