Deligne-Lusztig Varieties Talk 3 (Ashwin)

Intro/Reductive Groups

Let G be a reductive group defined over $\mathbb{F}_q$, $G = G(\overline{\mathbb{F}}_p)$

Def A Borel subgroup is a maximal connected solvable subgroup variety of G

Rk A Theorem of Lie-Kolchin says that connected solvable smooth algebraic groups have only 1-dim irred. representation.

Pf Treat H abelian, induct $\dim H/[H,H] < \dim H$

If I take $H \to GL_n$, this can be conjugated to upper-triangular matrices.

Ex B Borel in GL_n, $B \hookrightarrow GL_n$ is a rep^n, so has to land in gB_ng^{-1}, $g \in GL_n$, B_n = uppertriangular matrices

Remark For $B \subseteq G$, I can construct a quotient G/B as an algebraic variety.

Prop G/B is projective

Pf For GL_n, one can do this directly
For general group, $\exists$ $G \to GL(V)$ and a line $L \subset V$ s.t. $B = \mathrm{stab}(L)$

For GL_n, $\{ gB_ng^{-1} : g \in G\}$ are all the Borels.

Prop Any 2 Borels $B, B' \subset G$ are conjugate in G

Pf $B' \circlearrowright G/B$ action
$b' \cdot gB \to b'gB$

Thm (Borel's Fixed point theorem)
Every action of a smooth connected solvable alg. group on a projective variety has a fixed point

$\exists gB$ s.t. $B'gB = gB$
$\Rightarrow B' = gBg^{-1}$

Prop $B = N_G(B)$

Cor The map $G/B \to \mathcal{B} := \left\{ \begin{matrix} \text{the set of Borels} \\ \text{in } G \end{matrix} \right\}$

$$gB \to gBg^{-1}$$

is a bijection.

Ex GL_n? We established that the Borels are just gB_ng^{-1}

But if we let $e_1, \dots, e_n$ to be the std. basis $\mathbb{F}_p^n$.

$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, \dots, e_{n-1} \rangle \subset \overline{\mathbb{F}}_p^n$ flag

B_n stabilizes this flag

Then if I look at gB_ng^{-1}

$0 \subset \langle ge_1 \rangle \subset \langle ge_1, ge_2 \rangle \subset \dots \subset \langle ge_1, \dots, ge_{n-1} \rangle \subset \overline{\mathbb{F}}_p^n$

gB_ng^{-1} fixes this flag.

every step adds in dim.

$$G/B_n \to \mathcal{B} \xrightarrow{\sim} \left\{ \begin{matrix} \text{complete flags} \\ \text{in } \mathbb{F}_p^n \end{matrix} \right\}$$

$$\underset{\substack{\shortparallel \\ gBg^{-1}}}{B} \mapsto F \overset{i}{=} \langle ge_1, \dots, ge_i \rangle$$

$$\mathrm{Stab}(F) \longleftarrow F$$

(+ Schubert varieties)

Deligne-Lusztig varieties give certain stratifications of $X := G/B$

Weyl group

Let T be a maximal torus, i.e. maxl alg. subgroup of G

s.t. $T \simeq (\mathbb{G}_m)^n$

Def $W_G = N_G(T)/T$

Rk W_G doesn't depend on T.

Note T smooth, connected, solvable, so $\exists\, B \supset T$ (B Borel).

If T' another maxl torus, $T' \subset B'$

But $gB'g^{-1} = B$. So $gT'g^{-1} \subseteq B$ is another maximal torus inside B. But one can show $gT'g^{-1}$ and T are conjugate by an element in B_u = unipotent radical in B.

GL_2 : $\left\{\begin{pmatrix} a & c-a \\ & c \end{pmatrix}\right\}$ $\qquad B_u = \left\{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\right\}$

Prop (Bruhat Decomposition)

$$G = \bigsqcup_{w \in W_G} BwB \qquad (\text{For any Borel } B)$$

Pick $T \subseteq B$ $\quad w \in W_G = N_G(T)/T$, lift to $\tilde{w} \in N_G(T)$

$$N_G(T)/T \longrightarrow B\backslash G/B$$
$$w \mapsto \tilde{w} \mapsto B\tilde{w}B$$

is a bijection.

Exercise : double coset

$$B\backslash G/B \longrightarrow G\backslash(X \times X) \qquad \text{orbit space}$$

coset G/B $\;=\; X$

$$BgB \longmapsto [(eB, gB)]$$

is a bijection

For GL_n, $W = S_n$

$B\tilde{w}B$

lift to a permutation matrix (row reductions)

If $B, B' \subset GL_n$ are 2 Borels, they always contain a common maximal torus.

$$F_B = 0 \subset F_B^1 \subset F_B^2 \subset \cdots \subset F_B^n = \overline{\mathbb{F}}_p^{\,n}$$
$$F_{B'} = 0 \subset F_{B'}^1 \subset F_{B'}^2 \subset \cdots \subset F_{B'}^n = \overline{\mathbb{F}}_p^{\,n}$$

Given any two you can find a basis

Prop You can find

$$\overline{\mathbb{F}}_p^{\,n} = L_1 \oplus \cdots \oplus L_n \qquad \text{basis}$$

s.t.

$$F_B^i = L_1 \oplus \cdots \oplus L_i$$
$$F_{B'}^i = L_{\sigma(1)} \oplus \cdots \oplus L_{\sigma(i)}$$

$\sigma \in S_n$.

$G \circlearrowright X \times X = G/B \times G/B$

Morphism

$$G \times (G/B \times G/B) \to G/B \times G/B$$

The orbits, i.e.

image $G \times \{(x_1, x_2)\} \to X \times X$ is locally closed (open in its closure)

<u>Fact</u> orbits are smooth. $w \in W$, $O(w)$ is smooth

G acts transitively on O_w

$w \in W_G = G \backslash (X \times X)$ → orbit space

<u>Fact</u> $\dim O(w) = \dim X + \ell(w)$ length of w

W is a coxeter group, so gen. by simple reflections

$w = e$, $O(w) = X \hookrightarrow X \times X$

$x \longmapsto (x, x)$

There is a natural partial order on W,

$$\overline{O(w)}^{X \times X} = \bigcup_{w' \leq w} O(w')$$

$F : X \to X$ q-Frobenius $q = p^t$

<u>Def</u> Deligne-Lusztig variety

$$X(w) := \Gamma_F \cap O(w) \subset X \times X \qquad \text{smooth}$$

$\Gamma_F = \{(B, F(B))\}$, $X(w) \to X$

<u>Def</u> Schubert variety

$\pi_1 : X \times X \to X$

$$S(w) = \overline{(X \times 1) \cap O(w)}$$

like the graph of the identity

$$S(w) = \overline{\{x \in X : \text{rel}(X, \text{std Borel}) = w\}}$$

<u>Ex</u>

$G = GL_2$ $\quad W = S_2 = \{1, w\}$

$X = \mathbb{P}^1$

$X \times X = \mathbb{P}^1 \times \mathbb{P}^1$

$X(e) = \{B : B \cong F(B)\}$ B defined over $\mathbb{F}_q$

$= X^F$ $\qquad X^F$