Math 245B Notes

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These are course notes for Math245B, Topics in Algebraic Geometry, Winter 2023, taught by Pol van Hoften.

Contents

1 Lecture 1: 1/9/23

The course will be on Deligne–Lusztig Theory. In short, we will attempt to understand the $\mathbf{C}$-representation theory of finite groups of Lie types; e.g. $\mathrm{GL}_3(\mathbf{F}_8), \mathrm{SP}_8(\mathbf{F}_{27}), \mathrm{SO}_5(\mathbf{F}_3)$. Goal: "construct" all the representations of these groups.

Example 1.1. *Consider* $\mathrm{SL}_2(\mathbf{F}_p)$ *for* $p > 2$ *(or* $\mathbf{F}_q$ *for* $q = p^r$*). Inside this group we have* $B(\mathbf{F}_p)$*, the* Borel *subgroup of upper triangular groups, and inside* B *we have* $T(\mathbf{F}_p)$*, the (abelian) subgroup of diagonal matrices. Given a character* $\theta : T \to \mathbf{C}^\times$*, we consider it as a representation of* $B(\mathbf{F}_p)$ *via the quotient* $B(\mathbf{F}_p) \to T(\mathbf{F}_p)$ *(mod out by the normal subgroup that is upper triangular matrices with 1's on the diagonal), and then* $Ind_{B(\mathbf{F}_p)}^{\mathrm{SL}_2(\mathbf{F}_p)}\theta$ *is a* SL_2*-representation.*

If θ *is the trivial representation 1, then* $Ind_{B(\mathbf{F}_p)}^{\mathrm{SL}_2(\mathbf{F}_p)}\theta$ *is just the functions from* $\mathbf{P}^1(\mathbf{F}_p) \cong \mathrm{SL}_2(\mathbf{F}_p)/B(\mathbf{F}_p)$ *to* $\mathbf{C}$.

There is a short exact sequence

$$0 \to \mathbf{C} \xrightarrow{cst} Ind_{B(\mathbf{F}_p)}^{\mathrm{SL}_2(\mathbf{F}_p)} 1 \to st \to 1,$$

where st *is the* Steinberg *representation. Exercise: prove it is irreducible.*

Next time: If $\theta^2 \neq 1$, then $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{SL}_2(\mathbf{F}_p)}\theta \cong \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{SL}_2(\mathbf{F}_p)}\theta^{-1}$.

Fact: If $p > 2$ and $q = p^r$, then we get $\frac{q+5}{2}$ irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$ from inducing up from B.

Exercise: $\mathrm{SL}_2(\mathbf{F}_q)$ has $q + 4$ conjugacy classes. In lieu of doing the computation (using Jordan canonical forms and taking care that we only conjugate by elements in SL_2, not GL_2), we just give the following classification:

Representative	# of Elements	# of Classes
$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	1
$-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	1	1
$J_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\frac{q^2-1}{2}$	1
$J_2 = \begin{bmatrix} 1 & \epsilon \\ 0 & 1 \end{bmatrix}$	$\frac{q^2-1}{2}$	1
$J_3 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$	$\frac{q^2-1}{2}$	1
$J_4 = \begin{bmatrix} -1 & \epsilon \\ 0 & -1 \end{bmatrix}$	$\frac{q^2-1}{2}$	1
$c_x = \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}, x \neq \pm 1$	q^2+q	$\frac{q-3}{2}$
$d_{x,y} = \begin{bmatrix} x & y \\ \epsilon y & x \end{bmatrix}, x \neq \pm 1, x^2 - \epsilon y^2 = 1$	q^2-q	$\frac{q-1}{2}$

,

where ϵ is a generator of $\mathbf{F}_q^\times$. Note that c_x and $c_{x^{-1}}$ determine the same conjugacy class (hence exactly $\frac{q-3}{2}$ classes labeled c), and so do $d_{x,y}$ and $d_{x,-y}$. We can identify $d_{x,y}$ with $\zeta = x + y\sqrt{\epsilon}$ in the cyclic subgroup $\{\zeta \in (\mathbf{F}_{q^2})^\times : \zeta^{q+1} = 1\}$, where we skip over the elements ± 1, and ζ, ζ^{-1} determine the same class (hence exactly $\frac{q+1-2}{2} = \frac{q-3}{2}$ classes labeled d). Moreover, $|\mathrm{SL}_2(\mathbf{F}_q)| = q(q-1)(q+1)$, so the above list accounts for all elements. There are indeed a total of $6 + \frac{q-3}{2} + \frac{q-1}{2} = q+4$ conjugacy classes.

Hence the other half of the representations must come from a different construction. MacDonald's conjecture says that these are related to characters to $T'(\mathbf{F}_q) \subseteq SL_2(\mathbf{F}_q)$. Idea: take another maximal torus $\mathbf{F}_{q^2}^\times \subseteq \mathrm{GL}_2(\mathbf{F}_q)$ (think of $\mathbf{F}_{q^2}$ as a 2-dimensional vector space, and its elements acting on this space), and $\mathbf{F}_{q^2}^\times \cap \mathrm{SL}_2 =: \mu_{q+1}$. Want to induct up. The issue is that there is no analogue of $B(\mathbf{F}_q)$ containing $T'(\mathbf{F}_q)$.

Drinfeld: look at the curve C: $xy^q - yx^1 = 1$ inside $\mathbf{A}^2_{\mathbf{F}_q}$. This has commuting actions of $\mathrm{SL}_2(\mathbf{F}_q)$ and μ_{q+1}, given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot (x,y) = (ax+by, cx+dy)$ and $\zeta \cdot (x,y) = (\zeta x, \zeta y)$. Then for $\theta : \mu_{q+1} \to \mathbf{C}^\times \cong \overline{\mathbf{Q}_\ell}$, look at $H^1_{et}(C_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}_\ell})[\theta]$. Drinfeld says these should explain the rest of the irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$.

Remark: C is a μ_{q+1}-cover of $\mathbf{P}^1_{\mathbf{F}_q} - \mathbf{P}^1(\mathbf{F}_q)$, see [1, Sec. 2.2.3]

In the first three weeks of this course we will prove this result, mostly following [1].

1.1 Representation Theory Preliminaries

Definition 1.2. *Let G be a finite group, and k a field. Then a k-linear* representation *of G is a pair (V, π) where V is a finite dimensional k-vector space, and $\pi : G \times V \to V$ is an action of G that is k-linear. Can also think of it as a homomorphism $\rho : G \to \mathrm{GL}(V)$.*

A *morphism* of representations (V, π) and (V', π') is a map $f : V \to V'$ such that

$$\begin{tikzcd} G \times V \arrow[r, "1\times f"] \arrow[d, "\pi"] & G \times V' \arrow[d, "\pi'"] \\ V \arrow[r, "f"] & V' \end{tikzcd}$$

We will write $Rep_k(G)$ for the category of k-linear representations of G.

Fact: This category is abelian, and the forgetful functor $Rep_k(G) \to Vect_k$ commutes with limits and colimits. In other words, the categorical kernels and quotients are just given by the kernels and quotients of the maps on vector spaces, equipped with the natural action of G.

Fact: there is a tensor product of representations, which is the usual thing on underlying vector spaces.

Theorem 1.3 (Maschke)**.** *If $|G|$ is invertible in k, then $Rep_k(G)$ is semisimple; i.e. all representations are a direct sum of irreducible representations.*

Definition 1.4. *Given (V, π, ρ) a representation, there is a function $\chi_V = \chi_\pi = \chi_\rho : G \to k$, the* character*, which is given by $g \mapsto tr(\rho(g))$.*

Observe that χ_V is conjugation-invariant (as trace is).

Theorem 1.5 (Schur Orthogonality)**.** *If $|G| \in k^\times$, and V, V' are representations, then $\langle \chi_V, \chi_{V'} \rangle \coloneqq \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\chi_{V'}(g^{-1}) = \dim Hom_G(V, V')$ holds* in k *(consider the right hand side inside k).*

Note that in $\mathbf{C}$, this is $\frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_{V'}(g)}$ (complex eigenvalues lie on the unit circle, where inverse equals conjugate).

Proof. Sketch: The LHS is

$$\frac{1}{|G|} \sum_{g \in G} \chi_V(g)\chi_{V'}(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(g|\mathrm{Hom}(V, V')),$$

(the right hand side means the trace of g acting on the representation $\mathrm{Hom}(V, V')$), and for any representation W, it is not hard to see that $\frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(g|W) = \dim W^G$, the space of G-fixed elements of W. $\square$

Fact: If $|G|$ is invertible in k and k algebraically closed, consider the k-vector space of conjugation-invariant maps $G \to k$. Then the inclusion of the set of irreducible characters into this space is an equality.

2 Lecture 2: 1/11/23

Today: Mackey theory and $\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)}\theta$.

Fix a finite group G and an algebraically closed field k such that $|G| \in k^\times$ (can take $k = \mathbf{C}$, but for now we can work with general $k = \overline{k}$).

If $H \subseteq G$ is a subgroup, there is a functor $Rep_k(G) \to Rep_k(H)$ given by restriction. It has both a left and right adjoint (the same), which we will describe. It is the *induced representation* $\mathrm{Ind}_H^G : Rep_k(H) \to Rep_k(G)$, given by $V \mapsto \{f : G \to V : f(hg) = \rho_V(h)f(g)\ \forall h \in H, g \in G\}$. The adjointness was proven by Frobenius (Frobenius reciprocity), in terms of characters.

Remark 2.1. *Dimension of $Ind_H^G V$ is $\dim V \cdot [G : H]$.*

Notation: For a representation V of H we will use χ_V^G to denote the character of $Ind_H^G V$. For a representation W of G we will use $(\chi_W)_H$ for the character of the restriction of W to H.

Theorem 2.2 (Frobenius reciprocity). *$\langle \chi_V^G, \chi_V^G\rangle_G = \langle \chi_V, (\chi_V^G)_H\rangle_H$. More generally, for χ a character of G and θ a character of H, we have $\langle \theta^G, \chi\rangle_G = \langle \theta, \chi_H\rangle_H$ (χ_H being the restriction of χ to H).*

Recall that $\mathrm{Ind}_H^G V$ is irreducible if and only if $\langle \chi_V^G, \chi_V^G\rangle_G = 1$.

Definition 2.3. *For $g \in G$, write H^g for gHg^{-1}. For $\rho : H \to \mathrm{GL}(V)$, write $\rho^g = gHg^{-1} \to GL(V)$ given by $ghg^{-1} \mapsto \rho(h)$.*

Then:

Theorem 2.4 (Mackey decomposition). *$Res_H^G Ind_H^G \rho = \bigoplus_{[g]\in H\backslash G/H} Ind_{H\cap H^g}^H Res_{H\cap H^g}^{H^g} \rho^g$.*

Corollary 2.5. *$Ind_H^G V$ is irreducible if and only if V is irreducible (with representation ρ and character χ) and $Res_{H\cap H^g}^{H^g}\rho^g$ and $Res_{H\cap H^g}^{H^g}\rho$ share no common irreducible factors (other than when $g = 1$).*

Proof. Using Frobenius reciprocity, we have

$$\begin{aligned}
\langle \chi_V^G, \chi_V^G\rangle_G &= \langle \chi_V, (\chi_V^G)_H\rangle_H \\
&= \langle \chi_V, \sum_{[g]\in H\backslash G/H} \chi_{\mathrm{Ind}_{H\cap H^g}^H \mathrm{Res}_{H\cap H^g}^{H^g}\rho^g}\rangle \\
&= \sum_{[g]\in H\backslash G/H} \langle \chi_V, \chi_{\mathrm{Ind}_{H\cap H^g}^H \mathrm{Res}_{H\cap H^g}^{H^g}\rho^g}\rangle \\
&= \sum_{[g]\in H\backslash G/H} \langle \mathrm{Res}_{H\cap H^g}^{H^g}\chi, \mathrm{Res}_{H\cap H^g}^{H^g}\chi^g\rangle.
\end{aligned}$$

We want to know when this is 1. It is certainly at least 1 (coming from when $g = 1$), so is equal to 1 precisely when the summands for $[g] \neq 1$ vanish. $\square$

Let's apply this to $G = \mathrm{SL}_2(\mathbf{F}_q)$, $H = B(\mathbf{F}_q)$. Let $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}_2(\mathbf{F}_q)$, and the conjugation of a generic matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{F}_q)$ is $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$. So conjugation by S preserves $T(\mathbf{F}_q)$ (diagonal matrices), and acts as -1 on it (it swaps a and d, which must be inverses as we're in $\mathrm{SL}_2(\mathbf{F}_q)$). Also, this conjugation makes an upper triangular matrix lower triangular, and vice versa. Hence $B(\mathbf{F}_q) \cap SB(\mathbf{F}_q)S^{-1} = T(\mathbf{F}_q)$.

Lemma 2.6 (Bruhat decomposition)**.** $\mathrm{SL}_2(\mathbf{F}_q) = B(\mathbf{F}_q) \cup B(\mathbf{F}_q)SB(\mathbf{F}_q)$.

Proof. Suppose $g := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{F}_q)$ is not in $B(\mathbf{F}_q)$. Then $c \neq 0$, and one checks that $S^{-1} \begin{bmatrix} 1 & -\frac{a}{c} \\ 0 & 1 \end{bmatrix} g \in B(\mathbf{F}_q)$. Rearrange terms to get the answer. $\square$

The upshot is that if we start with $\theta_1, \theta_2 : T(\mathbf{F}_q) \to \mathbf{C}^\times$, and consider them as representations of the quotient map $B(\mathbf{F}_q) \to T(\mathbf{F}_q)$ (see Lecture 1), then we have

$$\langle \mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)} \theta_1, \mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)} \theta_2 \rangle_{\mathrm{SL}_2} = \langle \theta_1, \theta_2 \rangle_T + \langle \theta_1, \theta_2^{-1} \rangle_T.$$

Proof. By Frobenius reciprocity and Mackey's formula, the LHS is equal to

$$\langle \theta_1, \mathrm{Res}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)} \mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)} \theta_2 \rangle_B = \langle \theta_1, \theta_2 \rangle_T + \langle \theta_1, \theta_2^S \rangle_T = \langle \theta_1, \theta_2 \rangle_T + \langle \theta_1, \theta_2^{-1} \rangle_T.$$

In particular, note that the double cosets of B in SL_2 are represented by either 1 or S, so that $B \cap B^g$ (in Mackey's formula) for $g \in \{1, S\}$ is either $T(\mathbf{F}_q)$ or B. $\square$

Corollary: if $\theta_1 = \theta_2 = \theta$, we find that $\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)} \theta$ is irreducible if $\theta \neq \theta^{-1}$. If θ_1 is in $\{\theta_2, \theta_2^{-1}\}$, then $\mathrm{Ind}\theta_1, \mathrm{Ind}\theta_2$ share no common factors.

So now, if $p > 2$ and q is a power of p, then there are $q - 3$ characters θ with $\theta \neq \theta^{-1}$, so $\frac{q-3}{2}$ irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$. We have $\mathrm{Ind}1 = 1 + st$ (the "Steinberg representation"), and Ind_α (for $\alpha \neq 1$, $\alpha^2 = 1$) splits into irreducibles as $R(\alpha)_+ + R(\alpha)_-$. Fact: $R(\alpha)_+ \not\cong R(\alpha)_-$, but they have the same dimension. This gives $\frac{q-3}{2} + 4 = \frac{q+5}{2}$ irreducible representations.

Definition 2.7. *A representation of* $\mathrm{SL}_2(\mathbf{F}_q)$ *that does* not *contain any of the previous* $\frac{q+5}{2}$ *irreducible representations as a summand is called* cuspidal*.*

Exercise (if you know what the words mean): Consider the natural quotient map $\mathrm{SL}_2(\mathbf{Z}_p) \to \mathrm{SL}_2(\mathbf{F}_p)$ inside $\mathrm{SL}_2(\mathbf{Q}_p)$. Let $\mathrm{SL}_2(\mathbf{Z}_p)$ act on V via a cuspidal representation. Then $c\text{--}\mathrm{Ind}_{\mathrm{SL}_2(\mathbf{Z}_p)}^{\mathrm{SL}_2(\mathbf{Q}_p)} V$ is cuspidal.

3 Lecture 3: 1/13/23

Today: introduction to ℓ-adic etale cohomology.

Let X be a smooth projective variety over $\mathbf{F}_p$. Define

$$Z_X \coloneqq \exp\left(\sum_{n\geq 1} |X(\mathbf{F}_{q^n})| \frac{T^n}{n}\right) \in \mathbf{Q}[[T]].$$

Example: if $X = \mathrm{Spec}(\mathbf{F}_q)$, then $Z_X = \exp\left(\sum_{n\geq 1} \frac{T^n}{n}\right) = \exp(-\log(1-T)) = \frac{1}{1-T}$ (ignoring issues like exp of log is 1, but this actually works).

Another example: if $X = \mathbf{P}^1_{\mathbf{F}_q}$, then $Z_X = \frac{1}{(1-T)(1-qT)}$.

Another example: if X is an elliptic curve, there are $\alpha, \beta \in \overline{\mathbf{Q}}$ such that $Z_X = \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-qT)}$.

One of the Weil conjectures: Z_X is always a rational function (proved by Dwork in the 1960s).

Proof. Idea: we are counting fixed points of Frob^n on $X_{\overline{\mathbf{F}_q}}$. Here is a useful fact (Lefschetz fixed-point theorem): if M is a compact oriented manifold, and $\psi : M \to M$ continuous with isolated fixed points, then the number of fixed points of ψ is $\sum_i (-1)^i \mathrm{tr}(\psi_*, H^i_{sing}(M, \mathbf{R}))$. Exponential generating function of $\mathrm{Fix}(\psi^m)$ is nice. □

The following linear algebra lemma was stated orally in class, but never written down.

Lemma 3.1. *Let V be a finite dimensional vector space over a field K and let $f : V \to V$ be an endomorphism. Then*

$$\exp\left(\sum_{n\geq 1} \mathrm{Tr}(f^n, V) \frac{T^n}{n}\right) \in K[[T]] \tag{1}$$

is given by the characteristic polynomial of f.

Proof. exercise □

Question: is there an "algebraic definition" of singular cohomology of nice X over $\mathbf{C}$? We know $H^0_{sing}(X(\mathbf{C}), \mathbf{Q})$ is $\mathbf{Q}$[connected components] (vector space with basis corresponding to the connected components). Also, $H^1_{sing}(X(\mathbf{C}), \mathbf{Z}) = \pi_1(X(\mathbf{C}))_{ab}$, and $C^\times$ has a $\mathbf{Z}$-cover given by the exponential map, which is not algebraic. However, Riemann existence theorem proves that all *finite* covering spaces are algebraic, so that $H^1_{sing}(X(\mathbf{C}), \mathbf{Z}/n\mathbf{Z})$ "has an algebraic definition".

Serre has a simple argument that shows that there cannot exist a cohomology theory for nice varieties over $\overline{\mathbf{F}}_p$ in $\mathbf{Q}$-vector spaces such that H^1(Elliptic curve) is 2-dimensional. The issue is that there is an elliptic curve E over $\overline{\mathbf{F}}_p$ such that $\hom(E, E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a quaternion division algebra over $\mathbb{Q}$, and such algebras cannot act nontrivially on a two-dimensional $\mathbb{Q}$-vector space. However one can show that the n-th power map $[n] \in \hom(E, E)$ has to induce multiplication by $[n]$ on cohomology.

So we could hope to define a cohomology theory with values in $\mathbf{Z}/\ell^n\mathbf{Z}$ for $\ell \neq p$. Should hope to get a theory with $\varprojlim \mathbf{Z}/\ell^n\mathbf{Z} = \mathbf{Z}_\ell$ or in $\mathbf{Z}_\ell[1/\ell] = \mathbf{Q}_\ell$. This is possible (Grothendieck-Deligne-Artin).

Facts: there is a contravariant functor (for all $\ell \geq 0$, $\ell \neq p$) $H^i_{et}(-, \mathbf{Q}_\ell)$ from the category of nice varieties over $\overline{\mathbf{F}}_p$ to the category of finite-dimensional $\mathbf{Q}_\ell$ vector spaces. Here are its properties:

1. It is 0 unless $0 \leq i \leq 2\dim(X)$ (since turning a complex manifold into a real manifold doubles the dimension).
2. $H^0_{et}(X, \mathbf{Q}_\ell)$ is just the vector space over $\mathbf{Q}_\ell$ with basis corresponding to the connected components.
3. If X lifts to $\widetilde{X}$ over $\mathbf{C}$, then
$$H^i_{sing}(\widetilde{X}(\mathbf{C}), \mathbf{Q}_\ell) = H^i_{sing}(\widetilde{X}(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \cong H^i_{et}(X, \mathbf{Q}_\ell).$$
4. Poincare duality: assuming X equidimensional of dimension d, then $H^i_{et}(X) \cong H^{2d-i}_{et}(X)^\vee$.
5. If X is defined over $\mathbf{F}_q$, $|X(\mathbf{F}_{q^m})| = \sum_i (-1)^i \mathrm{tr}(\mathrm{Frob}_{q^m}, H^i_{et}(X_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell))$.
6. If $\psi : X \to X$ has isolated fixed points, then $|\mathrm{Fix}(\psi)| = \sum_i (-1)^i \mathrm{tr}(\psi_*, H^i_{et}(M, \mathbf{Q}_\ell))$.

There is an extension: a contravariant functor $H^i_c(\_, \mathbf{Q}_\ell)$ ("c" stands for "compact") from the category of (all) varieties over $\overline{\mathbf{F}}_p$ with proper maps, to the category of finite-dimensional $\mathbf{Q}_\ell$ vector spaces, such that:

1. $H^i_c(X, \mathbf{Q}_\ell) = H^i(X, \mathbf{Q}_\ell)$ if X is proper/projective.
2. Vanishes unless $i \in [0, 2\dim(X)]$.
3. If X is smooth and affine, then (Artin vanishing) $H^i_c(X, \mathbf{Q}_\ell) = 0$ for $0 \leq i \leq \dim(X)$.
4. if $Z \subseteq X$ is closed and $U = X - Z$, then there is a long exact sequence
$$\ldots \to H^i_c(U, \mathbf{Q}_\ell) \xrightarrow{\text{extension by 0}} H^i_c(X, \mathbf{Q}_\ell) \xrightarrow{\text{restriction}} H^i_c(Z, \mathbf{Q}_\ell) \to H^{i+1}_c(U, \mathbf{Q}_\ell) \to \ldots$$
5. Same as previous (5).
6. Same as previous (6).

Let C be the Drinfeld curve $(xy^q - x^q y = 1)$ over F_q equipped with actions of $\mathrm{SL}_2(\mathbf{F}_q)$ and μ_{q+1}. Let θ be a character of μ_{q+1} with values in $\mathbf{Q}_\ell$. Then:

Definition 3.2 (Deligne-Lusztig induction)**.** *Write $R(\theta)$ for $H^0_c(C_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell)[\theta] - H^1_c(C_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell)[\theta] + H^2_c(C_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell)[\theta]$, where $[\theta]$ denotes $Hom_{\mu_{q+1}}(\theta, \_)$.*

No lecture Monday 1/16/23 (MLK day).

4 Lecture 4: 1/18/23

Recall the *Drinfeld curve C* given by $xy^q - x^q y = 1$ inside $\mathbf{A}^2_{\mathbf{F}_q}$, where $q = p^r$. This has commuting actions of $\mathrm{SL}_2(\mathbf{F}_q)$ and μ_{q+1}, given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot (x, y) = (ax + by, cx + dy)$ and $\zeta \cdot (x, y) = (\zeta x, \zeta y)$ (see Lecture 1).

Observation: $C(\mathbf{F}_q) = \varnothing$, since $x = x^q$ and $y = y^q$.

We define, for a character $\theta_{q+1} \to \overline{\mathbf{Q}}_l^\times$ (or $\mathbf{C}^\times$), the *virtual representation* $R'(\theta) := H^2_c(C_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}}_l)[\theta] - H^1_c(C_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}}_l)[\theta]$. Here, for $W \in \mathrm{Rep}(\mu_{q+1})$, we write $W[\theta] = \{w \in W : \zeta w = \theta(\zeta) w\}$.

We start by computing $R'(1)$. We have

$$H^i_c(C_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}}_l) = H^i_c((C_{\overline{\mathbf{F}_q}}/\mu_{q+1}, \overline{\mathbf{Q}}_l)$$

as $\mathrm{SL}_2(\mathbf{F}_q)$-representations (using that the actions commute).

Lemma 4.1. *The map $C \to \mathbf{P}^1_{\overline{\mathbf{F}_q}} - \mathbf{P}^1(\mathbf{F}_q)$ realizes the target as the quotient C/μ_{q+1}.*

Proof. We need that:

1. $[\zeta x, \zeta y] = [x, y]$ in $\mathbf{P}^1_{\mathbf{F}_q}$ (clear).
2. The map is surjective on $\overline{\mathbf{F}_q}$-points.
3. If $(\lambda, \lambda x)$ and $(\lambda', \lambda' x)$ are points of $C_{\overline{\mathbf{F}_q}}$, then $\lambda = \zeta \lambda'$ for some $\zeta \in \mu_{q+1}$.

For (2), note that given $[1, x] \in \mathbf{P}^1_{\overline{\mathbf{F}_q}} - \mathbf{P}^1(\mathbf{F}_q)$, then $x^q \neq x$. We want to find $\lambda \in \overline{\mathbf{F}_q}^\times$ such that with $y = \lambda$, we have $[\lambda, \lambda x]$ implying $\lambda^{q+1} x^q - \lambda^{q+1} x = 1$. We can solve this using linear algebra ($q + 1$'st roots exist in $\overline{\mathbf{F}_q}^\times$).

For (3), note that $\lambda^{q+1} x^q - \lambda^{q+1} x = (\lambda')^{q+1} x^q - (\lambda')^{q+1} x$, so that $\lambda^{q+1} = (\lambda')^{q+1}$.

So we conclude that $C(\overline{\mathbf{F}_q})/\mu_{q+1} = (\mathbf{P}^1_{\overline{\mathbf{F}_q}} - \mathbf{P}^1(\mathbf{F}_q))(\overline{\mathbf{F}_q})$. $\square$

Now, we compute $H^1_c(U_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}}_l)$ (with $U = \mathbf{P}^1_{\overline{\mathbf{F}_q}} - \mathbf{P}^1(\mathbf{F}_q)$) via the long exact sequence (see Lecture 3) for $X = \mathbf{P}^1_{\overline{\mathbf{F}_q}}$, $Z = \mathbf{P}^1(\mathbf{F}_q)$, $U = X - Z$. Note that $H^0_c(Z_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}}_l) = \mathrm{Fun}(\mathbf{P}^1(\mathbf{F}_q), \overline{\mathbf{Q}}_l) = 1 \oplus st$.

Another fact: $H^i_c(\mathbf{P}^1_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}}_l) = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \\ 1 & i = 2 \end{cases}$ as $\mathrm{SL}_2(\mathbf{F}_q)$-representations.

Going back to the long exact sequence, we get

$$0 \to H^0_c(\mathbf{P}^1) \to H^0_c(Z) \to H^1_c(U) \to H^1_c(\mathbf{P}^1) \to 0 \to H^2_c(\mathbf{P}^1) \xrightarrow{\simeq} H^2_c(U) \to 0.$$

Note that $H^0_c(\mathbf{P}^1)$ is 1, $H^0_c(Z)$ is $1 + st$, and the map is injective and SL_2-equivariant. Hence we conclude it maps 1 to 1. The upshot is that $R(1) = 1 - st$ and $R'(1) = st - 1$.

Observation: for $\zeta \in \mu_{q+1}, \zeta \neq 1$, we have $C^{\zeta}_{\overline{\mathbf{F}_q}} = \varnothing$ (clear).

The trace formula tells us that

$$\mathrm{tr}(\zeta, H^2_c(C)) - \mathrm{tr}(\zeta, H^1_c(C)) = 0.$$

This also characterises the regular representation of μ_{q+1}. So the character of the (virtual) representation $H^1_c(C) - H^2_c(C)$ is a multiple of the regular representation of μ_{q+1}. On applying $[\theta]$, for $\theta \neq 1$, we get an actual character. Upshot: the "degree" of $H^1_c(C)[\theta]$ equals the "degree" of $H^1_c(C)[1] - H^2_c(C)[1]$. So $H^1_c(C)[\theta]$ has dimension $q - 1$.

Theorem 4.2. *If $\theta \neq 1$, then $H^1_c(C_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}}_l)[\theta]$ is cuspidal (recall Definition 2.7).*

Proof. Let U be the subgroup of matrices of the form $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ inside $\mathrm{SL}_2(\mathbf{F}_q)$. Consider the functors

$$Rep_{\overline{\mathbf{Q}}_l} T \to Rep_{\overline{\mathbf{Q}}_l} B \xrightarrow{\text{induce}} Rep_{\overline{\mathbf{Q}}_l} \mathrm{SL}_2(\mathbf{F}_q).$$

To go backwards we restrict to B, and take U-coinvariants (quotient by submodule generated by all $x - ux$ for $u \in U$).

For us, it suffices to show that $(H^1_c(C)[\theta])_U = (H^1_c(C)[\theta])^U = 0$. So we need to understand $H^1_c(C_{\overline{\mathbf{F}_q}}/U, \overline{\mathbf{Q}}_l)$ with its action of μ_{q+1}. What is the quotient by U? Looking back at the action of $\mathrm{SL}_2(\mathbf{F}_q)$ on C, a good guess for the quotient is $C \to \mathbf{A}^1 - \{0\}$ given by $(x, y) \mapsto y$ (which is actually correct, as we will show next time). $\square$

5 Lecture 5: 1/20/23

Recall the Drinfeld curve C
$$\{xy^q - yx^q = 1\} \subset \mathbf{A}^2_{\mathbf{F}_q}$$
equipped with actions of $\mathrm{SL}_2(\mathbf{F}_q)$ and μ_{q+1}. Last time we proved that $H^2_C(C, \overline{\mathbf{Q}_l}) = 1$ as a representation of $\mathrm{SL}_2(\mathbf{F}_q) \times \mu_{q+1}$. For $\theta : \mu_{q+1} \to \overline{\mathbf{Q}_l}^\times$ non-trivial, we have $H^1_C(C)[\theta]$ is a $q-1$ dimensional representation on $\mathrm{SL}_2(\mathbf{F}_q)$. Our goal was to show this representation is cuspital. It turns out that this is equivalent to proving that
$$(H^1_C(C)[\theta])_U = (H^1_C(C)[\theta])^U = (H^1_C(C))^U[\theta] = 0$$
So now we compute $H^1_C(C/U)$ as a representation of μ_{q+1}.

Lemma 5.1. *The map $C \to \mathbf{A}^2 - \{0\}$ induces an isomorphism $C/U \xrightarrow{\sim} \mathbf{A}^2 - \{0\}$.*

Proof. The map is U-invariant since U acts as $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}(x, y) = (x + by, y)$. For surjectivity, given $y \neq 0$ we can always solve the equation
$$xy^q - yx^q - 1 = 0$$
in $\overline{\mathbf{F}_q}$ since it's algebraically closed. The final step is to show that any two solutions (x_1, y) and (x_2, y) are related by the action by U, i.e. we want to find $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ such that $x_2 = x_1 + by$. We see that b must be $\frac{x_2 - x_1}{y} \in \overline{\mathbf{F}_q}$, and we need to show it's in $\mathbf{F}_q$. Subtracting the equations and dividing by y^{q+1}, we get
$$\left(\frac{x_2}{y}\right)^q - \frac{x_2}{y} = \left(\frac{x_1}{y}\right)^q - \frac{x_1}{y}$$
so indeed
$$\left(\frac{x_2 - x_1}{y}\right)^q = \frac{x_2 - x_1}{y}.$$
This shows $C/U \xrightarrow{\sim} \mathbf{A}^2 - \{0\}$ is an isomorphism. □

Recall that $H^i_C(\mathbf{A}^n) = 1$ if $i = 2n$ and 0 otherwise, and the map $\{0\} \hookrightarrow \mathbf{A}^1$ is μ_{q+1}-equivariant. Thus by considering the long exact sequence (of the pair $\{0\} \subset \mathbf{A}^1$), $H^1(\mathbf{A}^1 - \{0\})$ is the trivial representation of μ_{q+1}.

Corollary 5.2. *If $\theta \neq 1$, then $H^1_C(C)[\theta]$ is cuspidal.*

As this point, we have already found $\frac{q+5}{2}$ irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$ (there are $q+4$ in total, assuming $p>2$). We are missing $\frac{q+3}{2}$ of them. Notice that $\frac{q+3}{2}=\frac{q-1}{2}+2$. We might hope that the 2 is contributed by the unique representation of order 2 of μ_{q+1}, and the remaining $q-1$ nontrivial representations of μ_{q+1} ones pair up. [The trivial representation of μ_{q+1} does not produce any new characters, as we have seen.]

Observe that the map $F: C\to C$ given by $(x,y)\mapsto(x^q,y^q)$ is $\mathrm{SL}_2(\mathbf{F}_q)$-equivariant, but it is not μ_{q+1}-equivariant because $F(\zeta x,\zeta y)=\zeta^q F(x,y)$, and in μ_{q+1} we know $\zeta^q=\zeta^{-1}$. So F induces a $\mathrm{SL}_2(\mathbf{F}_q)$-equivariant isomorphism $H^1_C(C)\to H^1_C(C)$ taking $H^2_C(C)[\theta]$ isomorphically to $H^2_C(C)[\theta^{-1}]$. (It is a fact that a Frobenius map induces isomorphism on etale cohomology.) The situation is now analogous to the induction procedure we went through with $B(\mathbf{F}_q)$ and $T(\mathbf{F}_q)$:

Theorem 5.3 (Geometric Mackey Formula)**.** *Let θ_1,θ_2 be non-trivial representations. Then*

$$\langle H^1_C(C)[\theta_1],H^1_C(C)[\theta_2]\rangle_{\mathrm{SL}_2(\mathbf{F}_q)}=\langle\theta_1,\theta_2\rangle_{\mu_{q+1}}+\langle\theta_1,\theta_2^{-1}\rangle_{\mu_{q+1}}$$

Fact: $H^1_C(C)[\theta]^\vee=H^1_C(C)[\theta]=H^1_C(C)[\theta^{-1}]$ where the dual is as $\mathrm{SL}_2(\mathbf{F}_q)$-vector spaces.

We compute

$$\begin{aligned}\langle R'(\theta_1),R'(\theta_2)\rangle&=\langle 1,R'(\theta_1)\otimes R'(\theta_2)\rangle\\&=\dim(H^1_C(C)[\theta_1]\otimes H^1_C(C)[\theta_2])^{\mathrm{SL}_2(\mathbf{F}_q)}.\end{aligned}$$

The idea then is to use the Kunneth formula. From now on we write $H^*_C(X)$ for $\sum_{i=1}^\infty(-1)^iH^i_C(X)$. So we want to understand the dimension of the virtual character (of $\mu_{q+1}\times\mu_{q+1}$)

$$H^*_C(C\times C)[\theta_1\times\theta_2]^{\mathrm{SL}_2(\mathbf{F}_q)}.$$

It amounts to understanding the virtual character $H^1_C(\frac{C\times C}{\mathrm{SL}_2(\mathbf{F}_q)})$.

Write $Z=C\times C\subset\mathbf{A}^4_{\mathbf{F}_q}$ (in variables x,y,z,w), and write $Z=Z_0\cup Z_{\neq 0}$ where $Z_0=\{xw-yz=0\}$ and $Z_{\neq0}$ is its complement.

Lemma 5.4. *$Z_0\subset C\times C$ is $\mu_{q+1}\times\mu_{q+1}\times\mathrm{SL}_2(\mathbf{F}_q)$ stable.*

Proof. Only the stability under the $\mathrm{SL}_2(\mathbf{F}_q)$ action needs a proof. We have

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}(x,y,z,w)=(ax+by,cx+dy,az+bw,cz+dw),$$

and it is easy to observe that

$$(ax+by)(cz+dy)-(cx+dy)(az+bw)=(bc-ad)(yz-wx).$$

□

6 Lecture 6: 1/23/23

Recall the Drinfeld curve C with actions of $\mathrm{SL}_2(\mathbf{F}_q)$ and μ_{q+1}. Moreover, we have seen that for $\theta : \mu_{q+1} \to \overline{\mathbf{Q}}_l^\times$ then $H^1_c(C)[\theta]$ is cuspidal of dimension $q-1$. Recall that we also have:

Theorem 6.1 (Mackey formula)**.**

$$\langle H^1_c(C)[\theta_1], H^1_c(C)[\theta_2]\rangle = \langle \theta_1, \theta_2\rangle + \langle \theta_1, \theta_2^{-1}\rangle.$$

From Lecture 5, we also saw that this is closely related to computing $H^*_c\left(C \times C/\mathrm{SL}_2(\mathbf{F}_q)\right)$ as a representation of $\mu_{q+1} \times \mu_{q+1}$.

Let's start by computing $C/\mathrm{SL}_2(\mathbf{F}_q)$.

Lemma 6.2. *The map $\varphi : C \to \mathbf{A}^1_{\mathbf{F}_q}$ sending (x, y) to $(xy^{q^2} - yx^{q^2})$ identifies $C/\mathrm{SL}_2(\mathbf{F}_q) \simeq \mathbf{A}^1_{\mathbf{F}_q}$.*

Proof. First show this map is G-invariant. Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{F}_q)$. We have

$$\begin{aligned}
&(ax + by)(cx^{q^2} + dy^{q^2}) - (cx + dy)(ax^{q^2} + by^{q^2}) \\
&= (adxy^{q^2} + bcyx^{q^2}) - (cbxy^{q^2} + adyx^{q^2}) \\
&= (ad - bc)xy^{q^2} + (bc - ad)yx^{q^2} \\
&= xy^{q^2} - yx^{q^2}.
\end{aligned}$$

We next want to show that the action of $\mathrm{SL}_2(\mathbf{F}_q)$ on C is free. If $g \in \mathrm{SL}_2(\mathbf{F}_q)$ has a fixed point, then it has an eigenvalue equal to 1. Without loss of generality, we can say $g = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$, and then $g \cdot (x, y) = (x + by, dy)$, implying $b = 0, d = 1$, as needed.

Next, we need to show that for any $a \in \mathbf{A}^1(\mathbf{F}_q)$, that $\varphi^{-1}(a)$ contains $|\mathrm{SL}_2(\mathbf{F}_q)| = q(q^2-1)$ elements. Write $(z, t) = (x, y/x)$, then we are solving $(t^{q^2} - t)(z^{q^2+1}) = a$ and $(t^q - 1)(z^{q+1}) = 1$. Use $t^{q^2} - t = (t^q - t)^q + (t^q - t)$ (we're in characteristic p), so that these two equations are satisfied if and only if $(t^q - t)(z^{q+1}) = 1$ and $\frac{1}{z^{q+1}} + \frac{1}{z^{q^2+q}} = \frac{a}{z^{q^2+1}}$. This happens if and only if $t^q - t = \frac{1}{z^{q+1}}$ and $z^{q^2-1} - az^{q+1} + 1 = 0$.

So we are done if these polynomials always have distinct roots (there are $q^2 - 1$ roots for z, and then q roots for t for each z). Can check this by taking the derivative.

Here is a fact that completes the proof: the map φ is etale (hence smooth). Thus $C/\mathrm{SL}_2(\mathbf{F}_q) \simeq \mathbf{A}^1$. ☐

We now turn to discussing $C \times C$. Recall this sits inside $\mathbf{A}^2_{\mathbf{F}_q}$ (with coordinates (x, y, z, t) with diagonal $\mathrm{SL}_2(\mathbf{F}_q)$ action and $\mu_{q+1} \times \mu_{q+1}$-action $(\zeta_1, \zeta_2) \cdot (x, y, z, t) = (\zeta_1 x, \zeta_1 y, \zeta_2 z, \zeta_2 t)$.

In Lecture 5, we introduced $C \times C = Z_{\neq 0} \cup Z_0$ where Z_0 is cut out by $\{xt - yz = 0\}$ (and similarly for $Z_{\neq 0}$). We checked that is stable under all the above actions.

Define now $V \subseteq (\mathbf{A}^1 - \{0\}) \times \mathbf{A}^2$ by $u^{q+1} - ab = 1$ for coordinates (u, a, b).

Lemma 6.3. *The map $Z_{\neq 0} \to V$ given by $(x, y, z, t) \mapsto (xt - zy, xt^q - yz^q, x^q t - y^q z)$ induces an isomorphism $Z_{\neq 0}/G \simeq V$.*

Proof. See [1] (may have some computational errors below). Here is an outline:

1. Check that φ actually lands in V. We have

$$\begin{aligned}(xt^q - yz^q)(x^q t - y^q z) &= x^{q+1}t^{q+1} + y^{q+1}z^{q+1} - xt^q y^q z - yz^q x^q t \\ &= (x^q t^q - y^q z^q)(xt - yz) - xt^q y^q z - yz^q x^q t + x^q t^q yz + y^q z^q xt \\ &= (xt - yz)^{q+1} - (xy^q - yx^q)(zt^q - tz^q)\end{aligned}$$

   It is true that $(xy^q - yx^q)(zt^q - tz^q) = -1$. I am not sure...

2. Check that the map is $\mathrm{SL}_2(\mathbf{F}_q)$-invariant. For the first coordinate, we have

$$(ax + by)(cz + dt) - (cx + dy)(za + dt) = (ad - bc)(xt - yz) = xt - yz.$$

   For the second coordinate, we have

$$(ax + by)(cz^q + dt^q) - (cx + dy)(az^q + dt^q) = (ad - bc)(xt^q - yz^q) = xt^q - yz^q.$$

3. Fix $(u, a, b) = V(\mathbf{F}_q)$. Then $\varphi^{-1}(u, a, b)$ consists of those (x, y, z, t) satisfying the 5 equations:

   (a) $xy^q - yx^q = 1$

   (b) $zt^q - tz^q = 1$

   (c) $xt - yz = u$

   (d) $xt^q - yz^q = a$

   (e) $x^q t - y^q z = b$.

   We rewrite (c) and (e) as

$$\begin{bmatrix} x & -y \\ x^q & -y^q \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} u \\ b \end{bmatrix},$$

   and invert the matrix to get equations for u, b in terms of t, z. In particular, we have $z = ux^q - bx$ and $t = uy^q - by$.

   The upshot is that the system reduces to i $xy^q - yx^q = 1$

ii $z = ux^q - bx$

iii $xy^{q^2} - yx^{q^2} = \frac{a+b^q}{u^q}$

iv $t = uy^q - by$.

From before, there are exactly $(q^2-1)q$ pairs of (x, y) satisfying (i) and (iii), and z and t are determined by u, x, y.

4. Finally, a tangent space computation shows that the map is etale. Conclude as in Lemma 6.2.

$\square$

Moreover, V has an action of $\mu_{q+1} \times \mu_{q+1}$ by

$$(\zeta_1, \zeta_2) \cdot (u, a, b) = (\zeta_1\zeta_2 u, \zeta_1\zeta_2^{-1}a, \zeta_1^{-1}\zeta_2 b),$$

which commutes with the $G_m = \overline{\mathbf{F}_q}^\times$ action given by $\lambda(u, a, b) = (u, \lambda a, \lambda^{-1}b)$.

We end with the following fact (Torus-equivariant localization):

$$H_c^*(V) = H_c^*(V^{G_m})[z].$$

We will discuss this more next time (some disagreement in class).

7 Lecture 7: 1/25/23

Last time we were trying to the geometric Mackey formula for $H^1_C(C)[\theta]$ for $\theta : \mu_{q+1} \to \overline{\mathbf{Q}_l}^\times$. Geometrically, this amounts to understanding

$$H^*_C\left(\frac{C \times C}{\mathrm{SL}_2(\mathbf{F}_q)}\right)$$

as a virtual representation of $\mu_{q+1} \times \mu_{q+1}$. We decomposed $C \times C = Z_0 \cup Z_{\neq 0}$ into $\mathrm{SL}_2(\mathbf{F}_q) \times \mu_{q+1} \times \mu_{q+1}$ stable parts, and showed that

$$Z_{\neq 0}/G \xrightarrow{\sim} V = \{u^{q+1} - ab = 1\} \subset (\mathbf{A}^q - \{0\}) \times \mathbf{A}^2$$

with $(\zeta_1, \zeta_2)(u, a, b) = (\zeta_1\zeta_2 u, \zeta_1\zeta_2^{-1} a, \zeta_1^{-1}\zeta_2 b)$. The question was how to compute $H^*_C(V)$ as a virtual representation of $\mu_{q+1} \times \mu_{q+1}$. This is equivalent to computing the trace $\mathrm{tr}\left((\zeta_1, \zeta_2), H^*_C(V)\right)$ for all $(\zeta_1, \zeta_2) \in \mu_{q+1} \times \mu_{q+1}$. The idea is to use the G_m-action $\lambda(u, a, b) = (u, \lambda^{-1}a, \lambda b)$ and compare to $\mathrm{tr}(-, H^*_C(V^{G_m}))$. What is the $-$ suppose to be? Here are some facts:

1. Since V is affine, there exists $t \in \overline{\mathbf{F}_q}^\times = G_m(\overline{\mathbf{F}_q})$ such that $V^{G_m} = V^t$.
2. Suppose γ is a finite order automorphism of a variety of V, $\gamma = su$ such that u has p-power order, s has prime-to-p-power order, $us = vs$. (not sure what is going on here) Then
$$\mathrm{tr}(\gamma, H^*_C(V)) = \mathrm{tr}(u, H^*_C(V^s))$$

Lemma 7.1. *Let Γ be a finite group. Suppose that $\Gamma \times G_m$ acts on an affine variety V, then $tr(\gamma, H^*_C(V)) = tr(\gamma, H^*_C(V^{G_m}))$. When Γ is the trivial group, this is due to B-B (how to spell?).*

Proof. Choose $t \in G_m(\overline{\mathbf{F}_q})$ such that $v^t = V^{G_m}$. Let $\gamma = su$ as before. Then

$$\mathrm{tr}(\gamma, H^*_C(V^{G_m})) = \mathrm{tr}(\gamma, H^*_C(V^t)) = \mathrm{tr}(u, (V^t)^s) = \mathrm{tr}(u, (V^s)^t)$$

Now using fact 2 again for $\gamma = ut$, this is equal to

$$\mathrm{tr}(ut, V^s) = \mathrm{tr}(s, V^u) = \mathrm{tr}(su, V) = \mathrm{tr}(\gamma, V)$$

□

Let's go back to the previous situation: $V = \{u^{q+1} - ab = 1\} \subset (\mathbf{A}^q - \{0\}) \times \mathbf{A}^2$, G_m acting on V by $\lambda(u, a, b) = (u, \lambda a, \lambda^{-1} b)$. So the fixed points are

$$V^{G_m} = \mu_{q+1} \times \{0\} \times \{0\}$$

with $\mu_{q+1} \times \mu_{q+1}$ acting by $(\zeta_1, \zeta_2)(\zeta) = \zeta_1\zeta_2\zeta$.

To compute $Z_0 \subset C \times C \subset \mathbf{A}^4$ cut out by (some equations), we need

Lemma 7.2. *The map $\phi : \mu_{q+1} \times C \to Z_0$ given by $(\zeta, x, y) \mapsto (x, y, \zeta x, \zeta y)$ is a $\mathrm{SL}_2(\mathbf{F}_q)$-equivariant isomorphism.*

Proof. We can easily see the action lands in Z_0 by checking the defining equations. Given $(x, y) \in C(\overline{\mathbf{F}_q})$, we want to show that there are at most $q+1$ options for (z, t) such that $(x, y, z, t) \in Z_0(\overline{\mathbf{F}_q})$. Such (x, y) must satisfy $t = \frac{yz}{x}$ and z must satisfy $z^{q+1}\left(\frac{y}{x}\right)^q - z^{q+1}\left(\frac{y}{x}\right) = 1$. This shows that ϕ is a bijection on $\overline{\mathbf{F}_q}$-points. □

Corollary 7.3. $Z_0/G \cong \mu_{q+1} \times \mathbf{A}^1$ *with* $\mu_{q+1} \times \mu_{q+1}$ *acting by* $(\zeta_1, \zeta_2)(\zeta, z) = (\zeta_1^{-1}\zeta_2\zeta, \zeta_1^2 z)$.

We can now prove the geometric Mackey formula.

Theorem 7.4 (Mackey)**.** *Let θ_1, θ_2 be non-trivial characters of μ_{q+1}. Then*

$$\langle H^1_C(C)[\theta_1], H^1_C(C)[\theta_2]\rangle_{\mathrm{SL}_2(\mathbf{F}_q)} = \langle \theta_1, \theta_2\rangle_{\mu_{q+1}} + \langle \theta_1, \theta_2^{-1}\rangle_{\mu_{q+1}}.$$

Proof. We have seen that

$$\langle H^1_C(C)[\theta_1], H^1_C(C)[\theta_2]\rangle_{\mathrm{SL}_2(F_q)} = \dim H^*_C((C \times C)^{\mathrm{SL}_2(\mathbf{F}_q)})[\theta_1 \times \theta_2].$$

The right side decomposes as

$$\begin{aligned}
&\dim H^*_C((Z_0)^{\mathrm{SL}_2(\mathbf{F}_q)})[\theta_1 \times \theta_2] + \dim H^*_C((Z_{\neq 0})^{\mathrm{SL}_2(\mathbf{F}_q)})[\theta_1 \times \theta_2] \\
&= \dim H^*_C(Z_0/\mathrm{SL}_2(\mathbf{F}_q))[\theta_1 \times \theta_2] + \dim H^*_C(Z_{\neq 0}/\mathrm{SL}_2(\mathbf{F}_q))[\theta_1 \times \theta_2] \\
&= \dim \mathrm{Ind}^{\mu_{q+1} \times \mu_{q+1}}_{\mu^1_{q+1}}[\theta_1 \times \theta_2]1 + \dim \mathrm{Ind}^{\mu_{q+1} \times \mu_{q+1}}_{\mu^2_{q+1}}[\theta_1 \times \theta_2]1 \\
&= \langle 1, \theta_1 \otimes \theta_2\rangle_{\mu_{q+1}} + \langle 1, \theta_1 \otimes \theta_2^{-1}\rangle_{\mu_{q+1}}.
\end{aligned}$$

□

Corollary 7.5. *1. $H^1_C(C)[\theta]$ is irreducible of dimension $q-1$ if $\theta^2 \neq 1$.*

2. $H^1_C(C)[\theta_0]$ has two irreducible non-trivial summands $?_+$ and $?_-$.

By counting, we have now found all irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$ ($p > 2$). One can compute that both $?_+$ and $?_-$ have dimension $\frac{q-1}{2}$.

Remark 7.6. *One should think of the usual induction $\mathrm{Ind}^{\mathrm{SL}_2(\mathbf{F}_q)}_{B(\mathbf{F}_q)}\alpha$ as a special case of DL induction with a 0-dimensional DL variety $H^*_C\left(\frac{\mathrm{SL}_2(\mathbf{F}_q)}{U(\mathbf{F}_q)}\right)[\alpha]$.*

8 Lecture 8: 1/27/23

Today: how to we generalize the Drinfeld curve? A very rough sketch (no proofs today).

For SL_2, there are 2 varieties with a $\mathrm{SL}_2(\mathbf{F}_q)$ action and also an action of $T'(\mathbf{F}_q)$, $T(\mathbf{F}_q)$. Recall that C is:

1. A $\mathrm{SL}_2(\mathbf{F}_q)$-cover of $\mathbf{A}^1$.
2. A $U(\mathbf{F}_q)$-cover of $\mathbf{A}^1 - \{0\}$.
3. $\mu_{q+1} = T'(\mathbf{F}_q)$-cover of $\mathbf{P}^1_{\overline{\mathbf{F}_q}} - \mathbf{P}^1(\mathbf{F}_q)$.

We will seek to generalize (3).

How do we generalize the action of SL_2 on $\mathbf{P}^1$? Say for GL_3, we could take $\mathbf{P}^2$. We know that $\mathbf{P}^2(\overline{\mathbf{F}_q})$ is the set of lines $L \subseteq \overline{\mathbf{F}_q}^{\oplus 3}$ (with an obvious action by $\mathrm{GL}_2(\overline{\mathbf{F}_q})$). But we also have the set of flags:

$$FL(\overline{\mathbf{F}_q}) \coloneqq \{0 \subsetneq L_1 \subsetneq L_2 \subsetneq \overline{\mathbf{F}_q}^{\oplus 3}\}.$$

$\mathrm{GL}_3(\overline{\mathbf{F}_q})$ acts on FL, but since the action extends to GL_3, it is not interesting on cohomology. So our next idea might be to look at $FL - FL(\mathbf{F}_q)$.

Observe that $FL = \mathrm{GL}_3/B$, where B is the subgroup of upper triangular matrices. A matrix in B with columns e_1, e_2, e_3 corresponds to the flag $\{0 \subsetneq e_1 \subsetneq e_1 \oplus e_2 \subsetneq FL(\overline{\mathbf{F}_q})\}$. Hence $FL(\mathbf{F}_q) = \mathrm{GL}_3(\mathbf{F}_q)/B(\mathbf{F}_q)$ and $H^*_c(FL(\mathbf{F}_q)) = \mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_3(\mathbf{F}_q)} 1$.

A good theory of "relative positions of flags" can be done in two ways: either $L_1 \not\subset L_2$ or $L_1' \not\subset L_2'$. So we want a "relative map" $FL(\overline{\mathbf{F}_q}) \times FL(\overline{\mathbf{F}_q})$ to somewhere (TBD). First, we note that there is *no* algebraic map $\mathbf{P}^1(\overline{\mathbf{F}_q}) \times \mathbf{P}^1(\overline{\mathbf{F}_q}) \to \{0, 1\}$.

Better way to see this: use the Bruhat decomposition $\mathrm{GL}_2 = B \cup BSB$. Hence there are two left B-orbits on G/B. Moreover, there are 2 left G-orbits on $\mathbf{P}^1 \times \mathbf{P}^1$. $G - (G/B \times G/B) = B \backslash G/B = 1 \cup S$. This isn't right/missing many details, I got lost around here. In general, for GL_3, we want to look at $\mathrm{GL}_3 \backslash FL \times FL$.

Fact/exercise: Let S_n be the symmetric group, and conflate $\sigma \in S_n$ with the corresponding permutation matrix in GL_n. Let B be the set of upper triangular matrices in GL_n Then $\mathrm{GL}_n = \bigcup_{\sigma \in S_n} B\sigma B$.

Geometrically, this translates into $\mathrm{GL}_n/B \times \mathrm{GL}_n/B = \bigcup_{\sigma \in S_n} O(\sigma)$, where $O(\sigma)$ is the GL_n-orbit of $(1, \sigma)$. We have that the dimension of $O(\sigma)$ is $l(\sigma) + \dim \mathrm{GL}_n/B$, l denoting the length of the permutation σ.

So we get two decompositions of GL_n/B:

1. $\mathrm{GL}_n/B \to \mathrm{GL}_n/B \times \mathrm{GL}_n/B$ via $x \mapsto (x, 1)$ and take inverse image of $O(\sigma) \rightsquigarrow (\mathrm{GL}_n/B)^\sigma$.
2. $\mathrm{GL}_n/B \to \mathrm{GL}_n/B \times \mathrm{GL}_n/B \rightsquigarrow (\mathrm{GL}_n/B)(\sigma)$ via Frobenius and pull back $\mathcal{O}(s)$. This leads to Deligne-Lusztig varieties.

Observe that $X(\omega) = (\mathrm{GL}_n/B)(\omega)$ are $\mathrm{GL}_n(\mathbf{F}_q)$-stable. We are almost "in business", but we need to construct $Y(\omega) \to X(\omega)$, where $Y(\omega)$ has an action by $T_\omega(\mathbf{F}_q) \times \mathrm{GL}_n(\mathbf{F}_q)$.

Facts:

1. All the $X(\omega)$ are smooth of pure dimension $l(\sigma)$ (like the X^σ).
2. Their closures have the same singularities as the closures of the X^σ.
3. $X(\sigma)$ is usually not connected.

No class next week (Pol is gone).

9 Lecture 9: 2/6/23

This week: Algebraic groups. We want to define/motivate "finite groups of Lie type".

Let k be a perfect field (probably algebraically closed in a bit).

Definition 9.1. *A variety is a separated, geometrically reduced over k of finite type.*

Definition 9.2. *A* group scheme *is a quadruple (G,m,e,i) where G is a k-scheme locally of finite type, $m : G \times_k G \to G$, $i : G \to G$, and $e : Spec(k) \to G$ are k-morphisms such that the "group axioms hold".*

This is the same thing as saying that for all k-schemes T, the operations m, e, i make $G(T)$ into a group with inverse operation i_T and ideneity e_T.

Basic group theory tells us that e and i are uniquely determined by m.

Definition 9.3. *G as above is an* algebraic group *if G is a variety over k.*

Here are some examples:

1. $\mathbb{G}_m \coloneqq \mathrm{Spec} k[x, x^{-1}]$, which represents the functor $R \mapsto (R^\times, \cdot)$ for k-algebras R.
2. $\mathbb{G}_a \coloneqq \mathrm{Spec} k[x]$, which which represents the functor $R \mapsto (R^+, +)$.
3. Elliptic curves.
4. If Γ is a finite group, then we can take $G = \coprod_{\gamma \in \Gamma} \mathrm{Spec}(k)$ as a group scheme. It is an algebraic group denoted $\underline{\Gamma}$.
5. If Γ is a finite abelian group of order invertible in k, then $\mathrm{Spec}(k[\Gamma])$ is an algebraic group. Need the order to be invertible so that the scheme is reduced (for instance, $\mathbf{F}_p[\mathbf{Z}/p\mathbf{Z}] = \mathbf{F}_p[x]/(x^p - 1)$ gives μ_p, nonreduced in characteristic p.

6. $G = \mathrm{GL}_n = \mathrm{Spec}(k[X_{ij} : 1 \le i, j \le n][1/D])$, where $D = \det(X_{ij})$. This represents the functor $R \mapsto \mathrm{GL}_n(R)$.

We will not be interested in projective (proper) algebraic groups (boring representation theory, because they are commutative). We also have to exclude disconnected algebraic groups, like $\underline{\Gamma}$.

More subtly, the representation theory of $\mathbb{G}_a$ is also not so interesting.

Note: we mean the representation theory of $\mathbb{G}_a(\mathbf{F}_q)$ when $k = \mathbf{F}_q$.

Conrad's addendum: we should really mean k-homomorphisms $G \to \mathrm{GL}_n$ by "representation theory".

Conrad's addendum 2: It turns out any k-homomorphism $G \to \mathrm{GL}_n$ can be conjugated inside $\mathrm{GL}_n(k)$ to land inside U_n (closed subgroup of upper-triangular unipotent matrices), so is rarely "completely reducible". That failure is one reason why the representation theory of $\mathscr{G}_a$ as a k-group is bad.

Fact: all affine algebraic groups can be realized as closed subgroups of GL_n over k. For a finite group this is obvious by Cayley's theorem. But for a group like $\mathrm{PGL}_2 = \mathrm{GL}_2 / \mathbb{G}_m$, it is not so obvious how this is done. Proof idea: G "acts" on its own coordinate ring ($G = \mathrm{Spec}(R)$), and this will be an increasing union of G-stable finite-dimensional k-vector spaces. So should get some finite piece.

Such algebraic groups are called "linear algebraic groups".

Warning: non-reduced group schemes appear in nature, and are useful. For example, $Z_{\mathrm{SL}_2} = \mu_2 \subsetneq \mathrm{SL}_2$ in characteristic 2 (Z is the center).

Fact: quotients of an algebraic group G by a closed subgroup $H \subset G$ "exist". They are again groups if H is normal. In other words, $G \to G/H$ is H-invariant and universal for H-invariant maps of schemes. Note that although $G(\overline{k})/H(\overline{k}) = (G/H)(\overline{k})$ if $\overline{k} \supset k$ is algebraically closed, but *not in general true* for all k-points.

Fancy language: $G \to G/H$ is an H-torsor in the etale topology.

Example: $\mathrm{GL}_n / \mathbb{G}_m = \mathrm{PGL}_n = \mathrm{SL}_n / \mu_n$. But it's not true that $\mathrm{SL}_2(\mathbf{Q})/\mu_2(\mathbf{Q}) \neq \mathrm{PGL}_2(\mathbf{Q})$, because the RHS has "determinant in $\mathbf{Q}^*/(\mathbf{Q}^*)^2$", but the LH has "determinant 1". In fact, there is a long exact sequence

$$1 \to \mu_2(\mathbf{Q}) \to \mathrm{SL}_2(\mathbf{Q}) \to \mathrm{PGL}_2(\mathbf{Q}) \to \mathbf{Q}^*/(\mathbf{Q}^*)^2 \to 1.$$

Another example: $\mathrm{GL}_2 / B_2 \cong \mathbf{P}^1$, where B_2 is the subgroup of upper triangular matrices in GL_2. But here $\mathrm{GL}_2(\mathbf{Q})/B_2(\mathbf{Q})$ really is $\mathbf{P}^1(\mathbf{Q})$.

An algebraic group G over k has:

1. A maximal quotient G^{ab}, fitting inside a short exact sequence

$$1 \to G^{der} \to G \to G^{ab} \to 1,$$

where G^{der} is the "derived group" of G. For instance,

$$1 \to \mathrm{SL}_n \to \mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_{\gtrdot} \to 1.$$

Conrad's addendum 3: The right way to think of G^{der} is a smooth closed k-subgroup for which $G^{der}(\overline{k})$ is the commutator subgroup of $G(\overline{k})$. For example, SL_n turns out to be its own derived k-subgroup. Same is true for PGL_n, but the commutator subgroup of $\mathrm{PGL}_n(k)$ is typically a proper subgroup (it's the image of $\mathrm{SL}_n(k)$).

2. The connected component $G^0 \subseteq G$ containing e (the "identity" or "neutral" component) is a *normal* subgroup, and the quotient G/G^0 is finite.

3. Given $H \subseteq G$ a closed subgroup, there is a centralizer $C_G(H) \subseteq G$, which represents

$$R \mapsto \left(\bigcap_{R \to R'} C_{G(R)}(H(R')) \right) \subseteq G(R)$$

.

We now turn to finite groups of Lie type. They are motivatived by classical compact connected real lie groups such as $SO_n(\mathbf{R})$, $SU_n(\mathbf{R})$ by loose analogy.

Definition 9.4. *An algebraic group G is called* reductive *if G does not contain a normal subgroup $H \subseteq G$ isomorphic to $\mathbb{G}_a^{\oplus m}$. for all m.*

If we have one, we should take the quotient (doesn't really change the representation theory).

1. $\mathbb{G}_m$ is reductive.

2. But if we take B_2, this is not reductive since $N_2 = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}$ is isomorphic to $\mathbb{G}_a$. Note that B_2/N_2 is the diagonal matrices, isomorphic to $\mathbb{G}_m^2$.

3. $G = \mathrm{GL}_n$ is reductive (very hard exercise).

Fact: when $k = \mathbf{C}$, the reductive groups G are precisely those such that $Lie(G) = Lie(H) \otimes_{\mathbf{R}} \mathbf{C}$, where H is a compact real Lie group.

We expect reductive groups, or at least connected reductive groups, to have simple representation theory. For instance, maps $G \to \mathrm{GL}_n$ of algebraic groups for $k = \mathbf{C}$ are very simple and have a good classification.

Definition 9.5 (Strictly for this course only!)**.** *A* finite group of Lie type *is a finite group isomorphic to $G(\mathbf{F}_q)$, for some $G/\mathbf{F}_q$ connected reductive group (perhaps not a good definition for the classification of finite simple groups, but better for us).*

10 Lecture 10

Let k be a perfect field (which we often assume to be algebraically closed). Recall that a connected reductive group is an affine, reduced, finite type k-group scheme G such that G contains no normal subgroups isomorphic to $\mathbb{G}_{a,k}^{\oplus m}$ for all m. [For nonperfect fields, the correct condition is that the basechange of G to an algebraic closure of k is reductive.] Examples:

1. $G = \mathbb{G}_{m,k}^{\oplus n}$. Need to check that there are no isomorphisms $\mathbb{G}_{a,k} \to \mathbb{G}_{m,k}$.

2. $\mathrm{GL}_{n,k}$. The subgroup $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is isomorphic to $\mathbb{G}_{a,k}$ but not normal.

Definition 10.1. *A torus T is an algebraic group over k scuh that $T_{\overline{k}} \cong \mathscr{G}_m^{\oplus n}$.*

e.g. $\mathscr{G}_m$

Definition 10.2. *Let G be a connected reductive group. A maximal torus T is a closed subgroup with T a torus, maximal with this property.*

e.g. the diagonal matrices in GL_3.
Fact:

- Maximal tori exist, and if $T \subset G$ is a maximal torus, then so is $T_{\overline{k}} \subset G_{\overline{k}}$.
- When k is algebraically closed, any two maximal tori are $G(k)$-conjugate.
- $C_G(T) = T$ when $T \subset G$ is a maximal torus.

Example: Let L over k be a finite etale k-algebra of rank n (i.e. $L = \prod_{i=1}^m k_i$, where k_i/k is a degree j_i (separable since k is perfect) field extension and $\sum j_i = n$). Then we consider

$$\mathrm{Res}_{L/k}\mathbb{G}_m$$

which has functor of points

$$R \mapsto (L \otimes_k R)^\times.$$

This is a torus, because

$$L \otimes_k \overline{k} \cong \prod_{i=1}^n \overline{k}$$

and so $(\mathrm{Res}_{L/k}\mathbb{G}_m) \otimes_k \overline{k} = \mathbb{G}_{n,\overline{k}}^{\oplus n}$.

If we choose a k-basis of L and we let L act on itself $L = k^{\oplus n}$ by left translation, then there is an induced morphism

$$\mathrm{Res}_{L/k}\mathbb{G}_{m,L} \to \mathrm{GL}_{n,k}.$$

Fact: This procedure produces all maximal tori of $GL_{n,k}$, so the Galois theory of k enters the picture.

Definition 10.3. *A representation of G is a pair (V,ρ) where V is a k-vector space and ρ is a natural transformation (of functors on k-algebras)*

$$(R \mapsto G(R)) \mapsto (R \mapsto \mathrm{Aut}_r(V \otimes_k R)).$$

A morphism $f : (V,\rho) \to (V',\rho')$ is a k-linear map $V \to V'$ such that for all R and $g \in G(R)$, $v \in V \otimes_k R$,

$$f_R(\rho(g).v) = \rho'(g) f_R(v).$$

This gives an abelian category of finite dimensional representations.

Example: The adjoint representation. Let $R[\epsilon] = R[x]/(x^2)$, then we define

$$\mathrm{Lie}\, G(R) = \ker(G(k[\epsilon]) \to G(R))$$

E.g. when $G = GL_n$, we get

$$1 + \epsilon M_{n\times n}(R) \subset \mathrm{GL}_n(k[\epsilon])$$

with the group structure

$$(1+\epsilon A)(1+\epsilon B) = 1 + \epsilon(A+B).$$

Fact: $\mathrm{Lie}\, G \subset R$ is canonically an R-module, and $(\mathrm{Lie}\, G)(k) \otimes_k R \cong (\mathrm{Lie}\, G)(k)$. The left hand is also the tangent space to G at $e \in G(k)$. Then the adjoint representation is

$$\begin{aligned}(R \mapsto G(R)) &\mapsto (\mathrm{Aut}_R(LieG(R))) \\ g &\mapsto \text{conjugation by } g \in G(R) \subset G(R[\epsilon])\end{aligned}$$

Often $\mathfrak{g}$ is used to denote $\mathrm{Lie}\, G$, which is a Lie algebra.

Fact: The category of finite dimensional representations of $\mathbb{G}_m$ is semisimple, with simple objects the one-dimensional representations, corresponding to

$$\begin{aligned}\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) &\cong \mathbf{Z} \\ (x \mapsto x^n) &\mapsto n\end{aligned}$$

For a torus T, we write $X^*(T) = \mathrm{Hom}_{\overline{k}}(T_{\overline{k}}, \mathbb{G}_m)$ for the character lattice. Dually, the co-character lattice $X_*(T) = \mathrm{Hom}_{\overline{k}}(\mathbb{G}_m, T_{\overline{k}})$. They are finite free $\mathbf{Z}$-modules equipped with actions of $\mathrm{Gal}(\overline{k}/k)$, which factor through $\mathrm{Gal}(\overline{k}/k) \to \mathrm{Gal}(k'/k)$ for some finite extension k' of k.

There is a pairing (composition)

$$\begin{aligned}X^*(T) \times X_*(T) &\to \mathrm{Hom}_{\overline{k}}(\mathbb{G}_m, \mathbb{G}_m) = \mathbf{Z} \\ (\chi, \mu) &\mapsto \langle \chi, \mu \rangle = \chi \circ \mu\end{aligned}$$

Definition 10.4. *Let $T \subset G$ be a maximal torus, and assume $k = \overline{k}$. Consider* $\operatorname{Lie} G|_T$ *as a T-representation. Then*

$$\operatorname{Lie} G|_T = \operatorname{Lie} T \oplus \bigoplus_{\alpha \in \Phi} (\operatorname{Lie} G)_\alpha$$

$\Phi \subset X^*(T)$ is the subset of those characters α such that $\operatorname{Lie} G_\alpha \neq 0$*. These characters are called the* roots *of G with respect to T.*

Fact: Each $\operatorname{Lie} G_\alpha$ is one-dimensional. Fact: For each $\alpha \in \Phi$, there is a unique $\alpha^\vee \in X_*(T)$ such that

$$\alpha, \alpha^\vee = 2$$

and $S_\alpha(\Phi) = \Phi$ and $S_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$, where

$$S_\alpha : X^*(T) \to X^*(T)$$

is given by $x \mapsto x - \langle x, \alpha^\vee \rangle \cdot \alpha$, and similarly

$$S_{\alpha^\vee} : X_*(T) \to X_*(T)$$

is given by $y \mapsto y - \langle \alpha, y \rangle \cdot \alpha^\vee$.

The upshot is that $(X^*(T), X_*(T), \Phi, \Phi^\vee)$ is a root datum, and connected reductive groups G over $k = \overline{k}$ are classified by root data(?).

A strange observation is that $(X_*(T), X^*(T), \Phi^\vee, \Phi)$ is also a root datum, so we get a dual group $\widehat{G}$ over k with this root datum. For example, under this operation, SL_n goes to PGL_n, SP_{2n} goes to SO_{2n+1}, and GL_n is self-dual. This construction is the heart of all of "Langlands duality". The example of SP_{2n} and SO_{2n+1} is illustrative.

Bruhal Decomposition:

Definition 10.5. *A borel subgroup $B \subset G$ is a subgroup such that G/B is a projective variety, and B is minimal with this property.*

Fact: When $k = \overline{k}$ or $k = \mathbf{F}_q$, Borel subgroups exists and they are all $G(k)$-conjugates. They are maximal solvable connected subgroups of G over $\overline{k}$.

E.g. for GL_n, B is the subgroup of upper triangular matrices. GL_n / B is the variety of full flags.

Fact: Every maximal torus is contained in a Borel subgroup. This is only true for $k = \overline{k}$. It fails already for GL_2 over a finite field (as we have seen for $\operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{G}_m \subset \mathrm{GL}_{2,\mathbb{F}_q}$)

Now we get ($T \subset BG$ maximal torus contained in a Borel subgroup B, and $k = \overline{k}$)

$$\Phi_B^+ \subset \Phi$$

of those $\alpha \in \Phi$ such that $(\operatorname{Lie} B)_\alpha \neq G$.

Fact: If $\alpha \neq \beta \in \Phi^+$, then $\alpha + \beta \in \Phi_B^+$. Every $\alpha \in \Phi$ is in Φ_B^+ or $-\alpha \in \Phi_B^+$. We can define $\Delta_B \subset \Phi_B^+$ as those $\alpha \in \Phi_B^+$ that are not of the form $\alpha_1\alpha_2$ for $\alpha_1 \neq \alpha_2 \in \Phi_B^+$. These are called the simple positive roots.

The Weyl groups. Let $T \subset G$ be a maximal torus. Let $W = N_G(T)/T$ which is a finite group, and W acts on T, hence on $X^*(T)$. It acts faithfully on $X^*(T)$.

Fact: $W \hookrightarrow \mathrm{Aut}(X^*(T))$ is generated by $S_\alpha : X^*(T) \to X^*(T)$ for $\alpha \in \Delta$. This exhibits W as a Coxeter group.

Fact: $G = \coprod_{w \in W} BwB$, where $w \in N_G(T)(k)$ is a representative of $w \in W$.

11 Lecture 11

Today we continue the discussion last time with some examples.

Let $G = \mathrm{SL}_{n+1}$ and T be the diagonal torus. Let B be the upper triangular matrices. $T = T' \cap \mathrm{SL}_{n+1}$ where $T' \subset \mathrm{GL}_{n+1}$ is isomorphic to $\mathbb{G}_m^{\oplus(n+1)}$. We can also think of T as the kernel of

$$T' \to G_m^{\oplus(n+1)}(x_0, \cdots, x_n) \mapsto x_0 \cdots x_n$$

So we can identify $* : X^*(T') = \mathbf{Z}^{n+1} \to X^*(T)$, where if e_i corresponds to the character sending $(t_0, \cdots, t_n)$ to t_i, then the kernel of $*$ is $e_0 + \cdots + e_n$. So $X^* \cong \mathbf{Z}^{n+1}/\mathbf{Z}$, where $\mathbf{Z}$ is embedded diagonally.

$\mathrm{Lie}\,\mathrm{SL}_{n+1} \subset \mathrm{Lie}\,\mathrm{GL}_{n+1}$ (the latter is just $(n+1)$-by-$(n+1)$ matrices), and one can check that T preserves the linear subspaces U_{ij}, which is those matrices with only (i,j)-th entry nonzero. And $(t_0, \cdots, t_n)$ acts on $U_{i,j}$ by the scalar $\frac{t_i}{t_j}$. Hence we see that the roots oare the characters $\alpha_{i,j} = e_i - e_j$ for $i \neq j$.

The positive roots are those that are upper triangular, so $i < j$. Namely, $\Phi^+ = \{\alpha_{ij} \mid i < j\}$. The simple roots are

$$\Delta = \{e_i - e_{i+1}\}_{0 \le i \le n-1}$$

One can check the coroots $\alpha_{i,j}^\vee$ dual to α_{ij} is given by the cocharacter

$$y \mapsto (1, \cdots, 1, y, \cdots, \frac{1}{y}, 1, \cdots 1)$$

where the y appears on the i-th slot and $\frac{1}{y}$ appears on the j-th slot. We see that indeed $\alpha_{ij} \circ \alpha_{ij}^\vee$ sends y to y^2.

For the case of SP_{2n} and SO_{2n}, see Conrad's notes.

From next week on, we will be working with a connected reductive group G_0 over $\mathbf{F}_q$ and $G = G_0 \otimes \overline{\mathbf{F}_q}$. We will describe a procedure that takes as inputs:

1. $T_0 \subset G_0$ a maximal torus
2. $B \supset T (T = T_0 \otimes \overline{\mathbf{F}_q})$
3. $\Theta : T_0(\mathbf{F}_q) \to \overline{\mathbf{Q}_l}^\times$

and outputs a virtual representation $R_{T_0,B}^\Theta$ of $G_0(\mathbf{F}_q)$ over $\overline{\mathbf{Q}_l}$. Today, we investigate how many choices of $(T_0, B \subset T)$ there are. (For SL_2 over $\mathbf{F}_q$, there are 2 choices up to conjugation.)

Fix a Borel pair $\mathbb{T} \subset \mathbb{B}_0$ (think of this as a base point). Then $(G_0/B_0 \times G_0/B_0)_{\overline{\mathbf{F}_q}}$ has finitely many G-orbits, indexed by $\underline{W}(\overline{\mathbf{F}_q})$ where $\underline{W} = N_G(\mathbb{T}_0)/\mathbb{T}_0$. This is a base point in the sense that the set of varieties of Borel subgroups can be identified with $G_0/\mathbf{B}_0$ by sending g to $g\mathbf{B}_0 g^{-1}$.

Warm-up:

1. There is only one choice of F-stable Borel pair $\mathbf{T}_0 \subset \mathbf{B}_0$ up to conjugacy.

2. There is only one choice of $\mathbf{B}_0$ up to conjugacy.

Proof. The variety of Borel subgroups can be identified with $G_0/\mathbf{B}_0$ using our base point. Now $(G_0/\mathbf{B}_0)(\mathbf{F}_q) = G_0(\mathbf{F}_q)/\mathbf{B}_0(\mathbf{F}_q)$ because $\mathbf{B}_0$ is connected and we are over a finite field. This shows the second claim since it says all $B_0 \subset G_0$ is $G_0(\mathbf{F}_q)$-conjugate to $\mathbf{B}_0$.

The variety of Borel pairs $(T_0 \subset B_0 \subset G_0)$ is a homogenous space for G_0, and using the base point $(\mathbf{T_0} \subset \mathbf{B_0})$, we can identify it with $G_0/(N_{G_0}(\mathbf{T_0}) \cap N_{G_0}(\mathbf{B}_0))$. It is a fact that the intersection $N_{G_0}(\mathbf{T_0}) \cap N_{G_0}(\mathbf{B}_0)$ is equal to $\mathbf{T}_0$, and so the variety of Borel pairs is identified with $G_0/\mathbf{T}_0$. Using the same argument as above, we see that $G_0/\mathbf{T}_0(\mathbf{F}_q) = G_0(\mathbf{F}_q)/\mathbf{T}_0(\mathbf{F}_q)$. Thus every Borel pair $(T_0 \subset B_0)$ is $G_0(\mathbf{F}_q)$-conjugate to $(\mathbf{T}_0 \subset \mathbf{B}_0)$. $\square$

Things are more complicated for the variety of maximal tori. Indeed this can be identified with the homogeneous space $G_0/N_{G_0}(\mathbf{T}_0)$, using $\mathbf{T}_0$ as a base point, and so we expect to see the (non-abelian) Galois cohomology set

$$H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q, N_{G_0}(\mathbf{T}_0)(\overline{\mathbb{F}}_q))$$

as an obstruction. Since $\mathbf{T}_0$ is connected the Galois cohomology $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q, \mathbf{T}_0)(\overline{\mathbb{F}}_q))$ vanishes by Lang's lemma. Therefore the long exact sequence in Galois cohomology for

$$1 \to \mathbf{T}_0 \to N_{G_0}(\mathbf{T}_0) \to \underline{W} \to 1$$

tells us that the natural map

$$H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), N_{G_0}(\mathbf{T}_0)(\overline{\mathbb{F}}_q)) \to H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), W(\overline{\mathbb{F}}_q))$$

is an isomorphism.

Concretely, if E is the variety of maximal tori T and $\widetilde{E}$ is the variety of Borel pairs $T \subset B$ there is a map

$$\widetilde{E} \xrightarrow{\pi} E$$

sending the point $\tilde{e}_0 = (\mathbf{T}_0 \subset \mathbf{B}_0)$ to $e_0 = \mathbf{T}_0$. This can be identified as the map $G_0/\mathbf{T}_0 \to G_0/N_{G_0}(\mathbf{T}_0)$ using chosen the base points. Let us write $Y = \pi^{-1}(E(\mathbb{F}_q))$, considered as an $\mathbb{F}_q$-scheme. We want to understand $A = G_0(\mathbf{F}_q)\backslash E(\mathbf{F}_q)$ and $\widetilde{A} = G_0(\mathbf{F}_q)\backslash Y(\overline{\mathbb{F}}_q)$.

Proposition 11.1. *The choice of $\widetilde{e_0}$ induces two bijections:*

$$\widetilde{A} \xrightarrow{\sim} \underline{W}(\overline{\mathbb{F}}_q)$$

and

$$A \xrightarrow{\sim} \frac{\underline{W}(\overline{\mathbf{F}_q})}{Ad_F \underline{W}(\overline{\mathbf{F}_q})}$$

Here, Ad_F denotes the action $w.(w') = ww'F(w)^{-1}$ of $\underline{W}$ on itself and the the fractional notation denotes taking the quotient.

Note: $\mathrm{Ad}_F \underline{W}(\overline{\mathbf{F}_q}) = H^1_{\mathrm{et}}(\mathbf{F}_q, \underline{W})$. So a fancy proof of the second bijection is to look at the set of isomorphism classes over $\mathbf{F}_q$-points of the stack $[G_0 \backslash E] = [\frac{1}{N_{G_0}(\mathbf{T}_0)}]$, so this set is just

$$H^1_{\mathrm{et}}(\mathbf{F}_q, N_{G_0}(\mathbf{T}_0)) = H^1_{\mathrm{et}}(\mathbf{F}_q, N_{G_0}(\mathbf{T}_0)/\mathbf{T}_0) = H^1_{\mathrm{et}}(\mathbf{F}_q, \underline{W}).$$

[This étale cohomology is the same as the Galois cohomology from above].

Proof of the Proposition. We are trying to compute the quotient $G_0(\mathbb{F}_q)\backslash Y(\overline{\mathbb{F}}_q)$, where $Y \subset \tilde{E}$ is the inverse image of $E(\mathbb{F}_q)$ under π.

The space $Y := \pi^{-1}(E(\mathbf{F}_q))$ can be identified, using the base point $\tilde{e_0}$, with

$$\{g \in G \mid \pi(g.\tilde{e_0}) \in E(\mathbf{F}_q)\}/\mathbf{T}_0 = \{g \in G \mid g^{-1}F(g) \in N_{G_0}(\mathbf{T}_0)\}/\mathbf{T}_0$$

because $F(g^{-1}\mathbf{T}_0 g) = g^{-1}\mathbf{T}_0 g$ is equivalent to $g^{-1}F(g) \in N_{G_0}(\mathbf{T}_0)$. The upshot of this discussion is that the following diagram is a pullback diagram

$$\begin{array}{ccc} Y & \subset & G_0/\mathbf{T}_0 \\ \big\downarrow & & \big\downarrow {\scriptstyle g \mapsto g^{-1}F(g)} \\ \frac{N_{G_0}(\mathbf{T}_0)}{\mathbf{T}_0} & \subset & G_0/\mathbf{T}_0. \end{array}$$

In other words, Y is the inverse image of $\frac{N_{G_0}(\mathbf{T}_0)}{\mathbf{T}_0} \subset \frac{G_0}{\mathbf{T}_0}$ under the map $\frac{G_0}{\mathbf{T}_0} \to \frac{G_0}{\mathbf{T}_0}$ induced by the Lang isogeny $g \mapsto g^{-1}F(g)$.

The reason we did all this is that $G_0(\mathbb{F}_q) \subset G_0$ is the kernel of the Lang isogeny $g \mapsto g^{-1}F(g)$. Thus the map induced by the Lang isogeny induces an isomorphism

$$G_0(\mathbb{F}_q)\backslash Y \simeq \frac{N_{G_0}(\mathbf{T}_0)}{\mathbf{T}_0} =: \underline{W}.$$

In particular, there is an isomorphism

$$G_0(\mathbb{F}_q)\backslash Y(\overline{\mathbb{F}}_q) \simeq \underline{W}(\overline{\mathbb{F}}_q),$$

which proves the first part of the Proposition. The proof of the second part is omitted. $\square$

12 Lecture 12: 2/13/23

Today: Deligne-Lusztig varieties & Induction.

Notation (fixed forever):

- q is a prime power.
- G_0 over $\mathbf{F}_q$ is a connected reductive group.
- $T_0 \subseteq B_0 \subseteq G_0$ a Borel pair, $U_0 \subseteq B_0$ the unipotent radical. By "unipotent radical" we mean $U_0 \subseteq B_0$ is maximal such that $B_0/U_0 \cong T_0$. Note that $T_0 \subseteq B_0$ is unique up to $G_0(\mathbf{F}_q)$-conjugacy (see previous lectures). If we drop the subscript 0, we mean base change to $\overline{\mathbf{F}_q}$.
- $Y \coloneqq G_0/U_0$ is a cover of $X \coloneqq G_0/B_0$ via the natural map $Y \to X$.
- $\underline{W}$ (considered as a scheme) is the normalizer $N_{G_0}(T_0)/T_0$. For $\underline{W}(\overline{\mathbf{F}_q})$ we have a G-orbit $\mathcal{O}(w) \subseteq X \times X$ (orbit of $(1, \dot{w})$, $\dot{w}$ being a lift of w to $N_{G_0}(T_0)(\overline{\mathbf{F}_q})$). Write $X(w)$ for the intersection of $\mathcal{O}(w)$ with Γ_F, where F is Frob_q, and Γ is its graph inside $X \times X$.

Note that when $G_0 = \mathrm{GL}_n$ and $T_0 \subseteq B_0$ are the diagonal subgroup contained in the upper triangular subgroup, then

$$X = \{\text{variety of flags } V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = \overline{\mathbf{F}_q}^{\oplus n}\}$$

and

$$Y = \{\text{variety of flags } V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = \overline{\mathbf{F}_q}^{\oplus n} \text{ and nonzero vectors } w_i = V_i/V_{i-1}\ \forall i = 1, \ldots, n\}.$$

The map $Y \to X$ makes Y a torsor for the group of diagonal matrices, where $\mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ acts by scaling w_i by λ_i.

Observe that $X(w) \subseteq \mathcal{O}(w)$ is not G-stable (since the graph of Frobenius is not). But it is $G_0(\mathbf{F}_q)$-stable (since the graph of Frobenius is).

Lemma 12.1. *$X(w)$ is smooth and equidimensional of dimension $l(w)$ and nonempty (l being the length).*

Proof. We will take as fact that $\mathcal{O}(w)$ has dimension $X + l(w)$. Then (if $X(w)$ is nonempty) it suffices to show the intersection $\mathcal{O}(w) \cap \Gamma_F$ is transverse at all $x \in \mathcal{O}(w)(\mathbf{F}_q) \cap \Gamma_F(\overline{\mathbf{F}_q})$. In other words, need $T_x\Gamma_F + T_x\mathcal{O}(w) \subseteq T_x(X \times X)$ is an equality for all such x.

Note that $T_x\Gamma_F = (T_xX, 0)$, since Γ_F has zero differential (differentiate x^q in characteristic p), and $T_x\mathcal{O}(w) \xrightarrow{pr_2} T_xX$ is surjective, because $\mathcal{O}(w) \xrightarrow{pr_2} X$ is a smooth surjective map of G-homogeneous spaces.

All of this works given that $X(w)$ is nonempty. To prove this, we need to show that there exists $g \in G(\overline{F}_q)$ such that $(g, g\dot{w}) \in \mathcal{O}(w)$ is also in Γ_F. For this we choose g such that $g^{-1}F(g) = \dot{w}$ (possible by surjectivity of $L : G_0 \to G_0$). Then $g^{-1}(g, F(g)) = (1, g^{-1}F(g)) = (1, \dot{w})$. $\square$

Define $A_{\dot{w}} \to X \times X$ as the moduli space of $\{g \in G_0 : gB_0 = x, g\dot{w}B_0 = y\}$. The map sends the set of R-points $\mathrm{Spec}(R) \xrightarrow{(x,y)} X \times X$ to the set of $g \in G_0(R)$ such that $gB_0 = x$ and $g\dot{w}B_0 = y$. What is really going on is a diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\Delta"] & G/(U \cap \dot{w}U\dot{w}^{-1}) \arrow[r, "T"] \arrow[d] & \mathcal{O}(w) = G/(B \cap \dot{w}B\dot{w}^{-1}) \arrow[d, "(1,\dot{w})"] \\ G \times G \arrow[r] & G/U \times G/\dot{w}U\dot{w}^{-1} \arrow[r, "T \times T"] & G/B \times G/\dot{w}B\dot{w}^{-1} \end{tikzcd}.$$

The upshot is that $Y \times X \xrightarrow{\pi_1} X \times X$ and $X \times Y \xrightarrow{\pi_2} X \times X$ are both right T-torsors and over $\mathcal{O}(w) \subseteq X \times X$, there is an isomorphism $Y \times X|_{\mathcal{O}(w)} \xrightarrow{\dot{w}} X \times Y|_{\mathcal{O}(w)}$ which is T-equivariant via $T \xrightarrow{\mathrm{Ad}\,\dot{w}^{-1}} T$. Moreover, over $\Gamma_F \subseteq X \times X$, there is a Frobenius $Y \times X|_{\Gamma_F} \xrightarrow{F} X \times Y|_{\Gamma_F}$ which is T-equivariant for $T \xrightarrow{F} T$.

$$\begin{tikzcd} G \arrow[r] \arrow[d, "(1,F)"] & G/U \arrow[r, "T"] \arrow[d, "(1,F)"] & \Gamma_F \arrow[d] \\ G \times G \arrow[r] & G/U \times G/U \arrow[r, "T \times T"] & G/B \times G/B \end{tikzcd}$$

Definition 12.2. *$\widetilde{X}(\dot{w}) \to X(w)$ is the equalizer of F and $\widetilde{w}$ inside $Y \times X|_{X(w)}$.*

Let $T(w)^F$ be the equalizer of the maps $F, \mathrm{Ad}\,\dot{w}^{-1} : T \to T$. Then $T(w)^F$ acts on $\widetilde{X}(w) \to X$ and in fact this action turns $\widetilde{X}(\dot{w}) \to X(w)$ into a $T(w)^F$-torsor. One checks that $G_0(\mathbf{F}_q)$ acts on $\widetilde{X}(\dot{w})$ such that $\widetilde{X}(\dot{w}) \to X(w)$ is equivariant.

Definition 12.3. *For a character $\Theta : T(w)^F \to \overline{\mathbf{Q}}_l^\times$, we define*

$$R^\theta(\dot{w}) := \sum_i (-1)^i H^i_c(\widetilde{X}(\dot{w}), \overline{\mathbf{Q}}_l)[\theta]$$

as a virtual $G_0(\mathbf{F}_q)$-representation.

Remark: $R^1(\dot{w}) = H^*_c(X(w), \overline{Q}_l)$. Note that this definition is not exactly satisfactory because $T(w)^F$ is not a priori $T_0(\mathbf{F}_q)$ for some $T_0 \subset G_0$.

Second remark: This definition does not depend on $\dot{w}$, only on w.

13 Lecture 13

Let $G_0, \mathbb{T}_0 \subset B_0, U_0, \underline{W}$ as before. Let $T_0 \subset G_0$ be a maximal torus, $T \subset B$ Borel, and $U \subset B$ unipotent radical.

Definition 13.1. *Define* $S_{T_0,B} = \{g \subset G \mid g^{-1}F(g) \in F(U)\}$.

The Lang isogeny $S_{T,B} \to F(U)$ is a $G(\mathbf{F}_q)$-covering, with $h \in G_0(\mathbf{F}_q)$ acting via $h.g = hg$. In the SL_2 case, we get an $\mathrm{SL}_2(\mathbf{F}_q)$-covering of $\mathbf{A}^1$ (which should be the Drinfeld curve but it is hard to check).

Observe that $\mathbb{T}_0(\mathbf{F}_q)$ acts on $S_{T_0,B}$ on the right by

$$t.g = gt.$$

($\mathbf{T}_0$ normalizes U, so $F(\mathbb{T}_0)$ normalizes $F(U)$)

Now for a character $\Theta : T_0(\mathbf{F}_q) \to \overline{\mathbf{Q}_l}^{\times}$, we can define

$$R^{\Theta}_{T_0,B} = \sum_i (-1)^i H^i_c(S_{T_0,B}, \overline{\mathbf{Q}_l}^{\times})[\Theta]$$

considered as a virtual $G_0(\mathbf{F}_q)$ representation. Note that $S_{T_0,B}$ has dimension equal to $\dim U = \dim X$, while $\widetilde{X}(\bar{w})$ had dimension $\ell(w)$.

Observe that $U \cap F(U)$ acts on $S_{T_0,B}$ on the right, equivariant for $G_0(\mathbf{F}_q) \times T_0(\mathbf{F}_q)$. Let $\widetilde{X}_{T_0,B} = S_{T_0,B}/(U \cap F(U))$, and quotienting by the affine scheme $U \cap F(U)$ doesn't change the cohomology.

Now $\widetilde{X}_{T_0,B}$ has dimension $\ell(w)$, where $w = \mathrm{Rel}(B, FB)$, the relative position of B in FB. $R^{\Theta}_{T_0,B} = \sum_i (-1)^i H^i_c(\widetilde{X}_{T_0,B}, \overline{\mathbf{Q}_l}^{\times})[\Theta]$.

When $T_0 = \mathbb{T}_0$, we have

$$\widetilde{X}_{\mathbb{T}_0,B} = G_0(\mathbf{F}_q)/U_0(\mathbf{F}_q) \leftarrow S_{\mathbb{T}_0,B}.$$

When $G_0 = \mathrm{SL}_2$ and T_0 is not conjugate to $\mathbb{T}_0$, then $S_{T_0,B} = \widetilde{X}_{T_0,B}$.

This definition of $\widetilde{X}_{\mathbb{T}_0,B}$, but it is not easy to show that it's independent of B. We now go back to think about $\widetilde{X}(w) \leftarrow \widetilde{X}(\bar{w})$. Choose $x \in G(\overline{\mathbf{F}_q})$ such that $x(\mathbb{T}_0 \subset B_0)x^{-1} = (T \subset B)$. Then $FB = F(x)B_0F(x)^{-1}$ and so $x^{-1}F(x) = w$. Moreover,

$$\mathbb{T} \xrightarrow{\mathrm{Ad}x} T$$

is an isomorphism. Then $t \mapsto F(t)$ induces

$$t \mapsto x^{-1}F(xtx^{-1})x = \bar{w}F(t)\bar{w}^{-1}.$$

In other words, T_0 is just $\mathbb{T}(w)$.

Moreover, the map $g \mapsto g.x$ identifies

$$\widetilde{X}_{T_0,B} = \{g \in G \mid g^{-1}F(g) \in FU\}/(U \cap F(U))$$

with

$$\{h \in G \mid (hx^{-1})^{-1}F(hx^{-1}) \in FU\}/(\mathrm{Ad}x^{-1}U \cap \mathrm{Ad}x^{-1}FU).$$

The condition is the same as $h^{-1}F(h) \iff x^{-1}F(x)U_0 = wU_0$. Then we note $\mathrm{Ad}x^{-1}U = U_0$ and $\mathrm{Ad}x^{-1}FU = x^{-1}F(x)U_0F(x^{-1})x = \dot{w}U_0\dot{w}^{-1}$. Therefore the above is identified with

$$\{h \in G \mid h^{-1}F(h) \in \dot{w}U_0\}/U_0 \cap wU_0w^{-1}.$$

We have the following diagram

$$\begin{tikzcd}
\widetilde{Z} = \{g \in G \mid g^{-1}Fg \in \dot{w}U\}/(U \cap \mathrm{Ad}\dot{w}U) \arrow[r] \arrow[d, hook] & Z = \{g \in G \mid g^{-1}Fg \in \dot{w}B\}/(B \cap \mathrm{Ad}\dot{w}B) \arrow[d, hook] \\
G/(U \cap \mathrm{Ad}\dot{w}U) \arrow[r, "\pi"] \arrow[d] & G/(B \cap \mathrm{Ad}\dot{w}B) = \mathcal{O}(w) \arrow[d] \\
G/U \times G/\mathrm{Ad}\dot{w}U \arrow[r, "\mathbb{T}\times\mathbb{T}"] & G/B \times G/\mathrm{Ad}\dot{w}B_0
\end{tikzcd}$$

What is $X(w)$ inside $\mathcal{O}(w)$? We claim that the above diagram identifies Z with $X(w) \hookrightarrow \mathcal{O}(w)$, and $\widetilde{Z} \to G/(U \cap \mathrm{Ad}\dot{w}U)$ with $\widetilde{X}(w)$, equivariant for the actions of $G_0(\mathbf{F}_q)$ and $T(w)^F$. Indeed, $\widetilde{X}(w) \subset G/(U \cap \mathrm{Ad}\dot{w}U)$ is cut out by $gwU\dot{w}g^{-1} = F(g)UF(g^{-1})$. This is equivalent to $g^{-1}F(g) \in \dot{w}U_0$.

Corollary 13.2. *$R^{\Theta}_{T_0,B} = R^{\Theta}(w)$ for $w = Rel(B, FB)$.*

14 Lecture 14: 2/17/23

Let $T_0 \subset B_0 \subset G_0$ as before. Also have $\underline{W}$ and $X = G/B$ as before. We saw a definition (for $w \in \underline{W}(\overline{\mathbf{F}_q})$) on Monday (12):

$$R^\theta(w) = \sum_i (-1)^i H_c^i(\widetilde{X}(\dot{w}, \overline{\mathbf{Q}}_l)[\theta].$$

On Wednesday, we saw that for $T_0 \subset G_0$, $T \subset B$, and $\theta : T_0(\mathbf{F}_q) \to \overline{Q_l}^\times$, we have a definition

$$R^\theta_{T_0,B} = \sum_i (-1)^i H_c^i(\widetilde{X}_{T_0,B}, \overline{Q_l})[\theta].$$

We saw that there was an isomorphism $T(w)^F = T_0(\mathbf{F}_q)$ and a $G_0(\mathbf{F}_q) \times T(w)^F$-equivariant isomorphism $\widetilde{X}_{T_0,B} \to \widetilde{X}(\dot{w})$ ($\dot{w}$ is such that $\mathrm{Rel}(B, FB) = w$).

Today: we will show the independence of B of $R^1_{T_0,B} \rightsquigarrow R^1(w) = R^1(w')$ (if there is $w_1 \in \underline{W}(\overline{\mathbf{F}_q})$ such that $w_1 w F(w_1)^{-1} = w'$).

This happens if and only if $\widetilde{X}(w) \xrightarrow{T(w)^F} X(w)$, so $R^1(w) = \sum_i (-1)^i H_c^i(\widetilde{X}_{T_0,B}, \overline{Q_l}) =: H_c^*(X(w))$ as a $G_0(\mathbf{F}_q)$ virtual character.

Proof. Without loss of generality, we may assume that $l(w_1) = 1$ and that: either $l(w) = l(w')$ or $l(w') = l(w) + 2$ (if $l(w') = l(w) - 2$, just switch w and w'). We will only prove the second case for lack of time.

Facts that we need:

A. If $w = w_1 w_2$ and $l(w) = l(w_1) + l(w_2)$, then $(X \times X) \times_X (X \times X) \to X \times X$ via $(x, y), (y, z) \mapsto (x, z)$ induces an isomorphism $\mathcal{O}(w_1) \times_X \mathcal{O}(w_2) \xrightarrow{\sim} \mathcal{O}(w)$.

B. If $s \in \underline{W}(\mathbf{F}_q)$ has length 1, then $\mathcal{O}(s) \times_X \mathcal{O}(s) \xrightarrow{\sim} \mathcal{O}(s) \cup \mathcal{O}(1) = \overline{\mathcal{O}(s)}$.

Now let $w' \in swF(s)$, then we have two maps $X(w') \xrightarrow{\delta,\gamma} X$ (use Fact A twice), such that $(x, \gamma(x)) \in \mathcal{O}(s)$, $(\gamma(x), \delta(x)) \in \mathcal{O}(w)$, and $(\delta(x), F(x)) \in \mathcal{O}(F(s))$. We would like to compare $X(w')$ to $X(w)$, so maybe we could hope that $\gamma(x) \in X(w)$ or that $F\gamma(x) = \delta(x)$. But Fact B tells us that

$$(\delta(x), F(x)), (F(x), F(\gamma(x)) \in \mathcal{O}(F(s)) \times_X \mathcal{O}(F(s)) \Rightarrow (\delta(x), F(\gamma(x)) \in \overline{\mathcal{O}(F(s))} = \mathcal{O}(F(s)) \cup \mathcal{O}(1).$$

So instead, we define

$$X_1 = \{x \in X(w') : \delta(x) = F(\gamma(x))\},$$
$$X_2 = X(w') - X_1 = \{x \in X(w') : (\delta(x), F(\gamma(x))) \in \mathcal{O}(F(s))\}.$$

Then there is a map $X_1 \to X(w)$ given by $x \mapsto \gamma(x)$. We claim that the map γ is an $\mathbf{A}^1$-fibration. To see this, note that $\mathcal{O}(s) \xrightarrow{pr_2} X$ is an $\mathbf{A}^1$-fibration. So there is an $\mathbf{A}^1$-worth of choices of x such that $\gamma(x) = y$ and $(x, y) \in \mathcal{O}(s)$. Hence $(F(x), F(y)) \in \mathcal{O}(F(s))$ and so

$$(x, y, F(y), F(x)) = (x, \gamma(x), \delta(x), F(x)).$$

Then all these choices of x lie in $X(w')$.

The upshot is that the following diagram is *Cartesian*:

$$\begin{tikzcd} X_1 \arrow[r, "\gamma"] \arrow[d, "{(x,\gamma(x))}"] & X(w) \arrow[d] \\ \mathcal{O}(s) \arrow[r, "\mathbf{A}^1", "pr_2"'] & X \end{tikzcd}.$$

Hence $H_c^*(X_1) = H_c^*(X(w))$ as virtual $G_0(\mathbf{F}_q)$-representations (using the fact that pr_2 is an $\mathbf{A}^1$-fibration).

It remains to show that $H_c^*(X_2) = 0$. Recall that $(\delta(x), F(\gamma(x)) \in \mathcal{O}(F(s))$ for $x \in X_2$, and $l(F(sw)) = l(F(s))$. By Fact A, this implies $(\delta(x), F(\delta(x))) \in \mathcal{O}(F(w))$. Now, we study the map $\delta : X_2 \to X(F(sw))$. Define $X_2' \subseteq X_2 \times X(sw)$ as the subset $\{(x, y) : F(y) = \delta(x)\}$. So we get a Cartesian diagram

$$\begin{tikzcd} X_2' \arrow[r, "pr_2", "\delta'"'] \arrow[d, "pr_1"] & X(sw) \arrow[d, "F"] \\ X_2 \arrow[r, "\delta"] & X(F(sw)) \end{tikzcd}.$$

We claim that the map δ' is an $\mathbf{A}^1 - \{0\}$-fibration and $G_0(\mathbf{F}_q)$ acts trivially on fibers (complement of zero section of line bundle). Once we have the claim, then $H_c^*(X_2) = 0$ as a $G_0(\mathbf{F}_p)$-representation as

$$H_c^*(X_2) = H_c^*(X(F(sw)) - H_c^*(X(F(sw)) = 0.$$

We won't prove this claim (ran out of time), but it should be similar to the above. $\square$

No lecture next Monday 2/20/23 (Presidents Day).

15 A Character Formula for $R^{\theta}_{T_0,B}$

Recall that on Friday we showed that $R^1(w) = R^1(\dot{w})$ if there exists w_1 such that $\dot{W} = w_1 w F(w_1^{-1})$. This is equivalent to saying $R^{\theta}_{T_0,B}$ does not depend on B.

Definition 15.1. *For $T_0 \subset G_0$ over $\mathbf{F}_q$, we define the Green function Q_{T_0,G_0} is the restriction to the set of unipotent elements in $G_0(\mathbf{F}_q)$ of the trace function of $R^1_{T_0,B}$ for some choice of B.*

Theorem 15.2. *Let $x = su$ where s is unipotent and s semisimple. Let Z^0_s to the identity component of $Z(s)$, the centralizer of s. Then*

$$tr(x, R^{\theta}_{T_0,B}) = \frac{1}{|Z^0(s)(\mathbf{F}_q)|} \sum_{\substack{g_0 \in G_0(\mathbf{F}_q), \\ g_0 T_0 g_0^{-1}}} \in Z^0(s) Q_{AdgT_0,Z(s)}(u)\theta(g_0^{-1} s g_0)$$

To see why this formula make sense, we note that u commutes with s, so $u \in Z(s)(\mathbf{F}_q)$, and it is a fact that $u \in Z^0(s)$. So it makes sense to evaluate the Green function at u. Moreover, since $g_0 T_0 g_0^{-1} \in Z^0(s)$, we have $s \in g_0 T_0 g_0^{-1}$, so $g_0^{-1} s g_0 \in T_0(\mathbf{F}_q)$.

The formula is clearly independent of B (there is no B in the formula).

Proof. Recall that we have the diagram

$$\begin{array}{ccc} \widetilde{X}_{T_0,B} & = & \{g \in G \mid g^{-1}F(g) \in F(U)\}/U \cap FU \\ \Big\downarrow{\scriptstyle T_0(\mathbf{F}_q)} & & \\ \widetilde{X}_{T_0,B} & = & \{g \in G \mid g^{-1}F(g) \in F(B)\}/B \cap FB \end{array}$$

Let $I(s) = H_0(X^s_{T_0,B})$. Then s acts on $H^{-1}(y)$ for $y \in X^s_{T_0,B}$. This gives us an element $t(s,y) \in T_0(\mathbf{F}_q)$. For all $x \in H^{-1}(y)$ we have $s.x = x.t(s,y)$. Fact: $t(s,y)$ is constant on connected components of $X^s_{T_0,B}$, and

$$\mathrm{tr}(su, H^*_c(\widetilde{X}_{T_0,B})[\theta]) = \sum_{y \in I(s)} \mathrm{tr}(u, H^*_c(X^s_{T_0,B}))\theta(t(s,y))$$

We note that $B_1 = gBg^{-1} \in X_{T_0,B}$ is s-fixed if and only if $sB_1s^{-1} = B_1$ or $s \in B_1$. For such B_1, we can always find h such that $hBh^{-1} = B_1$ and $hTh^{-1} \subset Z^0(s)$ for $s \in hTh^{-1}$. The set of such h is a $Z^0(s)$-homogenous space in G, by Lang, it has a rational point (g_0). Thus this homogenous space is of the form $Z(s)^0 g_0$ for $g_0 \in G_0(\mathbf{F}_q)$.

Note for $g_0 \in G_0(\mathbf{F}_q)$ such that $g_0T_0g_0^{-1} \in Z^0(s)$, we look at

$$X_{\mathrm{Ad}g_0T_0,\mathrm{Ad}g_0B\cap Z^0(s)} = \{z \in Z^0(s) \mid z^{-1}F(z) \in F(g_0Bg_0^{-1})\}/(F(g_0Bg_0^{-1}) \cap g_0Bg_0^{-1})$$

with a map

$$\begin{aligned} X_{\mathrm{Ad}g_0T_0,\mathrm{Ad}g_0B\cap Z^0(s)} &\to X^s_{T_0,B} \\ z &\mapsto z.g_0 \end{aligned}$$

.

This is well-defined since $z^{-1}F(z) \in g_0F(B)g_0^{-1}$, implying $g_0^{-1}z^{-1}F(z)F(g_0) \in F(B)$. We are s fixed because $g_0Bg_0^{-1} \ni s$ so also $zg_0Bg_0^{-1}z^{-1} = s$.

The upshot is that there is a map

$$\coprod_{g_0\in G_0(\mathbf{F}_q),g_0Tg_0^{-1}\in Z^0(s)} X_{\mathrm{Ad}g_0T_0,\mathrm{Ad}g_0B\cap Z^0(s)} \to X^s_{T_0,B}$$

that is surjective on $\overline{\mathbf{F}_q}$-points, and we are over counting by a factor of $\frac{1}{|Z^0(s)(\mathbf{F}_q)|}$.

Claim 1: This map is an isomorphism after restricting one g_0 from each $Z^0(s)(\mathbf{F}_q)$-coset in $G_0(\mathbf{F}_q)$. (Instead of restricting, we can quotient by $Z^0(s)(\mathbf{F}_q)$)

Claim 2: The function $t(s, y)$ on the image of X_{g_0} is equal to $g_0^{-1}sg_0 \in T_0(\mathbf{F}_q)$.

For claim 1, there is no overlap between factors, and individual factor maps injectively by inspection. For claim 2, we have the diagram

$$\begin{tikzcd} \widetilde{X}_{\mathrm{Ad}g_0T_0,\mathrm{Ad}g_0B\cap Z^0(s)} \arrow[r] \arrow[d, "g_0T(\mathbf{F}_q)g_0^{-1}"'] & \widetilde{X}_{T_0,B} \arrow[d, "T(\mathbf{F}_q)"] \\ X_{\mathrm{Ad}g_0T_0,\mathrm{Ad}g_0B\cap Z^0(s)} \arrow[r] & X_{T_0,B} \end{tikzcd}$$

$\square$

16 Lecture 16: 2/24/23

Today: towards a Mackey formula for DL induction.

Let G_0 be as before. Our goal is to:

- Compute $\langle R^{\theta}_{T_0}, R^{\theta'}_{T'_0}\rangle$ in a reasonable way.
- Try to show that $R^{\theta}_{T_0}$ and $R^{\theta'}_{T'_0}$ are *disjoint* as virtual characters, under general assumptions.

Strategy: study the $T_0(\mathbf{F}_q) \times T'_0(\mathbf{F}_q)$-action on

$$H^i_c\left(G_0(\mathbf{F}_q)\backslash(S_{T_0,B} \times S_{T'_0,B)}), \overline{\mathbf{Q}_l}\right).$$

Keep in mind, for SL_2, we broke up this variety into two pieces corresponding to the 2 terms in the Mackey formula.

Fix $B \supset T$, $B' \supset T'$, with unipotent radicals U, U'. Then recall $S_{T_0,B} = \{g \in G : g^{-1}F(g) \in FU\}$. Similar for T'_0, B'. Want to write down the quotient

$$\begin{tikzcd} S_{T_0,B} \times S_{T'_0,B'} \arrow[r] \arrow[d, "G_0(\mathbf{F}_q)\times G_0(\mathbf{F}_q)"'] & \overline{S} \arrow[dl] \\ FU \times FU' & \end{tikzcd}$$

We define $\overline{S} = \{(x, x', y) \in FU \times FU' \times G : xF(y) = yx'\}$, and the map to $FU \times FU'$ is to forget y. The horizontal map in (16)

$$\begin{aligned} S_{T_0,B} \times S_{T_0,B} &\to \overline{S} \\ (g, g') &\mapsto (g'F(g), g^{-1}F(g), g^{-1}g') \end{aligned}$$

This is clearly invariant under left multiplication by (g_0, g_0) for $g_0 \in G_0(\mathbb{F}_q)$

Lemma 16.1. *α induces an isomorphism $S_{T_0,B} \times S_{T'_0,B)}/G_0(\mathbf{F}_q) \to \overline{S}$.*

Proof. If (g_1, g'_1) and (g_2, g'_2) both map to (x, x', y), then $g_1 = g_0g_2$, $g'_1 = g'_0g'_2$ for $g_0, g'_0 \in G_0(\mathbf{F}_q)$. Then $g_1^{-1}g'_1 = g_2^{-1}g'_2$ forces $g_0^{-1}g'_0 = 1$, proving injectivity.

For surjectivity, we want to fiddle around with choosing g_0g_1, g'_0g_2 such that $g_1^{-1}g_0^{-1}g'_0g_2 = y$. We're not going to fill in the rest of the proof. □

Stratification argument: define $N(T, T') = \{g \in G : gTG^{-1} = T'\}$. This is a left $N(T')$ and right $N(T)$-torsor. Define $\underline{W}(T, T')$ as $N(T, T')/T' = T/N(T, T')$, which is a left $\underline{W}(T')$ torsor and right $W(T)$-torsor. There is a Bruhat decomposition for $\underline{W}(T, T')$:

$$G = \bigcup_{w \in \underline{W}(T,T')(\overline{\mathbf{F}_q})} \underbrace{FB\dot{w}FB'}_{G_w},$$

and then via $\overline{S} \to G$ sending (x, x', y) to y, we get a "stratification" $\overline{S} = \bigcup_w \overline{S}_w$.

We now want to extend the action of $T_0(\mathbf{F}_q) \times T_0(\mathbf{F}'_q)$ to the action of an algebraic group $H_w \subseteq T \times T'$ containing $T_0(\mathbf{F}_q) \times T'_0(\mathbf{F}_q)$. Then $H_w^0 \cap (T_0(\mathbf{F}_q) \times T'_0(\mathbf{F}_q))$ will act trivially on $H_c^i(\overline{S}_w)$ for all i.

Definition 16.2. *Write* $H_w = \{(t, t') \in T \times T' : t'F(t')^{-1} = F(\dot{w})^{-1} t F(t)^{-1} F(\dot{w})\}$ *for some (equiv. any)* $\dot{w} \in N(T, T')(\overline{\mathbf{F}_q})$ *mapping to* w.

To make this act, we need to refine the Bruhat decomposition on $\overline{S}_w$.

Claim: $G'_w = \{(u, n, u', v) : U \times N(T, T') \times U' \times (U')^-\}$, where $(U')^-$ is the opposite of U' with respect to T'. Moreover, the map $G'_w \to G_w$ sending $(u, n, u', v) \mapsto unu'$ is an isomorphism. So there is a map $G_w \to G'_w$ via $y \mapsto (u_y, n_y, u'_y, v'_y)$.

Definition 16.3. *Let* (t, t') *act on* $\overline{S}_w$ *by*

$$(t, t') \cdot (x, x', y) = \left(t^{-1} x F(u_y) t F(t^{-1} u_y^{-1} t), (t')^{-1} x' F(u'_y)^{-1} t' F((t')^{-1} u'_y t'), t^{-1} y t'\right).$$

To check that this preserves $\overline{S}_w$, we need:

$$t^{-1} x F(u_y) t F(t^{-1} u_y^{-1} t) F(t^{-1}) F(y) F(t') = t^{-1} y t' (t')^{-1} x' F(u'_y)^{-1} t' F((t')^{-1} u'_y t').$$

Cancelling t^{-1} from the left and $F(t')$ from the right and using $t'(t')^{-1} = 1$ the desired inequality becomes

$$x F(u_y) t F(t^{-1} u_y^{-1} t) F(t^{-1}) F(y) = y x' F(u'_y)^{-1} t' F((t')^{-1} u_y),$$

and after applying $xF(y) = yx'$ this becomes

$$x F(u_y) t F(t^{-1} u_y^{-1} t) F(t^{-1}) F(y) = x F(y) F(u'_y)^{-1} t' F((t')^{-1} u_y).$$

Now we can cancel x from the left and write $y = u_y n_y u'_y$ everywhere to get

$$F(u_y) t F(t^{-1}) F(u_y^{-1}) F(t) F(t^{-1}) F(u_y) F(n_y) F(u'_y) = F(u_y) F(n_y) F(u_y) F(u'_y)^{-1} t' F(t')^{-1} F(u_y)$$

which simplifies to

$$F(u_y) t F(t^{-1}) F(n_y) F(u'_y) = F(u_y) F(n_y) t' F(t')^{-1} F(u_y).$$

It follows from the definition of H'_w that $F(n_y) t' F(t')^{-1} = t F(t)^{-1} F(n_y)$ and so the equality is true by arguing backwards.

17 Lecture 17: 2/27/23

Today: disjointness for Deligne-Lusztig Induction.

Recap: Last time (Lecture 16) we studied $\overline{S} = S_{T_0,B} \times S_{T_0',B'}/G_0(\mathbf{F}_q)$ and broke it up into $\overline{S} = \bigcup_{w \in \underline{W}(T,T')} \overline{S}_w$, and showed $T_0(\mathbf{F}_q) \times T_0'(\mathbf{F}_q) \subset H_w \subseteq T \times T'$ acts on $\overline{S}_w$.

Hence H_w^0 (connected component of identity) intersect $T_0(\mathbf{F}_q) \times T_0'(\mathbf{F}_q)$ acts trivially on $H_c^i(\overline{S}_w)$. What does this say about $(T_0, \theta), (T_0', \theta')$? (Note that $(\theta \times \theta')$ is trivial on $H_w^0 \cap (T_0(\mathbf{F}_q) \times T_0'(\mathbf{F}_q))$.)

Definition 17.1. *For $n \in \mathbb{N}$, then there is a (surjective) "norm" map*

$$N : T_0(\mathbf{F}_{q^n}) \to T_0(\mathbf{F}_q)$$

given by

$$\prod_{i=0}^{n-1} F^i$$

for F the Frobenius.

Definition 17.2. *Let (T_0, θ) and (T_0', θ') as before, and we say (T_0, θ) is* geometrically conjugate *to (T_0', θ') if there is n and $g \in G_0(\mathbf{F}_{q^n})$ such that $gT_0g^{-1} = T'$ and $g(\theta \circ N)g^{-1} = \theta' \circ N$.*

Let's go back to the case $G = \mathrm{SL}_2$, and T_0, T_0' two distinct tori (say $T_0(\mathbf{F}_q) = \mu_{q-1}$, $T_0'(\mathbf{F}_q) = \mu_{q+1}$). Then for even $n \in \mathbb{N}$, these tori are conjugate in SL_2 over $\mathbf{F}_{q^n}$:

$$\begin{tikzcd} & \mu_{q+1} \\ \mu_{q^2-1} \arrow[ur, "N"] \arrow[dr, "N"] & \\ & \mu_{q-1} \end{tikzcd}$$

So they are geometrically conjugate. Moreover θ, θ' must satisfy $\theta^2 = (\theta')^2 = 1$.

Theorem 17.3. *If (T_0, θ) and (T_0', θ') are* not *geometrically conjugate, then for all i, $H_c^i(\overline{S}_w)[\theta \times \theta'] = 0$. In particular, $R_{T_0}^{\theta}$ and $R_{T_0}^{\theta'}$ share no common factors as $G_0(\mathbf{F}_q)$-representations.*

Proof. We show that for all $w \in \underline{W}(T, T')$, we have $H_c^j(S_w)[\theta \times \theta'] = 0$ for all j.

Fix an isomorphism $\overline{F}_q^{\times} \cong \mathbf{Z}_{(p)}/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$. For a torus T_0 over $\mathbf{F}_q$, let $X_*(T) = \mathrm{Hom}_{\mathbf{F}_q}(\mathbb{G}_m, T)$ be the cocharacter lattice equipped with an endomorphism F (e.g. $T_0 = \mathbb{G}_m, X_*(T) = \mathbf{Z}$, F is the Frobenius $\cdot q$).

Then

$$T(\overline{\mathbf{F}_q}) = X_*(T) \otimes_{\mathbf{Z}} \overline{\mathbf{F}_q}^{\times} = X_*(T) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}/\mathbf{Z}.$$

We have an exact sequence

$$0 \to T_0(\mathbf{F}_q) \to= X_*(T) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}/\mathbf{Z} \xrightarrow{F-1} X_*(T) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}/\mathbf{Z} \to 0,$$

and then we apply the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & X_*(T) \arrow[r] \arrow[d, "F-1"] & X_*(T) \otimes \mathbf{Z}_{(p)} \arrow[r] \arrow[d, "F-1"] & X_*(T) \otimes \mathbf{Z}_{(p)}/\mathbf{Z} \arrow[r] \arrow[d, "F-1"] & 0 \\ 0 \arrow[r] & X_*(T) \arrow[r] & X_*(T) \otimes \mathbf{Z}_{(p)} \arrow[r] & X_*(T) \otimes \mathbf{Z}_{(p)}/\mathbf{Z} \arrow[r] & 0 \end{tikzcd}$$

to get

$$0 \to X_*(T) \xrightarrow{F-1} X_*(T) \to T_0(\mathbf{F}_q) \to 0$$

exact. For $n \in \mathbb{N}$, we get

$$\begin{tikzcd} 0 \arrow[r] & X_*(T) \arrow[r] \arrow[d, "N"] & X_*(T) \arrow[r] \arrow[d, "\mathrm{id}"] & T_0(\mathbf{F}_{q^n}) \arrow[r] \arrow[d, "N"] & 0 \\ 0 \arrow[r] & X_*(T) \arrow[r] & X_*(T) \arrow[r] & T_0(\mathbf{F}_q) \arrow[r] & 0 \end{tikzcd}$$

(should check that the right square actually commutes with the norm map $T_0(\mathbf{F}_q) \xrightarrow{N} T_0(\mathbf{F}_q)$).

The upshot is that (T, θ) and (T', θ') are geometrically conjugate if and only if there is $g \in G(\overline{F}_q)$ such that $gTG^{-1} = T'$ and $g\theta g^{-1} = \theta'$ as characters of $X_*(T')$.

Back to the geometry: recall $H_w \subset T \times T'$ was the kernel of the composite

$$T \times T' \xrightarrow{x^{-1}Fx} T \times T' \xrightarrow{t^{-1}\mathrm{Ad}F(\dot{w})^{-1}(t')} T.$$

So $X_*(F_w) = X_*(H_W^0)$ is the kernel of

$$X_*(T) \times X_*(T') \xrightarrow{F-1} X_*(T) \times X_*(T') \xrightarrow{\alpha := \mathrm{Ad}F(\dot{w})^{-1}(x') - x} X_*(T).$$

Now, if there is $w \in \underline{W}(T, T')(\overline{\mathbf{F}_q})$ such that $H_c^*(\overline{S}_w)[\theta \times \theta'] \neq 0$, then we have the torus $T(\mathbf{F}_q) \times T_0'(\mathbf{F}_q) \subset H \subset T \times T'$ defined by

$$\{(t, t') \mid t^{-1}F(t) = F(w)^{-1}t^{-1}F(t')F(w')\}.$$

We are trying to understand what it means for θ, θ' if

$$\theta \times \theta'|_{H_w^0 \cap (T_0(\mathbf{F}_q) \times T_0'(\mathbf{F}_q))}$$

is trivial. Write $K = X_*(F_w)$. Then $\theta \times \theta'$, considered as a character of $X_*(T) \times X_*(T')$ is trivial on K. If it were trivial on $\ker \alpha$, then $\theta \cdot \mathrm{Ad}F(w)^{-1}(\theta') = 1$, which implies that θ' is geometrically conjugate to θ^{-1}.

So now it remains to show the following:

Lemma 17.4. *If $\theta \times \theta'$ is trivial on $H_w^0 \cap (T_0(\mathbf{F}_q) \times T_0'(\mathbf{F}_q))$ then $\theta \times \theta'$ is in fact trivial on* $\ker\alpha$*.*

Proof. We know

$$\begin{array}{ccc} H_w^0(\mathbf{F}_{q^m}) & \longrightarrow & \overline{\mathbf{Q}_l}^\times \\ \downarrow & & \\ T_{\ker\alpha}(\mathbf{F}_{q^n}) & \longrightarrow & \overline{\mathbf{Q}_l}^\times \end{array}$$

and every q^n-point of $T_{\ker\alpha}(\mathbf{F}_{q^n})$ lifts to a q^m-point of H_w^0. (This has some problems.) □

□

Not sure what the following is proving.

We wanted to show that $R_T^\theta, R_T^{\theta'}$ are disjoint when θ, θ' are not geometrically conjugate. Instead, we proved that

$$H_c^1(\overline{S})[\theta \times \theta'] \neq 0$$

implies θ^{-1} is geometrically conjugate to θ'. We now need

$$(R^{\theta_{T_0}})^\vee = R_{T_0}^{\theta^{-1}}.$$

Proof. We just look at the character formula.

$$\operatorname{tr}(sv, (R^{\theta_{T_0}})^\vee) = \frac{1}{Z(s)(\mathbf{F}_q)} \sum_{g \in G_0(\mathbf{F}_q), gsg^{-1} \in T_0} \overline{Q_{gg^{-1}Z(s)}(v)\theta(g^{-1}sg)}$$

Now $\overline{\theta(g^{-1}sg)} = \theta^{-1}(g^{-1}sg)$, and $\overline{Q}_{-,-}(v) = Q_{-,-}(v)$ since Q is $\mathbf{Z}$. $\square$

Corollary 17.5. *If R_T^θ and $R_{T_0'}^{\theta'}$ share a common $G_0(\mathbf{F}_q)$-factor, then (T_0, θ) and (T_0', θ') are geometrically conjugate.*

Note that (T_0, θ) and (T_0', θ') are geometrically conjugate, if there exists $w \in W_q(\overline{F_q})$ such that $\mathrm{Ad}w \circ \theta = \theta'$ as characters of $X_*(T)$.

18 Lecture 18

Now we proceed to prove a Mackey theorem. Recall we have fixed the notations G_0, T_0, T_0', θ, θ', $\underline{W}(T, T')$.

Theorem 18.1.

$$\langle R_{T_0'}^\theta, R_{T_0'}^{\theta'} \rangle = \#\{w \in \underline{W}(T, T')(\mathbf{F}_q) \mid Adw \circ \theta = \theta'\}.$$

Corollary 18.2. *If θ is not equal to $Adw\theta$ for any $w \in W(T, T')(\mathbf{F}_q)$, then up to a sign R_{T_0} is some irreducible representation.*

A naive attempt of proving the theorem is to use the character formula. Let's see how that would go. By the character formula, the inner product is

$$\begin{aligned}
\langle R_{T_0'}^\theta, R_{T_0'}^{\theta'} \rangle &= \frac{1}{|G_0(\mathbf{F}_q)|} \sum_g \langle \operatorname{tr}(g, R_{T_0}^\theta), \operatorname{tr}(g^{-1}, R_{T_0'}^{\theta'}) \rangle \\
&= \frac{1}{|G_0(\mathbf{F}_q)|} \sum \frac{1}{|Z|} \sum_{g, g'} \theta(g^{-1}sg)\theta'({g'}^{-1}sg')^{-1} \\
&= \sum_{v \in Z^0(s)(\mathbf{F}_q)} Q_?(w) \cdot Q_?(w^{-1})
\end{aligned}$$

where the last line is some mysterious green functions. To understand the last term, we need to prove another theorem.

Theorem 18.3.

$$\frac{1}{|G_0(\mathbf{F}_q)|}\sum_{v\in G_0(\mathbf{F}_q)} Q_{T_0,G}(v)Q_{T_0',G}(v) = \frac{|N(T,T')(\mathbf{F}_q)|}{|T_0(\mathbf{F}_q)||T_0'(F_q)|}. \tag{2}$$

We will continue next time.

19 Lecture 19

Last time we had two theorems. To prove them, we want to use induction, which will reply on the following proposition.

Proposition 19.1 (Crucial Proposition)**.** *Let A be the left side of 2, and let B be the right side of 2. If 18.3 holds for $Z^0(s)$ for all non-central semisimple $s \in G_0(\mathbf{F}_q)$, then we have*

$$\langle R_{T_0'}^{\theta}, R_{T_0'}^{\theta'} \rangle = \#\{w \in \underline{W}(T,T')(\mathbf{F}_q) \mid Adw \circ \theta = \theta'\} + \alpha_{\theta,\theta'}$$

where

$$\alpha_{\theta,\theta'} = \frac{1}{|G_0(\mathbf{F}_q)|} \sum_{s \in Z_G(\mathbf{F}_q)} \theta(s)\theta'(s)^{-1}(A-B).$$

proof of Theorem 18.3. We use induction. The base case is $G_0 = \{e\}$ which is trivial. Now assume the hypotheses of 19.1 hold for G_0. Let $\overline{G} = G/Z_G$ and $\overline{T_0}$, $\overline{T_0'}$, etc. We claim that 18.3 is true for $\overline{G}, \overline{T_0}, \overline{T_0'}$ if and only if it is true for G, T_0, T_0'. First of all, $Q_{T_0,G}(u) = \mathrm{tr}(u, H_c^*(X_{T_0,B}))$ and the action of G on $X_G = X_{\overline{G}}$ factors through $\overline{G}$, so the action of $G_0(\mathbf{F}_q)$ on X factors through $\overline{G_0}(\mathbf{F}_q)$. Hence $Q_{T_0,G} = Q_{\overline{T_0},\overline{G}}$. It is a fact that $G_0(\mathbf{F}_q)$ and $\overline{G_0}(\mathbf{F}_q)$ have the same number of unipotent elements. Therefore, the left side changes by a constant factor when changing G to $\overline{G}$, etc. Now we use

$$1 \to Z_G \to G_0 \to \overline{G_0} \to 1$$

induces

$$1 \to Z_G(\mathbf{F}_q) \to G_0(\mathbf{F}_q) \to \overline{G_0}(\mathbf{F}_q) \to H^1(\mathbf{F}_q, Z_G) \to 1$$

and we have the exact sequences for T's. Therefore the left side changes by $\frac{|Z_G(\mathbf{F}_q)|}{|H^1(\mathbf{F}_q,Z_G)|}$, and so does the right side. This proves the claim.

Now there are two cases. Case 1 is that either $T_0(\mathbf{F}_q)$ or $T_0'(\mathbf{F}_q)$ has a non-trivial character. Case 2 is that $q = 2$ and $T_0 \cong \mathbb{G}_{m,\mathbf{F}_q} \cong T_0'$. The interesting case is case 1: WLOG let θ be a nontrivial character of $T(\mathbf{F}_q)$. Then $\langle R_{T_0}^{\theta}, R_{T_0'}^{1} \rangle = 0$. So using Proposition 19.1, we obtain

$$0 = \alpha_{\theta,1} = \sum_{s \in Z_G(\mathbf{F}_q)} \theta(s)1(s)^{-1}(A-B) = A - B.$$

Note that this shows $A - B$ is 0 regardless of what θ and θ' are, so Theorem 18.3 is true.

In case 2, T_0 and T_0' are maximal split, so $X_{T_0,B} = X(\mathbf{F}_q)$ and the proof is explicit. □

proof of Theorem 18.1. Use Proposition 19.1 and Theorem 18.3. □

It remains to prove the crucial proposition.

proof of Proposition 19.1. The character formula gives

$$\begin{aligned}\langle R^{\theta}_{T'_0}, R^{\theta'}_{T'_0}\rangle &= \frac{1}{|G_0(\mathbf{F}_q)|}\sum_g \langle \mathrm{tr}(g, R^{\theta}_{T_0}), \mathrm{tr}(g^{-1}, R^{\theta'}_{T'_0})\rangle \\ &= \frac{1}{|G_0(\mathbf{F}_q)|}\sum \frac{1}{|Z|}\sum_{g,g'} \theta(g^{-1}sg)\theta'(g'^{-1}sg')^{-1} \\ &= \sum_{v\in Z^0(s)(\mathbf{F}_q)} Q_{\mathrm{Ad}gT, Z^0(s)}(w)\cdot Q_{\mathrm{Ad}g'T, Z^0(s)}(w^{-1})\end{aligned}$$

By the assumption, we can write this as

$$\alpha_{\theta,\theta'} + \frac{1}{|G_0(\mathbf{F}_q)|}\sum_{s\in G_0(\mathbf{F}_q)} \frac{1}{|Z^0(s)(\mathbf{F}_q)|}\sum_{g,g'} \theta(g^{-1}sg)\theta'(g'^{-1}sg')^{-1}\frac{|N_{Z^0(s)}(\mathrm{Ad}gT_0, \mathrm{Ad}g'T'_0)(\mathbf{F}_q)|}{|T_0(\mathbf{F}_q)||T'_0(\mathbf{F}_q)|}.$$

To deal with the summation, we reparametrize the set

$$\{(g, g', n_1) \mid g^{-1}sg \in T_0, g'^{-1}sg' \in T'_0, n_1 \in N_{Z^0(s)}(\mathrm{Ad}g'T')(\mathbf{F}_q)\}$$

to

$$I = \{(g, n, n_1) \in G_0(\mathbf{F}_q) \times N_G(T, T')(\mathbf{F}_q) \times Z^0(g)(\mathbf{F}_q)\}$$

via the map $(g, g', n_1) \mapsto (g, g'^{-1}n_1 g, n_1)$, which is visibly injective (and actually bijective). Using this, we get

$$\begin{aligned}&\alpha_{\theta,\theta'} + \frac{1}{|G_0(\mathbf{F}_q)|}\sum_{s\in G_0(\mathbf{F}_q)} \frac{1}{|Z^0(s)(\mathbf{F}_q)|}\sum_{(g,n,n_1)\in I} \theta(g^{-1}sg)\theta'(n^{-1}g^{-1}sgn)^{-1}\frac{|Z^0(s)(\mathbf{F}_q)|}{|T_0(\mathbf{F}_q)||T'_0(\mathbf{F}_q)|} \\ &= \alpha_{\theta,\theta'} + \frac{1}{|G_0(\mathbf{F}_q)|}\sum_{\substack{t\in T_0(\mathbf{F}_q),\\ n\in N(T,T')(\mathbf{F}_q)}} \theta(t)\theta'(n^{-1}tn)^{-1}\frac{|G_0(\mathbf{F}_q)|}{|T_0(\mathbf{F}_q)||T'_0(\mathbf{F}_q)|} \\ &= \alpha_{\theta,\theta'} + \frac{1}{|T_0(\mathbf{F}_q)|}\sum_{n\in N(T,T')(\mathbf{F}_q)} \frac{1}{|T'_0(\mathbf{F}_q)|}\sum_{t\in T_0(\mathbf{F}_q)} \theta(t)\theta'(n^{-1}tn)^{-1} \\ &= \alpha_{\theta,\theta'} + \#\{w \in \underline{W}(T, T')(\mathbf{F}_q) \mid \mathrm{Ad}w \circ \theta = \theta'\}.\end{aligned}$$

$\square$

20 Lecture 20: 3/6/23

Today: Traces of $R_{T_0}^\theta$ on semisimple elements (part I).

We want to know $\mathrm{tr}(e, R_{T_0}^\theta)$. The character formula tells you what this is, in terms of $Q_{T_0,G_0}(e)$, the Euler characteristic of $X_{T_0,B}$ (for some $B \supset T$).

Now, recall the Steinberg representation: for $SL_2(\mathbf{F}_q)$, we write

$$\mathrm{St} = (\mathrm{Fun}(\mathbf{P}^1(\mathbf{F}_q) \to \overline{\mathbf{Q}_l})/\overline{\mathbf{Q}_l}.$$

For $\mathrm{GL}_3(\mathbf{F}_q)$, there are four maps

$$\begin{tikzcd} FL(V) \arrow[r] \arrow[d] & \mathbf{P}(\check{V}) \arrow[d] \\ \mathbf{P}(V) \arrow[r] & \{*\} \end{tikzcd}.$$

The left vertical map takes a flag $0 \subsetneq V_1 \subsetneq V_2 \subsetneq V$ to the 1-dimensional subspace V_1. The right vertical map takes it to the dual $\check{V_2}$ of V_2, which is one-dimensional.

So intuitively, we have

$$\mathrm{St}_{\mathrm{GL}_3} = (\mathrm{Fun}(\mathbf{P}^1(\mathbf{F}_q) \to \overline{\mathbf{Q}_l})/(\text{Functions pulled back from } \mathbf{P}(V)(\mathbf{F}_q) \text{ on } \mathbf{P}(\check{V})(\mathbf{F}_q)).$$

To make this rigorous, we also know that $FL = \mathrm{GL}_3 / B$ and

$$\text{Functions pulled back from } \mathbf{P}(V)(\mathbf{F}_q) \text{ on } \mathbf{P}(\check{V})(\mathbf{F}_q) = \mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_3(\mathbf{F}_q)} 1.$$

So

$$\mathrm{St}_{GL_3} = \mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_3(\mathbf{F}_q)} 1 - \mathrm{Ind}_{P_1(\mathbf{F}_q)}^{\mathrm{GL}_3(\mathbf{F}_q)} 1 - \mathrm{Ind}_{P_2(\mathbf{F}_q)}^{\mathrm{GL}_3(\mathbf{F}_q)} 1 + 1.$$

For general G_0 with $B_0 \subset G_0$, define

$$\mathrm{St}_{G_0} = \sum_{P_0 \supset B_0} (-1)^{r_0} \mathrm{Ind}_{P_0(\mathbf{F}_q)}^{G_0(\mathbf{F}_q)} 1,$$

where $r(P_0)$ is the *semisimple* $\mathbb{F}_q$*-rank* of P. This $r(P_0)$ is the dimension of a maximal $\mathbb{F}_q$-split torus in P/RP, or take $L \coloneqq P/U_P$ and then take the rank of L/ZL. Here, "maximal $\mathbb{F}_q$-split torus" just means the maximal $(\mathbb{G}_m)_{\mathbf{F}_q}^{\oplus k}$ that we can fit in L/ZL over $\mathbf{F}_q$.

Fact: St_{G_0} is actually an irreducible representation. In fact, it should be the quotient of the set of $\overline{\mathbf{Q}_l}$-valued functions on $(G_0/B_0)(\mathbb{F}_q)$ by the $G_0(\mathbb{F}_q)$-submodule generated by all the functions pulled back from $(G_0/P_0)(\mathbb{F}_q)$ for all parabolics $G_0 \supset P_0 \supset B_0$. Thus st_{G_0} is a quotient of the induction $\mathrm{ind}_{B_0(\mathbb{F}_q}^{G_0(\mathbb{F}_q)} 1$

Some notation: for $g \in G_0(\mathbf{F}_q)$, write $\mathrm{St}_{G_0}(g) = \mathrm{tr}(g, \mathrm{St}_{G_0})$.

Theorem 20.1.
$$Q_{T_0,G_0}(e) = (-1)^{\sigma(G_0)-\sigma(T_0)} \cdot \frac{|G_0(\mathbf{F}_q)|}{\mathrm{St}_{G_0}(e)\cdot|T_0(\mathbf{F}_q)|},$$
where $\sigma(H_0)$ is the $\mathbf{F}_q$-split rank.

As an example, in the case of $\mathrm{SL}_{2,\mathbf{F}_q}$, the theorem gives
$$1-q = (-1)\frac{q(q-1)(q+1)}{q(q+1)}.$$

We will prove this by induction on the dimension of G_0 (i.e. assume theorem for $Z^0(s)$ for all non-central, semisimple s). We start with the following lemma:

Lemma 20.2.
$$\mathrm{St}_{G_0}(s) = \begin{cases} (-1)^{\sigma(G_0)-\sigma(T_0)}\,\mathrm{St}_{Z^0(s)}(e) & \textit{if } s \textit{ is semisimple} \\ 0 & \textit{else} \end{cases}.$$

Proof. By properties of the trace, we have
$$\begin{aligned}\mathrm{St}_{G_0}(s) &= \sum_{P_0\supset B_0}(-1)^{r(P_0)}\mathrm{tr}(s, H_c^0(G_0/P_0(\mathbf{F}_q),\overline{\mathbf{Q}_l}))\\ &= \sum_{P_0\supset B_0}(-1)^{r(P_0)}\mathrm{tr}(e, H_c^0((G_0/P_0(\mathbf{F}_q))^s,\overline{\mathbf{Q}_l})).\end{aligned}$$

Now, we have
$$(G_0/P_0(\mathbf{F}_q))^s = \{g\in G_0(\mathbf{F}_q) : gP_0g^{-1}\ni s\}/P_0$$
and $gP_0g^{-1}\cap Z^0(s)$ is a parabolic of $Z^0(s)$. If we choose $g_0\in(G_0/P_0(\mathbf{F}_q))^s$ and define $Q_0 = g_0P_0g_0^{-1}\cap Z^0(s)$ then the map $g\mapsto gg_0^{-1}$ induces a bijection $(G_0/F_0)(\mathbf{F}_q)^s = (Z^0(s)/Q_0)(\mathbf{F}_q)$.

We now need the following fact: for a fixed $Q_0\subset Z^0(s)$, we have
$$\sum_{P_0\subset G_0 \text{ parabolic, such that } P_0\cap Z^0(s)=Q_0}(-1)^{r(P_0)} = (-1)^{r(Q_0)}\cdot(-1)^{\sigma(Z^0(s))-\sigma(G_0)}.$$

Putting these facts into the equation
$$\mathrm{St}_{G_0}(s) = \sum_{P_0\supset B_0}(-1)^{r(P_0)}\mathrm{tr}(e, H_c^0((G_0/P_0(\mathbf{F}_q))^s,\overline{\mathbf{Q}_l})),$$
we get the lemma for semisimple s. □

To prove Theorem 20.1, we induct on $\dim(G_0)$, so we may assume the theorem for $Z^0(s)$, where s is semisimple and noncentral. Then to prove the theorem for G, it is equivalent to prove it for $G/Z_0 = \overline{G}$.

Now, assume $T_0(\mathbf{F}_q)$ has a nontrivial character θ. Let T_0' be a maximal torus of G_0 contained in a Borel B_0. Then St occurs in $\mathrm{ind}_{B_0(\mathbb{F}_q}^{G_0(\mathbb{F}_q)} 1 = R_{T_0'}^1$. So then $\langle R_{T_0}^\theta, \mathrm{St}_{G_0}\rangle = 0$. The character formula now gives

$$\begin{aligned} 0 &= \frac{1}{|G_0|(\mathbf{F}_q)} \sum_{g_0 \in G_0(\mathbf{F}_q)} \mathrm{tr}(g_0, R_{T_0}^\theta) \cdot \underbrace{\mathrm{tr}(g_0^{-1}, \mathrm{St}_{G_0})}_{\mathrm{St}_{G_0}(g_0^{-1})} \\ &= \sum_{s \in G_0(\mathbf{F}_q),\ \text{semisimple}} \frac{\mathrm{St}_{G_0}(s)}{|Z^0(s)(\mathbf{F}_q)|} \cdot \sum_{g \in G_0(\mathbf{F}_q), g^{-1}sg \in T_0} Q_{gTg^{-1}, Z^0(s)}(e)\theta(g^{-1}sg) \end{aligned}$$

Now we use the induction hypothesis to write

$$Q_{gTg^{-1}, Z^0(s)}(e) = (-1)^{\sigma(Z^0(s)) - \sigma(T_0)} \frac{|Z^0(s)(\mathbb{F}_q)}{\mathrm{St}_{Z^0(s)}(e) \cdot |T_0(\mathbb{F}_q)|}$$

and then we get (the term for $s = e$ gives us $|G_0(\mathbb{F}_q)|$ summands of one, which cancels against the $1/|G_0(\mathbb{F}_q)|$ in the denominator)

$$0 = \mathrm{St}_{G_0}(e) Q_{T_0, G_0}(e) + \sum_{s \neq e,\ \text{semisimple}} \frac{\mathrm{St}_{G_0}(s)}{|Z^0(s)(\mathbf{F}_q)|} \sum_{g \in G_0(\mathbf{F}_q), g^{-1}sg \in T_0} (-1)^{\sigma(Z^0(s)) - \sigma(T_0)} \frac{\theta(g^{-1}sg) \cdot |Z^0(s)(\mathbb{F}_q)|}{\mathrm{St}_{Z^0(s)}(e)|T_0(\mathbf{F}_q)|}.$$

Applying the lemma and cancelling the $|Z^0(s)(\mathbb{F}_q)|$-terms we get

$$\begin{aligned} 0 &= \mathrm{St}_{G_0}(e) Q_{T_0, G_0}(e) + \sum_{s \neq e,\ \text{semisimple}} (-1)^{\sigma(G_0) - \sigma(Z^0(s))} \sum_{g \in G_0(\mathbf{F}_q), g^{-1}sg \in T_0} (-1)^{\sigma(Z^0(s)) - \sigma(T_0)} \frac{\theta(g^{-1}sg)}{|T_0(\mathbf{F}_q)|}, \\ &= \mathrm{St}_{G_0}(e) Q_{T_0, G_0}(e) + (-1)^{\sigma(G_0) - \sigma(T_0)} \frac{1}{|T_0(\mathbf{F}_q)|} \sum_{s \in G_0(\mathbf{F}_q), s \neq e} \sum_{g \in G_0(\mathbf{F}_q), g^{-1}sg \in T_0} \theta(g^{-1}sg). \end{aligned}$$

Now we are just summing $\theta(t)$ over all the elements $e \neq t \in T_0(\mathbb{F}_q)$, with each element appearing $|G_0(\mathbb{F}_q)|$-times, so we get

$$0 = \mathrm{St}_{G_0}(e) Q_{T_0, G_0}(e) + \frac{(-1)^{\sigma(G_0) - \sigma(T_0)} \cdot |G_0(\mathbf{F}_q)|}{|T_0(\mathbf{F}_q)|} \sum_{t \in T_0(\mathbf{F}_q), t \neq e} \theta(t).$$

Since the character θ is nontrivial we have

$$\sum_{t \in T_0(\mathbf{F}_q), t} \theta(t) = 0 \tag{3}$$

$$\sum_{t \in T_0(\mathbf{F}_q), t \neq e} \theta(t) = 1 \tag{4}$$

and putting this in we get

$$\mathrm{St}_{G_0}(e) Q_{T_0, G_0}(e) = + \frac{(-1)^{\sigma(G_0) - \sigma(T_0)} \cdot |G_0(\mathbf{F}_q)|}{|T_0(\mathbf{F}_q)|},$$

which completes the proof.

References

[1] Cédric Bonnafé, *Representations of* $\mathrm{SL}_2(\mathbf{F}_q)$, Algebra and Applications, vol. 13, Springer-Verlag London, Ltd., London, 2011. MR2732651