MONODROMY AND IRREDUCIBILITY OF IGUSA VARIETIES

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ABSTRACT. We determine the irreducible components of Igusa varieties for Shimura varieties of Hodge type and use that to determine the irreducible components of central leaves. In particular, we show that the discrete Hecke-orbit conjecture is false in general. Our method combines recent work of D'Addezio on monodromy of compatible local systems with a generalisation of a method of Hida, using the Honda-Tate theory for Shimura varieties of Hodge type developed by Kisin–Madapusi Pera–Shin. We also determine the irreducible components of Newton strata in Shimura varieties of Hodge type by combining our results with recent work of Zhou-Zhu.

1. Introduction

Let $N \geq 4$, let p be a prime number coprime to N, let $Y_1(N)$ be the modular curve of level $\Gamma_1(N)$ over \mathbb{F}_p and let $Y_1(N)^{\operatorname{ord}}$ be the ordinary locus. There is a tower of finite étale covers (see [18]) $\operatorname{Ig}_m \to Y_1(N)^{\operatorname{ord}}$ with Galois groups $(\mathbb{Z}/p^m\mathbb{Z})^{\times}$, and we let $\operatorname{Ig}_{\operatorname{ord}} \to Y_1(N)^{\operatorname{ord}}$ be their limit; it is a classical result due to Igusa that $\operatorname{Ig}_{\operatorname{ord}}$ is irreducible.

Igusa varieties exist more generally as profinite étale covers of central leaves in the special fibers of Shimura varieties of Hodge type (c.f. [14, Section 5]; see [27] for the PEL case). Understanding the irreducible components of these Igusa varieties has important consequences for the theory of p-adic automorphic forms. For example, in the work of Eischen-Mantovan [10] on p-adic automorphic forms for unitary Shimura varieties, the irreducibility of Igusa varieties is assumed throughout.

The irreducibility of Igusa varieties was proven for Siegel modular varieties by Chai-Oort [5], and their proof works more generally for Shimura varieties of PEL type when hypersymmetric points exist (c.f. [10,16]). We would like to point out that even in the μ -ordinary locus, hypersymmetric points often do not exist (see [38, Corollary 7.5.]). Hida [15] proves the irreducibility of the ordinary Igusa tower over Shimura varieties of PEL type A and C without using hypersymmetric points. Our results are the first to treat Hodge type Shimura varieties and Igusa varieties over general central leaves (but see Remark 1.1.2); they are even new for the μ -ordinary locus in many PEL type cases.

1.1. Main results. Let (G,X) be a Shimura datum of Hodge type, let p>2 be a prime number, let $U^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup and let $U_p \subset G(\mathbb{Q}_p)$ be a hyperspecial subgroup. Moreover, let Sh_G be the basechange to $\overline{\mathbb{F}}_p$ of the canonical integral model of the Shimura variety of level $U=U^pU_p$ over $\mathcal{O}_{E,v}$ for some prime $v\mid p$ of the reflex field E of (G,X). Let $\operatorname{Sh}_{G,b}$ be a non-basic Newton stratum and let Ig_b be the Igusa variety associated to b.

Theorem 1. Assume that G^{der} is simply connected, that G^{ad} is \mathbb{Q} -simple and that p does not divide the order of the Weyl group W_G of G. Then there is a bijection (where G^{ab} is the maximal abelian quotient)

$$\pi_0(\mathrm{Ig}_b) \cong \pi_0(\mathrm{Sh}_G) \times G^{ab}(\mathbb{Z}_p).$$

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In the case of the modular curve, the ordinary Igusa variety Ig_b is a $(\mathbb{Z}_p^{\times})^2$ -torsor over the ordinary locus, and our theorem says us that its connected components are in bijection with \mathbb{Z}_p^{\times} ; here $(\mathbb{Z}_p^{\times})^2$ acts on \mathbb{Z}_p^{\times} via the product map. This recovers the result of Igusa from the introduction, because the Igusa tower $\mathrm{Ig}_{\mathrm{ord}}$ introduced there is the inverse image of 1 under $\mathrm{Ig}_b \to \mathbb{Z}_p^{\times}$.

We can use Theorem 1 to determine the irreducible (equivalently, connected) components of central leaves. Let $C \subset \operatorname{Sh}_{G,b}$ be a central leaf and let $\operatorname{Ig}_b \to C$ be the Igusa variety over C, this is a pro-étale torsor for a compact open subgroup $H_C \subset J_b(\mathbb{Q}_p)$, where J_b is the twisted centraliser attached to b (see Section 2).

Corollary 1.1.1. There is a bijection

$$\pi_0(C) \simeq \pi_0(\operatorname{Sh}_{G,b}) \times \frac{G^{ab}(\mathbb{Z}_p)}{H_C^{ab}},$$

where H_C^{ab} is the image of H_C in $G^{ab}(\mathbb{Z}_p)$ via the natural map $J_b \to G^{ab}$. The group H_C^{ab} is of finite index in $G^{ab}(\mathbb{Z}_p)$ because it always contains the image of $Z_G(\mathbb{Z}_p)$.

The discrete part of the Hecke orbit conjecture (c.f. [6,38]) predicts that $\pi_0(C) \simeq \pi_0(\operatorname{Sh}_G)$. In an earlier version of [17], the first-named author claimed to have proven this conjecture; however, the supplied proof contained an error. Using Corollary 6.2.2, the conjecture comes down to the equality $H_C^{ab} = G^{ab}(\mathbb{Z}_p)$ for all central leaves C. We will show that if H_C is a parahoric subgroup, then $H_C^{ab} = G^{ab}(\mathbb{Z}_p)$, this is also used in our proof of Theorem 1. We will also give an example, due to Rong Zhou, that shows that this equality does not always hold. In particular, the discrete Hecke orbit conjecture is false in general.

Our second main result is about irreducible components of Newton strata.

Theorem 2. Assume that G^{der} is simply connected, that G^{ad} is \mathbb{Q} -simple and that p does not divide the order of the Weyl group W_G of G. Then the natural map

$$\pi_0(\operatorname{Sh}_{G,b}) \to \pi_0(\operatorname{Sh}_G)$$

is a bijection, and the number of irreducible components in each connected component of $\mathrm{Sh}_{G,b}$ is given by the representation theoretic constant

$$\operatorname{Dim} V_{\mu}(\lambda_b)_{rel}$$
,

introduced in [41]. In particular, when $G_{\mathbb{Q}_p}$ is split, the connected components of $\mathrm{Sh}_{G,b}$ are irreducible.

Our proof uses the Mantovan product-formula [27], due to Hamacher-Kim [14] in this generality, and results of Chen-Kisin-Viehmann [7] and Zhou-Zhu [41]. Theorems 1 and 2 were proven for Siegel modular varieties by Chai and Oort in their seminal paper [5]. Amusingly, they prove irreducibility of Newton strata first, irreducibility of central leaves second and irreducibility of Igusa varieties last. Their proof of the Hecke orbit conjecture depends on the existence of hypersymmetric points.

Our proofs also work when $G_{\mathbb{Q}_p}$ is only assumed to be quasi-split and split over a tamely ramified extension, under some additional assumptions, see Theorems 6.2.1 and 6.2.4 for precise versions of our main theorems in that generality. The assumption that p does not divide the order of W_G is relaxed there. For example, this assumption is not necessary for the Igusa tower over the μ -ordinary locus. When b is basic, the Igusa variety Ig_b is zero-dimensional and highly reducible, in which case it will have many more irreducible components than the Igusa varieties for non-basic Newton strata. This means that the theorem is false for products of Shimura varieties with b basic in one factor and non-basic in the other; this is where the assumption that $G^{\rm ad}$ is \mathbb{Q} -simple comes from.

Remark 1.1.2. In recent work [25], Kret and Shin also determine the irreducible components of Igusa varieties and deduce results about the discrete Hecke orbit conjecture.¹ Their proof uses harmonic analysis and automorphic forms and is completely different from ours. They compute the 0-th étale cohomology of Ig_b as a representation of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ using the Langlands-Kottwitz method ([25, Theorem A]), and then determine the irreducible components of Ig_b using that computation. It should be possible to recover their computation of the 0-th étale cohomology from Theorem 1; we plan to include this in future work.

1.2. Strategy.

1.2.1. Setup. The Igusa variety is a profinite étale cover $\mathrm{Ig}_b \to C$ of a central leaf C inside the Newton stratum $\mathrm{Sh}_{G,b}$. To be precise, it is a torsor for a compact open subgroup $H_C \subset J_b(\mathbb{Q}_p)$, where J_b is the twisted centraliser attached to b (see Section 2).

There are many different central leaves C inside the Newton stratum, and they all have the same Igusa variety Ig_b but the groups $H = H_C$ depend on C. For the purposes of our proof we will always choose C to be a distinguished central leaf, i.e., a central leaf that is also an Ekedahl-Oort stratum; these exist by Theorem D of [35]. Under the assumptions of our main theorem, these distinguished central leaves are irreducible. To be precise, [17, Theorem 2] tells us that

$$\pi_0(C) \to \pi_0(\operatorname{Sh}_G)$$

is a bijection. Therefore the main theorem would follow if we could show that the fibers of $\pi_0(\mathrm{Ig}_b) \to \pi_0(C)$ are in bijection with $G^{ab}(\mathbb{Z}_p)$, equivariant for the action of H.

The algebraic group J_b is an inner form of a Levi subgroup of G, and therefore has a surjective map $J_b \to G^{ab}$. Let J_b' be the kernel of this map and let $H' = H \cap J_b'(\mathbb{Q}_p)$. In the last section, we will show that H/H' is isomorphic to $G^{ab}(\mathbb{Z}_p)$ and therefore it suffices to show that the stabiliser of a component $\pi_0(\mathrm{Ig}_b)$ under the action of H is given by H'.

From now on we work with a fixed connected component $\operatorname{Sh}_G^{\circ}$ of the Shimura variety, and hence a fixed connected component C° of C. The H-torsor $\operatorname{Ig}_b^{\circ} \to C^{\circ}$ corresponds to a morphism

$$\rho_{\operatorname{Ig}}: \pi_1^{\operatorname{\acute{e}t}}(C^{\circ}) \to H$$

and showing that H' acts trivially on $\pi_0(\mathrm{Ig}_b)$ is the same as showing that the image of monodromy contains H'. Let N be this image and let \mathcal{N} be its Zariski closure inside $J_b(\overline{\mathbb{Q}}_p)$, which is an algebraic group over $\overline{\mathbb{Q}}_p$.

It will follow from [17, Theorem 2] and two results of D'Addezio [8,9] that $\mathcal{N} = J_b'$. This implies that N is a compact open subgroup of $J_b'(\mathbb{Q}_p)$, and therefore that $N \cap H'$ is of finite index in both N and H'. This proves the theorem up to a finite error, but it is a finite error over which we have absolutely no control.

1.2.2. Outline of the proof. In order to show that N = H', we will make use of the fact that the action of H on Ig_b extends to an action of $J_b(\mathbb{Q}_p)$. If we could show that the action of $J_b'(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Ig}_b)$ was trivial, then we would know that $N = H \cap J_b'(\mathbb{Q}_p) = H'$.

In Section 3, we show via a group-theoretic argument that the isotropic factors of $J_b^{\text{der}}(\mathbb{Q}_p)$ act trivially on $\pi_0(\mathrm{Ig}_b)$. The main ingredients are the equality $\mathcal{N}=J_b'$ mentioned above and the fact that the \mathbb{Q}_p -points of semisimple and simply connected isotropic groups have no nontrivial finite quotients.

¹Kret and Shin pointed out to us that there is a mistake in their proof of Lemma 8.1.1 in the current version of their paper. This lemma asserts that $H_C^{ab} = G^{ab}(\mathbb{Z}_p)$ for all central leaves C and would imply the discrete Hecke orbit conjecture.

We show in Section 4 that $T'(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Ig}_b)$, where T' is a maximal torus of J'_b . In fact, we can show this for many maximal tori of J'_b . This is done by generalising an argument of Hida [15] using the Honda-Tate theory for Shimura varieties of [20].

In Section 5.1, we show that the anisotropic factors of $J_b^{\text{der}}(\mathbb{Q}_p)$ are generated by $T_1^{\text{der}}(\mathbb{Q}_p)$ and $T_2^{\text{der}}(\mathbb{Q}_p)$ for some well chosen maximal tori $T_1, T_2 \subset J_b$. Our arguments here are completely explicit, relying on the classification of semisimple and simply connected anisotropic groups over local fields.

At this point we have shown that $J_b^{\operatorname{der}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\operatorname{Ig}_b)$, and that we can find many maximal tori of $J_b'(\mathbb{Q}_p)$ that act trivially on $\pi_0(\operatorname{Ig}_b)$. Our proof ends with Section 5.2 where we show that $J_b'(\mathbb{Q}_p)$ is generated by $J_b^{\operatorname{der}}(\mathbb{Q}_p)$ and $T_1(\mathbb{Q}_p), \dots, T_n(\mathbb{Q}_p)$ where T_1, \dots, T_n are (well-chosen) maximal tori of J_b .

In Section 6.1 we identify the quotient H_C/H'_C with $G^{ab}(\mathbb{Z}_p)$ for our particular choice of central leaf C, and in Section 6.2 we state our main theorem in maximum generality.

2. Shimura varieties and Igusa varieties

Let (G,X) be a Shimura datum of Hodge type with G^{der} simply connected and let p>2 be a prime number such that $G_{\mathbb{Q}_p}$ is quasi-split and splits over a tamely ramified extension. Let $U^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup and let $U_p \subset G(\mathbb{Q}_p)$ be a very special subgroup; in particular, U_p is hyperspecial when $G_{\mathbb{Q}_p}$ is unramified. Kisin [22] and Kisin-Pappas [21] (in the ramified case) showed the existence of the canonical integral model of the Shimura variety of level $U = U^p U_p$ over $\mathcal{O}_{E,v}$ for any prime $v \mid p$ of the reflex field E of $(G,X)^2$. Let Sh_G denote the base change to $\overline{\mathbb{F}}_p$ of this canonical model.

The canonical model Sh_G comes equipped with the Newton stratification, indexed by the finite poset $B(G,\mu)$ which has a unique minimal element called the *basic* element. For the rest of this paper, we fix a non-basic Newton stratum $\operatorname{Sh}_{G,b}$ on Sh_G . Let J_b denote the twisted centraliser of b, an algebraic group over \mathbb{Q}_p defined by

$$J_b(R) = \{ g \in G(R \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p) | gb\sigma(g^{-1}) = b \},$$

for any \mathbb{Q}_p -algebra R, where \mathbb{Q}_p is the completion of a maximal unramified extension of \mathbb{Q}_p . The algebraic group J_b is an inner form of a Levi subgroup of G and therefore there is a morphism (where G^{ab} is the maximal abelian quotient of G)

$$J_b \to G^{\mathrm{ab}}$$

of algebraic groups over \mathbb{Q}_p , we let J_b' be its kernel.

Let $(G, X) \hookrightarrow (GSp, S^{\pm})$ denote a choice of embedding of (G, X) into the Shimura datum for the symplectic group, chosen as in [14, Section 2.3]. This determines a finite morphism from Sh_G to Sh_{GSp} , the special fiber of the Shimura variety for (GSp, S^{\pm}) of level $U' = U'^p U'_p$ with U'_p hyperspecial.³ In particular, this gives us a 'universal' abelian variety $\pi: A \to Sh_G$.

2.1. Central leaves. Given a Newton stratum $Sh_{GSp,b}$ on Sh_{GSp} , Oort came up with the idea of studying central leaves, which are loci where the p-divisible group associated to the universal abelian variety A over Sh_{GSp} is geometrically fiberwise constant. To be more precise, given a

²In the ramified case, these are canonical in the sense of [30].

³When U_p is hyperspecial, the main result of [39] tells us that this is a closed immersion.

point $x \in \operatorname{Sh}_{\mathrm{GSp}}(\overline{\mathbb{F}}_p)$, the central leaf C(x) passing through x is defined as (where \simeq_g denotes an isomorphism of p-divisible groups respecting the polarisations up to a scalar)

$$C(x) = \{ y \in \operatorname{Sh}_{\operatorname{GSp}}(\overline{\mathbb{F}}_p) \mid A_y[p^{\infty}] \simeq_g A_y[p^{\infty}] \}.$$

Our proves that C(x) is the set of $\overline{\mathbb{F}}_p$ -points of a locally closed subvariety of $\operatorname{Sh}_{\mathrm{GSp},b}$, and that it is in fact smooth and equidimensional ([29], [27]). Central leaves are closed inside $\operatorname{Sh}_{\mathrm{GSp},b}$ and give rise to a 'foliation' by partitioning $\operatorname{Sh}_{\mathrm{GSp},b}$ into a disjoint union of central leaves of the same dimension related to each other by isogeny correspondences.

This construction has been generalised to Shimura varieties of Hodge type by Hamacher in [13, §2.3] (see also [12, Definition 5] and [19]); basically, ones takes the locally closed subvariety where the p-divisible group with Hodge tensors is geometrically fiberwise constant. Corollary 5.3.1 of [19] states that these central leaves, which we still denote by C(x), are smooth and equidimensional.

2.2. Igusa varieties. Let C(x) be a fixed central leaf in Sh_G and write $\mathbb{X} = A_x[p^{\infty}]$ for the p-divisible group with additional structures associated to x. Let C' denote the corresponding central leaf in $\operatorname{Sh}_{\operatorname{GSp}}$.

In the case of Siegel modular varieties, the perfect Igusa variety $Ig' \to C'$ is the scheme over C' whose set of R-points, for a test scheme R/C', consists of isomorphisms

$$A_R[p^\infty] \simeq \mathbb{X}_R$$

respecting the polarisations up to a scalar. By construction, Ig' has an action of the group Aut(X), where X is considered as a p-divisible group with polarisation. It follows from [3, Section 4.3] that Ig' actually has an action of the group QIsog(X) of self quasi-isogenies of X and that Ig' does not depend on x but only on the Newton stratum containing x. Moreover, it is proved in loc. cit. that Ig' is a perfect scheme and that the morphism

$$\alpha: \mathrm{Ig}' \to C^{\mathrm{perf}}$$

is a pro-étale $\operatorname{Aut}(\mathbb{X})$ -torsor. To be precise, $\operatorname{Aut}(\mathbb{X})$ is a profinite group, and so there is an affine group scheme $\operatorname{\underline{Aut}}(\mathbb{X})$ whose topological space of connected components is homeomorphic to $\operatorname{Aut}(\mathbb{X})$. The morphism α is a torsor for this group scheme, which is an inverse limit of finite étale group schemes. There is also a deperfection of the Igusa tower (c.f. [27]), but it is more difficult to construct. Since the underlying topological space of a scheme is the same as that of its perfection, we will ignore this difference here.

Hamacher and Kim [14, §5.1] generalise this construction to Shimura varieties of Hodge type. The Igusa variety Ig_b over C(x) is defined as the subscheme of $Ig' \times_{Sh_G}^{Perf} Sh_{GSp}^{Perf}$ consisting of those isomorphisms that respects the crystalline Tate-tensors on the p-divisible groups. In this case, $Ig_b \to C(x)$ is a pro-étale H-torsor, where $H := Aut(\mathbb{X}) \cap J_b(\mathbb{Q}_p)$ is a compact open subgroup of $J_b(\mathbb{Q}_p) \subset QIsog(\mathbb{X})$. The action of H extends to an action of $J_b(\mathbb{Q}_p)$, but this is not compatible with the projection $Ig_b \to C(x)$. As before, the scheme Ig_b with its action of $J_b(\mathbb{Q}_p)$ does not depend on the choice of x; however, H does. We will often choose x such that C(x) is an Ekedahl-Oort stratum (or EKOR stratum when G is ramified, see [34]), in which case H will be a parahoric subgroup of $J_b(\mathbb{Q}_p)$.

3. Geometric Monodromy

Assume that the adjoint group G^{ad} of G is \mathbb{Q} -simple from now on. Let $b \in B(G,\mu)$ be non-basic and let $C(x) \subset \operatorname{Sh}_{G,b}$ be a central leaf in the Newton stratum associated to b. Let $\ell \neq p$ be a prime number such that $U_{\ell}^p = G(\mathbb{Z}_{\ell})$ is hyperspecial and let $\pi : A \to \operatorname{Sh}_G$ be the universal abelian variety. The local system $R^1\pi_*\mathbb{Z}_{\ell}$ induces a pro-étale $G(\mathbb{Z}_{\ell})$ -torsor over Sh_G , which is just the tower of Shimura varieties over Sh_G obtained by shrinking the level at ℓ .

The induced $G^{ab}(\mathbb{Z}_{\ell})$ -torsor is pulled back from the discrete Shimura variety for (G^{ab}, X^{ab}) via the natural map and is therefore trivial. Therefore on each connected component of Sh_G , the image of monodromy is contained in $G^{\operatorname{der}}(\mathbb{Z}_{\ell})$. In fact the image is equal to $G^{\operatorname{der}}(\mathbb{Z}_{\ell})$ because the connected components of the Shimura variety are naturally identified with the discrete Shimura variety for (G^{ab}, X^{ab}) for each choice of level at ℓ . To proceed we will to assume the following:

Assumption 3.0.1. The natural map

$$C(x) \to \operatorname{Sh}_G$$

induces a bijection on connected components for all choices of prime-to-p level U^p .

If C(x) is a distinguished central leaf and if moreover $G_{\mathbb{Q}_p}$ splits over an unramified extension, this assumption holds true by [17, Theorem 2]. It also holds for distinguished central leaves in the ramified case under certain technical assumptions; this is made precise in the statement of our main results in Section 6.

Since we can apply the assumption after shrinking the level at ℓ , it follows that the ℓ -adic monodromy of $R^1\pi_*\mathbb{Z}_\ell$, over every connected component $C(x)^{\circ}$ of the leaf C(x), is equal to $G^{\operatorname{der}}(\mathbb{Z}_\ell)$ and therefore has Zariski closure equal to G^{der} .

Notation 3.0.2. We will write $J_b^{der} = J_{b,an}^{der} \times J_{b,iso}^{der}$ with the first factor anisotropic and the second factor isotropic, and use the same notations for subgroups of J_b^{der} .

Proposition 3.0.3. Suppose that Assumption 3.0.1 holds, then $J_{b,iso}^{der}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Ig}_b)$.

Let N be the image of $\pi_1^{\text{\'et}}(C(x)^\circ) \to H$ for a connected component $C(x)^\circ$ of C(x). We first prove the following lemma:

Lemma 3.0.4. The Zariski closure of N inside $J_b(\overline{\mathbb{Q}}_p)$ is equal to J'_b .

Proof. As noted above, the assumption tells us that the Zariski closure of the ℓ -adic monodromy of $R^1\pi_*\mathbb{Q}_\ell$ is equal to G^{der} . The central leaf C(x) is smooth by [19, Corollary 5.3.1] (c.f. [27, Proposition 1] in the PEL case and Proposition 2.6 of [12] in the unramified Hodge type case). Then [8, Theorem 1.2.1] shows that the geometric monodromy group of the overconvergent isocrystal \mathcal{E}^{\dagger} associated to $\pi:A\to C(x)^{\circ}$ is equal to the geometric monodromy group of $R^1\pi_*\mathbb{Q}_\ell$ and therefore equal to G^{der} . Theorem 1.1.1 of [9] shows that the geometric monodromy group of the underlying isocrystal \mathcal{E} is equal to the parabolic subgroup $P(\lambda)\subset G^{\mathrm{der}}$, where λ is a cocharacter of G^{der} coming from the slope filtration on \mathcal{E} . The $J_b(\mathbb{Q}_p)$ -torsor induced from the H-torsor given by the Igusa variety, is the 'associated graded' of the slope filtration on \mathcal{E} , and its geometric monodromy group is therefore isomorphic to the Levi subgroup $N(\lambda)$ of $P(\lambda)$, which is the same thing as J'_b .

Proof of Proposition 3.0.3. Because J_b^{der} is semi-simple, it follows that the Lie algebra of the p-adic Lie group N^{der} agrees with the Lie-algebra of the algebraic group J_b^{der} (see [1, Corollary 7.9]). Therefore N^{der} contains a compact open subgroup of $J_b^{\text{der}}(\mathbb{Q}_p)$ using the p-adic exponential map. It follows that N^{der} is of finite index in H^{der} because they are both compact open subgroups of $J_b^{\text{der}}(\mathbb{Q}_p)$. Therefore, the action of H^{der} on $\pi_0(\mathrm{Ig}_b)$ has finite orbits.

Let $J_b^{\operatorname{der}} = J_{b,\operatorname{an}}^{\operatorname{der}} \times J_{b,\operatorname{iso}}^{\operatorname{der}}$ and let $\operatorname{Ig}_b^{\operatorname{der}}$ be the $J_{b,\operatorname{iso}}^{\operatorname{der}}(\mathbb{Q}_p)$ -orbit of a connected component of Ig_b , then $\pi_0(\operatorname{Ig}_b^{\operatorname{der}})$ is automatically compact because $\pi_0(\operatorname{Ig}_b)$ is compact Hausdorff and $J_{b,\operatorname{iso}}^{\operatorname{der}}(\mathbb{Q}_p) \times \{1\} \subset$

 $J_b(\mathbb{Q}_p)$ is closed. We claim that $\pi_0(\mathrm{Ig}_b^{\mathrm{der}})$ is finite. It suffices to show that it is a union of finitely many $H_{\mathrm{iso}}^{\mathrm{der}}$ -orbits because H^{der} has finite orbits as explained above. Let $Q \subset J_{b,\mathrm{iso}}^{\mathrm{der}}(\mathbb{Q}_p)$ be the stabiliser of a connected component of $\mathrm{Ig}_b^{\mathrm{der}}$, then the action map induces a homeomorphism

$$\pi_0(\mathrm{Ig}_b^{\mathrm{der}}) \simeq J_{b,\mathrm{iso}}^{\mathrm{der}}(\mathbb{Q}_p)/Q.$$

The quotient map $J_{b,\mathrm{iso}}^{\mathrm{der}}(\mathbb{Q}_p) \to J_{b,\mathrm{iso}}^{\mathrm{der}}(\mathbb{Q}_p)/Q$ is open, hence $H_{\mathrm{iso}}^{\mathrm{der}}$ -orbits are open, and so there are finitely many of them by compactness. Proposition 12.14 of [26] tells us that J_b^{der} is simply connected because G^{der} is simply connected and it follows that $J_{b,\mathrm{iso}}^{\mathrm{der}}$ is simply connected. Therefore the group $J_{b,\mathrm{iso}}^{\mathrm{der}}(\mathbb{Q}_p)$ has no nontrivial finite quotients by [32, Theorem 7.1, Theorem 7.5]. Since $\pi_0(\mathrm{Ig}_b)$ is finite, the stabiliser in $J_{b,\mathrm{iso}}^{\mathrm{der}}(\mathbb{Q}_p)$ of a point in $\pi_0(\mathrm{Ig}_b^{\mathrm{der}})$ is a finite index subgroup and therefore equal to $J_{b,\mathrm{iso}}^{\mathrm{der}}(\mathbb{Q}_p)$.

This argument will not work for the anisotropic part of J_b^{der} , because it is not true that $J_{b,\text{an}}^{\text{der}}(\mathbb{Q}_p)$ has no nontrivial finite quotients. In fact it is a profinite group (because it is compact), and so it has many nontrivial finite quotients.

4. Construction of a maximal torus

4.1. **Prime-to-**p **Hecke operators.** By [14, Lemma 5.1.4], the prime-to-p Hecke operators act on Ig_b such that the projection $Ig_b \to Sh_{G,b}$ is equivariant. To be precise, the construction of Ig_b is compatible with changing the level $U^p \subset G(\mathbb{A}_f^p)$ and $G(\mathbb{A}_f^p)$ acts on the inverse limit over U^p of the Igusa variety. Let Σ be the finite set consisting of p, ∞ and places of \mathbb{Q} where G is anisotropic.

Proposition 4.1.1. The prime-to- Σ Hecke operators coming from G^{der} act trivially on $\pi_0(\mathrm{Ig}_b)$

Proof. This can be proven as in [38, Section 4], making use of the fact that $G^{\text{der}}(\mathbb{Q}_{\ell})$ has no finite quotients for $\ell \notin \Sigma$.

For a point $y \in C(x)$, we let I_y be the algebraic group over \mathbb{Q} of self-quasi isogenies of the abelian variety A_y that preserve the Hodge tensors. Choosing a quasi-isogeny between $A_y[p^{\infty}]$ and \mathbb{X} , leads to an inclusion $I_{y,\mathbb{Q}_p} \subset J_b$. This is not necessarily an isomorphism, but it is true that both groups have the same rank (see [23, Corollary 2.1.7], [40, Corollary 9.5] and [17, Proposition A.4.3] in the ramified case). It follows from loc. cit. that I_y is an inner form of a subgroup of G, and therefore that there is a morphism $I_y \to G^{ab}$; we let I'_y be its kernel.

Proposition 4.1.2. For each $y \in C(x)^{\circ}$, the subgroup $I'_y(\mathbb{Q}) \subset J_b(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Ig}_b)$.

Proof. The point is that $I'_y(\mathbb{Q}) \subset G^{\operatorname{der}}(\mathbb{A}_f^{\Sigma})$ acts trivially on C(x) while it acts via $I_y(\mathbb{Q}) \subset J_b(\mathbb{Q}_p)$ on Ig_b . Since $G^{\operatorname{der}}(\mathbb{A}_f^{\Sigma})$ acts trivially on $\pi_0(\operatorname{Ig}_b)$, the result follows.

Corollary 4.1.3. The closure of $I'_y(\mathbb{Q}) \subset J_b(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Ig}_b)$.

Proof. Let $z \in \pi_0(\mathrm{Ig}_b)$ be a connected component with stabiliser Q_z . Then $I'_y(\mathbb{Q}) \subset Q_z$ and the corollary would follow if we could show that Q_z was a closed subgroup. There is a homeomorphism $\pi_0(\mathrm{Ig}_b) \simeq J_b(\mathbb{Q}_p)/Q_z$ and since $\pi_0(\mathrm{Ig}_b)$ is Hausdorff, it follows that Q_z is a closed subgroup of $J_b(\mathbb{Q}_p)$.

Definition 4.1.4. We call y weakly hypersymmetric if $I_{y,\mathbb{Q}_p} \simeq J_b$. If in addition $I_y(\mathbb{Q})$ is dense in $I_y(\mathbb{Q}_p)$, we call y hypersymmetric.

When $G_{\mathbb{Q}_p}$ splits over an unramified extension, weakly hypersymmetric points are automatically hypersymmetric. Indeed, by [32, Theorem 7.8] and the discussion in [32, Section 7.3], weak approximation at a prime p holds if the group is unramified there. Therefore in the PEL case, this definition coincides with the definition given by Chai-Oort [4, Definition 6.4].

When the Newton stratum contains a hypersymmetric point, the arguments above show that $J'_b(\mathbb{Q}_p)$ acts trivially on Ig_b , without using the results of Section 3. Unfortunately, although Newton strata on Siegel modular varieties always contain hypersymmetric points [4], they are sparse in more general settings. For example, in the PEL case, not every Newton stratum contains a hypersymmetric point [42]; they might not exist even in the μ -ordinary stratum [38, Corollary 7.5.]. Hence, all we can say about I_{y,\mathbb{Q}_p} in general is that it contains a maximal torus of J_b . Using Honda-Tate theory, we will show that for every maximal torus of J_b we can find a point y in C(x) such that I_{y,\mathbb{Q}_p} contains that maximal torus, up to isomorphism of tori.

Remark 4.1.5. Basic points in Shimura varieties of Hodge type are known to be hypersymmetric [37, Lemma 7.2.14]. It would be interesting to characterise all Newton strata containing hypersymmetric points for Shimura varieties of Hodge type.

Remark 4.1.6. If we could show that $J'_b(\mathbb{Q}_p)$ is generated by the closures of $I'_y(\mathbb{Q})$ as y ranges over points in C(x), then the main theorem would follow. This is closely related to Hida's strategy in [15], where he shows that the group H is generated by $T(\mathbb{Z}_p)$ as T ranges over isomorphism classes of maximal tori of J_b , in the special case that $H = \mathrm{GL}_n(\mathcal{O}_F)$ for some p-adic local field F. We were not able to make this strategy work in general, which is why we need to make use of the result of Section 3.

4.2. **Honda-Tate theory.** Classical Honda-Tate theory describes isogeny classes of abelian varieties over finite fields containing \mathbb{F}_p in terms of q-Weil numbers. Equivalently, we can describe isogeny classes of abelian varieties A of dimension g by the characteristic polynomial of Frob_p acting on the ℓ -adic Tate module of A for some $\ell \neq p$ (this is an element of $\mathbb{Z}[X]$). This gives us a (semisimple) conjugacy class of matrices in $\operatorname{GL}_{2g}(\mathbb{Q})$ and its stabiliser is an inner form of the group I of self quasi-isogenies of A. Choosing a maximal torus $T \subset I$ corresponds, roughly speaking, to choosing a CM lift of the isogeny class.

For Shimura varieties of Hodge type like our (G, X), things are best phrased in terms of maximal tori $T \subset G$ such that some $h \in X$ factors through $T_{\mathbb{R}}$, which makes T into a special point. These special points give rise to points on Sh_G via reduction mod p. The main theorem of [20] shows that every Newton stratum in our Shimura variety of Hodge type contains many such points, and [17, Theorem A.4.5.(2)] shows that in fact every isogeny class contains the reduction of a special point.

Proposition 4.2.1. For each maximal torus $T' \subset J'_b$, there is a point y in the leaf C(x) such that $I'_{y,\mathbb{Q}}$ contains a torus \mathscr{T}' with $\mathscr{T}'_{\mathbb{Q}_p}$ isomorphic to T' under $I'_{y,\mathbb{Q}_p} \subset J'_b$. Moreover, we can choose \mathscr{T}' such that it has weak approximation, i.e., such that $\mathscr{T}'(\mathbb{Q})$ is dense in $\mathscr{T}'(\mathbb{Q}_p)$.

Proof. First of all we can uniquely extend T' to a maximal torus $T \subset J_b$, and since $G_{\mathbb{Q}_p}$ is quasisplit, we can transfer T to G. Then [20, Proposition 1.2.5] tells us that we can find a maximal torus $\mathscr{T} \subset G$ such that: There is an $h \in X$ that factors through $\mathscr{T}_{\mathbb{R}}$, i.e., $(\mathscr{T}, h) \subset (G, X)$ is a special point, and this special point induces an isogeny class $\mathscr{I} \subset \operatorname{Sh}_{G,b}(\overline{\mathbb{F}}_p)$ with automorphism group $I_{\mathbb{Q}}$ containing \mathscr{T} as a maximal torus, and such that $\mathscr{T}_{\mathbb{Q}_p}$ is $G(\mathbb{Q}_p)$ -conjugate to T and in particular \mathbb{Q}_p -isomorphic to T. After basechange to \mathbb{Q}_p , the inclusions

$$\mathscr{T}_{\mathbb{Q}_p} \subset I_{\mathbb{Q}_p} \subset J_b$$

then gives us a maximal torus of J_b , and thus a torus of J'_b which is \mathbb{Q}_p -isomorphic to T'.

The construction of \mathscr{T} in the proof of loc. cit. is quite flexible. They start by choosing any maximal torus \mathscr{T}_0 in G such that there is an $h \in X$ factoring through \mathscr{T}_0 and such that $\mathscr{T}_{0,\mathbb{Q}_p}$ is conjugate to T. Next, they choose $g \in G^{\operatorname{der}}(\overline{\mathbb{Q}})$ so that the cocycle $\sigma \mapsto g\sigma(g)^{-1}$ lies in $\mathscr{T}'_0(\overline{\mathbb{Q}})$, where $\mathscr{T}'_0 = \mathscr{T}_0 \cap G^{\operatorname{der}}$. By Lemma 1.2.1 of loc. cit., this cocycle is chosen so that its cohomology class is trivial in $H^1(\mathbb{Q}_p, \mathscr{T}'_0)$, and in fact the proof of the lemma shows that we can choose this cocycle such that it is trivial in $H^1(\mathbb{Q}_\ell, \mathscr{T}'_0)$ for finitely many ℓ 's. Then \mathscr{T} arises as $\operatorname{int}(g^{-1})(\mathscr{T}_{0,\overline{\mathbb{Q}}})$, which is defined over \mathbb{Q} .

By weak approximation for the variety of tori for G, which is isomorphic to the variety of tori for G^{der} , we can choose \mathscr{T}_0 with specified $G^{\text{der}}(\mathbb{Q}_\ell)$ -conjugacy class of \mathscr{T}_0' for finitely many ℓ 's. By the discussion above, we can choose g such that \mathscr{T}_0' has the same specified $G^{\text{der}}(\mathbb{Q}_\ell)$ -conjugacy classes. It now follows from [33, Theorem 1.(i)] and the result of V.E. Voskresenskii mentioned afterwards that this gives us enough flexibility to choose \mathscr{T}^{der} to have weak approximation.

The proposition now follows from the fact that our isogeny class \mathscr{I} intersects every central leaf in the Newton stratum $\operatorname{Sh}_{G,b}$, which follows from Rapoport-Zink uniformisation of isogeny classes ([23, Proposition 2.1.3] and [17, Theorem A.4.5.(1)] in the ramified case). To be precise, every isogeny class receives a map from a Rapoport-Zink space, and these Rapoport Zink spaces contain every isomorphism class of p-divisible groups with extra structure in the given Newton stratum. In other words, they intersect every central leaf in the Newton stratum.

Corollary 4.2.2. For every maximal torus $T' \subset J'_b$, we can find a maximal torus $\hat{T} \subset J'_b$ such that $\hat{T} \simeq T'$ as algebraic groups and such that $\hat{T}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Ig}_b)$.

Proof. This is a direct consequence of Proposition 4.1.2 and Proposition 4.2.1. \Box

5. Anisotropic groups

5.1. Action of the anisotropic part. It follows from [32, Theorem 6.5] that a semi-simple simply connected anisotropic group over \mathbb{Q}_p is isomorphic to a product of groups of the form $\operatorname{Res}_{K/\mathbb{Q}_p} \operatorname{SL}_1(D)$, where D/K is a central division algebra with K/\mathbb{Q}_p a finite extension and $\operatorname{SL}_1(D)$ is the kernel of the norm homomorphism $D^{\times} \to \mathbb{G}_{m,K}$.

Choose such a K and D, and assume that the rank of D is n^2 . Maximal tori of D^{\times} correspond to subfields of degree n of D, and each degree n field extension of K is such a subfield (see [28, Remark IV.4.4.(c)]). Let L be an unramified subfield of D degree n and K a totally ramified subfield of D degree n.

Lemma 5.1.1. Suppose that p is coprime to n. Then the group \mathcal{O}_D^{\times} is (topologically) generated by \mathcal{O}_L^{\times} and \mathcal{O}_F^{\times} .

Proof. The quotient $\mathcal{O}_D^{\times}/(1+\pi\mathcal{O}_D)$, where π is a uniformiser of D, is isomorphic to k_L^{\times} , where k_L is the residue field of L. Because \mathcal{O}_L^{\times} surjects onto k_L^{\times} , it suffices to show that the group (topologically) generated by \mathcal{O}_L^{\times} and \mathcal{O}_F^{\times} contains $1+\pi\mathcal{O}_D$. For this it is enough to show that the Lie algebra (topologically) generated by \mathcal{O}_L and \mathcal{O}_F is isomorphic to \mathcal{O}_D , because the exponential map converges on $\pi\mathcal{O}_D \subset \mathcal{O}_D = \text{Lie } \mathcal{O}_D^{\times}$.

Remark IV4.4.(b) of [28] tells us that D can be described as the (noncommutative) algebra $L\{\pi\}$ where $\pi^n = \varpi_L$ is a uniformiser in L and where

$$\pi \cdot a = \sigma^k(a) \cdot \pi,$$

for some $1 \leq k \leq n$ coprime to n with σ is the Frobenius automorphism of L/K. It follows in the usual way that \mathcal{O}_D is generated as an \mathcal{O}_K -algebra by \mathcal{O}_L and π . We then have that, as an \mathcal{O}_L module,

$$\mathcal{O}_D = \bigoplus_{j=0}^{n-1} \mathcal{O}_L \pi^j.$$

The Lie bracket is given by the commutator, which means that

$$[a, \pi^j] = (a - \sigma^{jk}(a))\pi.$$

Therefore, the \mathcal{O}_K -linear Lie algebra generated by \mathcal{O}_L and \mathcal{O}_F contains expressions of the form

$$a - \sigma^{jk}(a), b\pi^{jk}$$

for $b \in \mathcal{O}_K$ and $a \in \mathcal{O}_L$, and so it suffices to show that \mathcal{O}_L is generated by

$${a - \sigma^{jk}(a) \mid a \in \mathcal{O}_l} \cup \mathcal{O}_K.$$

We note that the \mathcal{O}_K module N generated by $\{a - \sigma^{jk}(a) \mid a \in \mathcal{O}_l\}$ sits in a short exact sequence

$$0 \to N \to \mathcal{O}_L \to (\mathcal{O}_L)_{\operatorname{Gal}(L/K)} \to 0,$$

where the subscript indicates taking coinvariants for the action of the Galois group given by the jk-th power of the natural action; this is well defined since $\operatorname{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z}$. If $p \nmid n$, then there is an isomorphism from invariants to coinvariants

$$\mathcal{O}_K = (\mathcal{O}_L)^{\operatorname{Gal}(L/K)} \simeq (\mathcal{O}_L)_{\operatorname{Gal}(L/K)}$$
,

which gives a direct sum decomposition $\mathcal{O}_L = N \oplus \mathcal{O}_K$, where the inclusion of $\mathcal{O}_K \to \mathcal{O}_L$ is the natural one up to multiplication by n.

Lemma 5.1.2. Suppose that p is coprime to n, then the group $SL_1(\mathcal{O}_D)$ is generated by

$$\mathrm{SL}_1(\mathcal{O}_L)$$
 and $\mathrm{SL}_1(\mathcal{O}_F)$,

with L and F as above.

Proof. It is enough to show that the Lie algebra of $SL_1(\mathcal{O}_D)$ is generated by the Lie algebras of $SL_1(\mathcal{O}_L)$ and $SL_1(\mathcal{O}_F)$ by the same argument as above. The short exact sequence of p-adic Lie groups

$$0 \to \mathrm{SL}_1(\mathcal{O}_D) \to \mathcal{O}_D^{\times} \to \mathcal{O}_K^{\times} \to 0$$

induces a short exact sequence of Lie algebras

$$0 \to \mathcal{O}_D^{\operatorname{Tr}_{D/K}=0} \to \mathcal{O}_D \to \mathcal{O}_K \to 0.$$

This sequence is split exact if p is coprime to n; a section of the trace map $\mathcal{O}_D \to \mathcal{O}_K$ is given by 1/n times the usual inclusion $\mathcal{O}_K \subset \mathcal{O}_D$. It follows that

$$\mathcal{O}_D \simeq \mathcal{O}_K \oplus \mathcal{O}_D^{\operatorname{Tr}_{D/K} = 0}$$

and similarly that

$$\mathcal{O}_L \simeq \mathcal{O}_L^{\operatorname{Tr}_{L/K}=0} \oplus \mathcal{O}_K$$

$$\mathcal{O}_F \simeq \mathcal{O}_F^{\operatorname{Tr}_{F/K}=0} \oplus \mathcal{O}_K.$$

Now the fact that \mathcal{O}_D is generated by \mathcal{O}_L and \mathcal{O}_F implies the analogous result for the trace zero part, concluding the proof.

Proposition 5.1.3. $J_{b,an}^{der}$ acts trivially on $\pi_0(\mathrm{Ig}_b)$.

Proof. Since $SL_1(\mathcal{O}_D) = SL_1(D)$, Lemma 5.1.2 implies that the \mathbb{Q}_p -points of each simple component of $J_{b,\mathrm{an}}^{\mathrm{der}}$ is generated by two maximal tori, which can be specified up to isomorphism. By Corollary 4.2.2, these tori may be chosen to act trivially on $\pi_0(\mathrm{Ig}_b)$.

In the last part of this section, we will collect a lemma for later use:

Lemma 5.1.4. Let Z be the center of $SL_1(D)$. Then any cohomology class $\alpha \in H^1(K, Z)$ maps to zero in either $H^1(K, SL_1(L))$ or $H^1(K, SL_1(F))$.

Proof. We know that $Z \simeq \mu_{n,K}$ and therefore $H^1(K,Z) = K^{\times}/K^{\times,n}$. A standard long exact sequence argument shows that $H^1(K, \mathrm{SL}_1(L)) = K^{\times}/\mathrm{Nm}_{L/K} L^{\times}$ and $H^1(K, \mathrm{SL}_1(F)) = K^{\times}/\mathrm{Nm}_{F/K} F^{\times}$. There are inclusions

$$K^{\times,n} \subset \operatorname{Nm}_{L/K} L^{\times}, \operatorname{Nm}_{F/K} F^{\times}$$

which give rise to the natural maps $H^1(K, Z) \to H^1(K, \mathrm{SL}_1(F))$ and $H^1(K, Z) \to H^1(K, \mathrm{SL}_1(L))$. The result follows because the group generated by $\mathrm{Nm}_{L/K} L^{\times}$ and $\mathrm{Nm}_{F/K} F^{\times}$ is equal to K^{\times} . \square

5.2. The product is surjective. So far we have seen that $J_b^{\text{der}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Ig}_b)$ and that we can find maximal tori of $J_b'(\mathbb{Q}_p)$ which act trivially on Ig_b . In this section we show that this implies that $J_b'(\mathbb{Q}_p)$ acts trivially, too. As a warm-up, we will deal with the case of quasi-split groups:

Lemma 5.2.1. Suppose that J_b is quasi-split, then $J_b'(\mathbb{Q}_p)$ is generated by $J_b^{der}(\mathbb{Q}_p)$ and $T(\mathbb{Q}_p)$ for a choice of maximal torus T.

Proof. Let T' be the centraliser of a maximal split torus T. The Galois module $X_*(T^{\text{der}})$ is induced by [2, Proposition 4.4.16] and the short exact sequence

$$1 \to T^{\operatorname{der}} \to J_b^{\operatorname{der}} \times T' \to J_b' \to 1$$

induces a short exact sequence on \mathbb{Q}_p points because $H^1(\mathbb{Q}_p, T^{\mathrm{der}})$ vanishes.

Proposition 5.2.2. For a general J'_b , not necessarily quasi-split, the group $J'_b(\mathbb{Q}_p)$ is generated by $J^{der}_b(\mathbb{Q}_p)$ and $T_1(\mathbb{Q}_p), \dots, T_n(\mathbb{Q}_p)$, for finitely many maximal tori of J'_b (which can be specified up to isomorphism).

The proof goes via a series of lemmas:

Lemma 5.2.3. The subgroup of $J'_b(\mathbb{Q}_p)$ generated by $J^{der}_b(\mathbb{Q}_p)$ and $T'(\mathbb{Q}_p)$, for a certain choice of T', contains the \mathbb{Z}_p -points of every parahoric subgroup of $J'_b(\mathbb{Q}_p)$.

Proof. We can take a torus T' that becomes the centraliser of a maximal split torus over the maximal unramified extension of \mathbb{Q}_p (these exist by [36, 1.10]). Remark 16 of the appendix to [31] tells us that the subgroup of $J'_b(\mathbb{Q}_p)$ generated by J_b^{der} and $T'(\mathbb{Z}_p)$ is equal to the group generated by all parahoric subgroups of $J'_b(\mathbb{Q}_p)$.

Proof of Proposition 5.2.2. The Iwasawa decomposition ([36, 3.3.3]) tells us that

$$J_h'(\mathbb{Q}_p) = KM(\mathbb{Q}_p)K$$

where K is a special maximal parahoric subgroup and M is a minimal \mathbb{Q}_p -rational Levi subgroup of J_b' . The group M is the centraliser of a maximal split torus S in J_b' , and we know that M is

an anisotropic group. The group generated by $J_b^{\text{der}}(\mathbb{Q}_p)$ and $T'(\mathbb{Q}_p)$ is normal, so if we choose tori that contain a maximal split torus of J_b' (which is something well defined up to isomorphism of tori), then we can conjugate them to contain S and therefore lie in M. The result now follows from Lemma 5.2.4.

Lemma 5.2.4. The group $M(\mathbb{Q}_p)$ is generated by $M^{der}(\mathbb{Q}_p)$ and $T_1(\mathbb{Q}_p), T_2(\mathbb{Q}_p)$ for maximal tori T_1, T_2 of M.

Proof. The derived subgroup M^{der} is simply connected because M is a Levi subgroup of J_b' and J_b^{der} is simply connected (see [26, Proposition 12.14]). Therefore there is a short exact sequence

$$1 \to M^{\mathrm{der}}(\mathbb{Q}_p) \to M(\mathbb{Q}_p) \to M^{\mathrm{ab}}(\mathbb{Q}_p) \to 1.$$

We see that it suffices to show that we can choose maximal tori such that the group generated by $T_1(\mathbb{Q}_p), \dots, T_n(\mathbb{Q}_p)$ surjects onto $M^{\mathrm{ab}}(\mathbb{Q}_p)$. Let Z_M and $Z_{M^{\mathrm{der}}}$ denote the centraliser of M and M^{der} , respectively. If T is any maximal torus of M then there are short exact sequences

which give rise to long exact sequences

$$1 \longrightarrow Z_{M^{\operatorname{der}}}(\mathbb{Q}_p) \longrightarrow Z_M(\mathbb{Q}_p) \longrightarrow G^{\operatorname{ab}}(\mathbb{Q}_p) \longrightarrow H^1(\mathbb{Q}_p, Z_{M^{\operatorname{der}}}) \longrightarrow 1$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$1 \longrightarrow T^{\operatorname{der}}(\mathbb{Q}_p) \longrightarrow T(\mathbb{Q}_p) \longrightarrow G^{\operatorname{ab}}(\mathbb{Q}_p) \longrightarrow H^1(\mathbb{Q}_p, T^{\operatorname{der}}) \longrightarrow H^1(\mathbb{Q}_p, T).$$

We see that it is enough to show that for every $\alpha \in H^1(\mathbb{Q}_p, Z_M^{\mathrm{der}})$, there is a maximal torus T_α of M such that α maps to zero in $H^1(\mathbb{Q}_p, T_\alpha^{\mathrm{der}})$. This is something we can solve entirely inside M^{der} , which is a product of restrictions of scalars of groups of the form $\mathrm{SL}_1(D)/K$, as in Section 5.1. For those groups, the existence of these maximal tori is proven in Lemma 5.1.4.

6. Proof of the main result

6.1. Identification of the quotient. In this section we identify the quotient H/H' where $H = H_C$ and C is a distinguished central leaf; we will show that $H/H' \simeq G^{ab}$.

After choosing a basis of the Dieudonné module of A_x , the Frobenius corresponds to an element $b \in G(\check{\mathbb{Q}}_p) \subset \operatorname{GL}_V(\check{\mathbb{Q}}_p)$. The group of automorphisms of $A_x[p^{\infty}]$ is precisely the twisted centraliser of this element b intersected with $\operatorname{GL}_V(\check{\mathbb{Z}}_p)$. Similarly, the group H is the intersection of $J_b(\mathbb{Q}_p) \subset G(\check{\mathbb{Q}}_p) \subset \operatorname{GL}_V(\check{\mathbb{Q}}_p)$ with $\mathcal{G}(\check{\mathbb{Z}}_p)$, where \mathcal{G}/\mathbb{Z}_p is the parahoric group scheme such that $\mathcal{G}(\mathbb{Z}_p) = U_p$. It follows that H is precisely the stabiliser in $J_b(\mathbb{Q}_p)$ of the element 1 in the affine Deligne-Lusztig variety $X(\mu,b)$ for G. If we have a different point x' such that b' is σ -conjugate to b, then the stabiliser $H_{x'}$ corresponds to the stabiliser of some element in $X(\mu,b)$, and every such stabiliser occurs as an $H_{x'}$.

If we choose b' to be a σ -straight element, then the corresponding stabiliser group is a parahoric subgroup of $J_b(\mathbb{Q}_p)$ (see the proof of Proposition 3.1.4 of [41]) and the corresponding central leaf C is called a *distinguished central leaf* because it is also equal to an Ekedahl-Oort stratum (see the proof of Theorem E of [34]).

This means that there is a smooth affine group scheme \mathcal{J}/\mathbb{Z}_p with connected special fiber such that $\mathcal{J}(\mathbb{Z}_p) = H$. This group scheme \mathcal{J} is the connected component of the identity of a Bruhat-Tits stabiliser group scheme $\tilde{\mathcal{J}}$. Proposition 2.4.9 of [24] tells us that the natural map $\tilde{\mathcal{J}}' \to \tilde{\mathcal{J}}$ of Bruhat-Tits stabiliser group schemes induced by $\beta: J_b' \to J_b$ is a closed immersion (the assumptions are satisfied because J_b splits over a tamely ramified extension). We define \mathcal{J}_{β}' to be the inverse image of \mathcal{J} under $\tilde{\mathcal{J}}' \to \tilde{\mathcal{J}}$, or equivalently the Zariski closure of J_b' in \mathcal{J} . This tells us that \mathcal{J}_{β}' is a smooth affine group scheme with $\mathcal{J}_{\beta}'(\mathbb{Z}_p) = H'$.

Lemma 6.1.1. There is a short exact sequence

$$(1) 1 \to \mathcal{J}'_{\beta} \to \mathcal{J} \to \mathcal{Z} \to 1,$$

where \mathcal{Z} is the connected Néron model of G^{ab} .

Proof. There is a natural map $\tilde{\mathcal{J}} \to \tilde{\mathcal{Z}}$, where $\tilde{\mathcal{Z}}$ is the finite type Néron model of G^{ab} . This map induces a map $\mathcal{J} \to \mathcal{Z}$ of identity components, which we claim to be surjective. To prove this claim, it suffices to prove that $\mathcal{T} \to \mathcal{Z}$ is surjective, where $\mathcal{T} \subset \mathcal{J}$ is the connected Néron model of some maximal torus T of J_b . Such a \mathcal{T} exists and in fact we can take T to be tamely ramified. In that case, [21, Proposition 1.1.4] gives the surjectivity of $\mathcal{T} \to \mathcal{Z}$.

We are left to show that \mathcal{J}'_{β} is the kernel of the map $\mathcal{J} \to \mathcal{Z}$. This can be checked on $\check{\mathbb{Z}}_p$ -points, and there we consider the following diagram

$$\mathcal{J}'_{\beta}(\check{\mathbb{Z}}_p) \longrightarrow \mathcal{J}(\check{\mathbb{Z}}_p) \longrightarrow \mathcal{Z}(\check{\mathbb{Z}}_p) \longrightarrow 1$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$1 \longrightarrow J'_b(\check{\mathbb{Q}}_p) \longrightarrow J_b(\check{\mathbb{Q}}_p) \longrightarrow G^{ab}(\check{\mathbb{Q}}_p) \longrightarrow 1.$$

The exactness of the left of the top row follows from the fact that the leftmost square is Cartesian, and so we are done. \Box

By construction, the connected components of the special fiber of \mathcal{J}'_{β} are given by the kernel of

$$\pi_1(J_b')_{I,\text{tors}} \to \pi_1(J_b)_{I,\text{tors}}.$$

Lemma 6.1.2. The special fiber of \mathcal{J}'_{β} is connected.

Proof. We know that J_b is an inner form of a standard Levi $M_b \subset G$ and similarly that J_b' is an inner form of a standard Levi $M_b' \subset G^{\operatorname{der}}$, we can arrange things such that $M_b' \subset M_b$. Using the isomorphism $\pi_1(J_b) \simeq \pi_1(M_b)$ and $\pi_1(J_b') \simeq \pi_1(M_b')$ it suffices to show that the kernel of

$$\pi_1(M_b')_{I,\text{tors}} \to \pi_1(M_b)_{I,\text{tors}}$$

is trivial. Because M_b is a standard Levi of G, it contains the centraliser T of a maximal split torus S of G and moreover T is also the centraliser of S in M_b . Let T', T^{der} be the intersections of T with M'_b and M^{der}_b respectively, then both $X_*(T')$ and $X_*(T^{\text{der}})$ are induced Galois modules by [2, Proposition 4.4.16] (recall that M^{der}_b is simply connected by [26, Proposition 12.14]). This

means that both $X_*(T^{\text{der}})_I$ and $X_*(T')_I$ are torsion free, and so there is a diagram of short exact sequences

$$0 \longrightarrow X_*(T^{\operatorname{der}})_I \longrightarrow X_*(T')_I \longrightarrow \pi_1(M'_b)_I \longrightarrow 0$$

$$\downarrow b \qquad \qquad \downarrow c$$

$$0 \longrightarrow X_*(T^{\operatorname{der}})_I \longrightarrow X_*(T)_I \longrightarrow \pi_1(M_b)_I \longrightarrow 0.$$

The snake lemma gives us an isomorphism $\ker c \simeq \ker b$ and since $X_*(T')_I$ is torsion free it follows that there is no torsion in the kernel of

$$\pi_1(M_b')_I \to \pi_1(M_b)_I.$$

Hence, the kernel of

$$\pi_1(M_b')_{I,\text{tors}} \to \pi_1(M_b)_{I,\text{tors}}$$

is trivial. \Box

Corollary 6.1.3. The natural map $H \to G^{ab}(\mathbb{Z}_p) := \mathbb{Z}(\mathbb{Z}_p)$ induces an isomorphism $H/H' \simeq G^{ab}(\mathbb{Z}_p)$.

Proof. This follows by taking \mathbb{Z}_p -points of the short exact sequence (1) and using Lang's Lemma to deduce that the cohomology of \mathcal{J}'_{β} vanishes.

6.2. Main theorems. In this section we will state the main theorems in full generality, giving Theorems 1 and 2 as special cases. The assumptions H1 and H2 below come into play because they are necessary for applying Theorem 2 and Theorem 5.4.3 of [17], respectively.

Let (G,X) be a Shimura datum of Hodge type and let p>2 be a prime such that $G_{\mathbb{Q}_p}$ is quasi-split and splits over a tamely ramified extension. Suppose moreover that G^{der} is simply connected and that G^{ad} is \mathbb{Q} -simple. Let $b\in B(G,\mu)$ be a non-basic element. Let Ig_b the Igusa variety associated to it, J_b its twisted centraliser and write its derived group as $J_b^{\operatorname{der}} = J_{b,\operatorname{an}}^{\operatorname{der}} \times J_{b,\operatorname{iso}}^{\operatorname{der}}$ with $J_{b,\operatorname{an}}^{\operatorname{der}}$ anisotropic and $J_{b,\operatorname{iso}}^{\operatorname{der}}$ isotropic. As before, the classification of reductive groups over \mathbb{Q}_p tells us that

$$J_{b,\mathrm{an}}^{\mathrm{der}} \simeq \prod_{i=1}^{k} \mathrm{Res}_{K_i/\mathbb{Q}_p} \operatorname{SL}_1(D_i),$$

where K_i/\mathbb{Q}_p is a finite extension and D_i/K_i is a central division algebra of rank n_i^2 . We will assume that p is coprime to n_i for all i. This is true if p does not divide the order of W(G), because then p does not divide the order of W(J(b)) since it is an inner form of a Levi and hence does not divide the order of the Weyl group of $J_{b,\mathrm{an}}^{\mathrm{der}}$, which is n!. Now consider the following two sets of hypotheses, where τ is the unique basic element of $B(G,\mu)$:

- H1 The group $G_{\mathbb{Q}_n}$ splits over an unramified extension.
- H2 The group $J_{\tau}^{\rm ad}$ has no anisotropic factors and either ${\rm Sh}_U$ is proper or [17, Conjecture 3.7.5] holds.

We can now state our main theorem:

Theorem 6.2.1. Let (G, X) be as above and suppose that either H1 or H2 holds. Then there is a bijection

$$\pi_0(\mathrm{Ig}_b) \simeq \pi_0(\mathrm{Sh}_G) \times G^{ab}(\mathbb{Z}_p),$$

where $G^{ab}(\mathbb{Z}_p)$ is the group of \mathbb{Z}_p -points of the connected Néron model of $G^{ab}(\mathbb{Q}_p)$.

Proof of Theorem 6.2.1. Theorem E of [34] tells us that we can arrange C(x) such that it is an EKOR stratum and Theorem 2 (under assumption H1) and [17, Theorem 5.4.3] (under assumption H2) tell us that the natural map

$$\pi_0(C(x)) \to \pi_0(\operatorname{Sh}_G)$$

is a bijection. Proposition 3.0.3 and Proposition 5.1.3 tell us that $J_b^{\text{der}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Ig}_b)$. Moreover, it follows from Proposition 5.2.2 that $J_b'(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Ig}_b)$ and hence $\pi_0(\mathrm{Ig}_b) \simeq H/H'$. We conclude by applying the fact that $H/H' \simeq G^{ab}(\mathbb{Z}_p)$, as proven in Section 6.1

Corollary 6.2.2. There is a bijection

$$\pi_0(C(x)) \simeq \pi_0(\operatorname{Sh}_{G,b}) \times \frac{G^{ab}(\mathbb{Z}_p)}{H_C^{ab}},$$

where H_C^{ab} is the image of H_C in $G^{ab}(\mathbb{Z}_p)$ via the natural map $J_b \to G^{ab}$. The group H_C^{ab} is of finite index in $G^{ab}(\mathbb{Z}_p)$ because it always contains the image of $Z_G(\mathbb{Z}_p)$.

Remark 6.2.3. It follows from the proof of Lemma 8.1.1 of [25] that there is a maximal torus $T \subset J_b(\mathbb{Q}_p)$ such that $T(\mathbb{Q}_p) \cap H$ acts transitively on H/H', confirming an assumption of Eischen-Mantovan, see of [10, Remark 3.3.4].

Now we state the generalisation of Theorem 2.

Theorem 6.2.4. Let (G, X) be as above and suppose that either H1 or H2 holds, then the natural map

$$\pi_0(\operatorname{Sh}_{G,b}) \to \pi_0(\operatorname{Sh}_G)$$

is an isomorphism. Moreover the number of irreducible components of each connected component of $Sh_{G,b}$ is given by the representation theoretic constant

$$\operatorname{Dim} V_{\underline{\mu}}^{\hat{H}}(\lambda_b)_{rel},$$

introduced in the appendix of [41].

Proof. Let $X_{\mu}(b)$ be the affine Deligne-Lusztig variety for G associated to b and μ , this is a perfect scheme over $\overline{\mathbb{F}}_p$. The Mantovan product formula tells us that there is a morphism

$$X_{\mu}(b) \times \operatorname{Ig}_{b} \to \operatorname{Sh}_{G,b}^{\operatorname{perf}},$$

which is a $J_b(\mathbb{Q}_p)$ -torsor. Therefore, in order to determine connected or irreducible components of $\mathrm{Sh}_{G,b}$, it suffices to determine the $J_b(\mathbb{Q}_p)$ -orbits of connected or irreducible components of $X_{\mu}(b) \times \mathrm{Ig}_b$.

Theorem 3.1.1 of [41] tells us that the stabiliser in $J_b(\mathbb{Q}_p)$ of an irreducible component of $X_{\mu}(b)$ is a parahoric subgroup. If $g \in J_b(\mathbb{Q}_p)$ stabilises an irreducible component Z of $X_{\mu}(b)$, then it also stabilises the connected component that Z lies in. Thus, the stabiliser of a connected component contains a parahoric subgroup.

By Theorem A.1.4. of [17], $J_b(\mathbb{Q}_p)$ acts transitively on $\pi_0(X_\mu(b))$. The stabiliser of any connected component surjects onto $G^{ab}(\mathbb{Z}_p)$ because it contains a parahoric subgroup, and therefore $J_b(\mathbb{Q}_p)$ acts transitively on the fibers of

$$\pi_0(X_\mu(b) \times \mathrm{Ig}_b) \to \pi_0(\mathrm{Sh}_G).$$

Since the natural map $\pi_0(\operatorname{Sh}_{G,b}) \to \pi_0(\operatorname{Sh}_G)$ is surjective by the proof of Axiom 5 in [40], the result on π_0 follows.

Theorem A.3.1 of [41] tells us that the number of $J_b(\mathbb{Q}_p)$ -orbits of irreducible components in $X_{\mu}(b)$ is given by

$$N := \operatorname{Dim} V_{\mu}^{\hat{H}}(\lambda_b)_{\mathrm{rel}}.$$

Since the stabiliser of each irreducible component is a parahoric subgroup which acts transitively on the fibers of $\pi_0(\mathrm{Ig}_b) \to \pi_0(\mathrm{Sh}_G)$, we find that the irreducible components of $\mathrm{Sh}_{G,b}$ are in bijection with

$$N \times \pi_0(\operatorname{Sh}_G) \simeq N \times \pi_0(\operatorname{Sh}_{G,b})$$

and the result follows.

6.3. A counterexample to the discrete Hecke orbit conjecture (due to Rong Zhou). Our counterexample is a unitary Shimura variety of PEL type. Let $F = F^+E$ be a CM field where F^+ is totally real of degree 4 and E is an imaginary quadratic field. Let V be a Hermitian F vector space of rank 2 with signature (1,1) at all infinite places of F^+ and let $(G,X) = (GU_V,X_V)$ be the corresponding Shimura datum of PEL type.

Let p be a prime which splits in E and which splits as $p = \mathfrak{p}_1\mathfrak{p}_2$ in F^+ where both \mathfrak{p}_1 and \mathfrak{p}_2 have residual degree 2. Then

$$G_{\mathbb{Q}_p} \simeq \operatorname{Res}_{K/\mathbb{Q}_p} \operatorname{GL}_2 \times \operatorname{Res}_{K/\mathbb{Q}_p} \operatorname{GL}_2 \times \mathbb{G}_m,$$

where K/\mathbb{Q}_p is the unique unramified quadratic extension. Let b be the element which is μ_1 ordinary in the first factor and μ_2 -basic in the second factor and μ_3 -basic in the third factor. Here $\mu = (\mu_1, \mu_2, \mu_3)$ according to the product decomposition of $G_{\mathbb{Q}_p}$.

We have seen above that $H = H_x$ is the precisely the stabiliser in $J_b(\mathbb{Q}_p)$ of an $\overline{\mathbb{F}}_p$ -point x of the affine Deligne-Lusztig variety $X(\mu, b)$. By the product decomposition of G we get a corresponding product decomposition of $X(\mu, b)$ and therefore

$$H_x = H_1 \times H_2 \times H_3$$
.

It suffices to show that H_2 does not surject onto the maximal abelian quotient of $\operatorname{Res}_{K/\mathbb{Q}_p}\operatorname{GL}_2$. In this case $J_{b_2}=\operatorname{Res}_{K/\mathbb{Q}_p}\operatorname{GL}_2$ and $X(\mu_2,b_2)$ is equidimensional of dimension 1.

Lemma 6.3.1. The irreducible components of $X(\mu_2, b_2)$ are isomorphic to \mathbb{P}^1 . The stabiliser of an irreducible component is a hyperspecial subgroup of $GL_2(K)$ isomorphic to $GL_2(\mathcal{O}_K)$ which acts on \mathbb{P}^1 via the natural map $GL_2(\mathcal{O}_K) \to GL_2(\mathbb{F}_{q^2})$.

Proof. It follows from the main result of [11] that $X(\mu_2, b_2)$ is a union of two one-dimensional Ekedahl-Oort strata and one zero-dimensional Ekedahl-Oort stratum. To prove the lemma, it suffices to work with the one-dimensional Ekedahl-Oort strata individually. It follows from paragraph 5.10 of [11] that these are each unions of (closures of) classical Deligne-Lusztig varieties for the group $\operatorname{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p}$. Therefore their irreducible components are isomorphic to \mathbb{P}^1 and the action of $\operatorname{GL}_2(\mathcal{O}_K)$ factors through the natural action of $\operatorname{GL}_2(\mathbb{F}_{q^2})$.

A direct computation shows that we can find a point $x \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$ such that its stabiliser in $GL_2(\mathbb{F}_{q^2})$ is given by the group of scalar matrices, and therefore does not surject onto \mathbb{F}_{p^2} via the determinant map. This implies that the stabiliser of x in $GL_2(\mathcal{O}_K)$ and therefore in $J_b(\mathbb{Q}_p)$ does not surject onto \mathcal{O}_K^{\times} via the determinant map.

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References

- [1] Armand Borel, *Linear algebraic groups*, Second, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.
- [2] F. Bruhat and J. Tits, Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée, Inst. Hautes Études Sci. Publ. Math. 60 (1984), 197–376.
- [3] Ana Caraiani and Peter Scholze, On the generic part of the cohomology of compact unitary Shimura varieties, Ann. of Math. (2) **186** (2017), no. 3, 649–766.
- [4] Ching-Li Chai and Frans Oort, *Hypersymmetric abelian varieties*, Pure and Applied Mathematics Quarterly 2 (2006), no. 1, 1–27.
- [5] ______, Monodromy and irreducibility of leaves, Ann. of Math. (2) 173 (2011), no. 3, 1359–1396.
- [6] ______, The Hecke orbit conjecture: a survey and outlook, Open problems in arithmetic algebraic geometry, [2019] ©2019, pp. 235–262. MR3971186
- [7] Miaofen Chen, Mark Kisin, and Eva Viehmann, Connected components of affine Deligne-Lusztig varieties in mixed characteristic, Compos. Math. 151 (2015), no. 9, 1697–1762.
- [8] Marco D'Addezio, The monodromy groups of lisse sheaves and overconvergent F-isocrystals, Selecta Math. (N.S.) **26** (2020), no. 3.
- [9] Marco D'Addezio, *Parabolicity conjecture of F-isocrystals*, arXiv e-prints (December 2020), available at 2012.12879.
- [10] Ellen Eischen and Elena Mantovan, p-adic families of automorphic forms in the μ -ordinary setting, Amer. J. Math. (2020). To appear.
- [11] Ulrich Görtz, Xuhua He, and Sian Nie, Fully Hodge-Newton decomposable Shimura varieties, Peking Math. J. 2 (2019), no. 2, 99–154. MR4060001
- [12] Paul Hamacher, The almost product structure of Newton strata in the deformation space of a Barsotti-Tate group with crystalline Tate tensors, Mathematische Zeitschrift 287 (2017), no. 3-4, 1255-1277.
- [13] _____, The product structure of Newton strata in the good reduction of Shimura varieties of Hodge type, Journal of Algebraic Geometry 28 (2019), no. 4, 721–749.
- [14] Paul Hamacher and Wansu Kim, ℓ-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients, Math. Ann. 375 (2019), no. 3-4, 973–1044.
- [15] Haruzo Hida, Irreducibility of the Igusa tower, Acta Math. Sin. (Engl. Ser.) 25 (2009), no. 1, 1–20.
- [16] ______, Irreducibility of the Igusa tower over unitary Shimura varieties, On certain L-functions, 2011, pp. 187–203.
- [17] Pol van Hoften, Mod p points on Shimura varieties of parahoric level (with an appendix by Rong Zhou), arXiv e-prints (October 2020), available at 2010.10496.
- [18] Jun-ichi Igusa, Kroneckerian model of fields of elliptic modular functions, Amer. J. Math. 81 (1959), 561–577. MR108498
- [19] Wansu Kim, On central leaves of Hodge-type Shimura varieties with parahoric level structure, Math. Z. 291 (2019), no. 1-2, 329–363.
- [20] M Kisin, K. Madapusi Pera, and S.W. Shin, Honda-Tate theory for Shimura varieties, 2018. preprint, available at https://math.berkeley.edu/~swshin/HT.pdf.
- [21] M. Kisin and G. Pappas, Integral models of Shimura varieties with parahoric level structure, Publ. Math. Inst. Hautes Études Sci. 128 (2018), 121–218.
- [22] Mark Kisin, Integral models for shimura varieties of abelian type, Journal of The American Mathematical Society 23 (2010), no. 4, 967–1012.
- [23] _____, mod p points on Shimura varieties of abelian type, J. Amer. Math. Soc. 30 (2017), no. 3, 819–914.
- [24] Mark Kisin and Rong Zhou, Independence of ℓ for Frobenius conjugacy classes attached to abelian varieties, 2021. In preparation.
- [25] Arno Kret and Sug-Woo Shin, H^0 of Igusa varieties via automorphic forms, 2021. In preparation.
- [26] Gunter Malle and Donna Testerman, Linear algebraic groups and finite groups of Lie type, Cambridge Studies in Advanced Mathematics, vol. 133, Cambridge University Press, Cambridge, 2011. MR2850737
- [27] Elena Mantovan, On the cohomology of certain PEL-type Shimura varieties, Duke Math. J. 129 (2005), no. 3, 573–610.

- [28] J.S. Milne, Class field theory (v4.03), 2020. Available at www.jmilne.org/math/.
- [29] Frans Oort, Foliations in moduli spaces of abelian varieties, Journal of the American Mathematical Society 17 (2004), no. 2, 267–296.
- [30] G. Pappas, On integral models of Shimura varieties, arXiv e-prints (March 2020), arXiv:2003.13040, available at 2003.13040.
- [31] G. Pappas and M. Rapoport, Twisted loop groups and their affine flag varieties, Adv. Math. 219 (2008), no. 1, 118–198. With an appendix by T. Haines and Rapoport. MR2435422
- [32] Vladimir Platonov and Andrei Rapinchuk, Algebraic groups and number theory, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.
- [33] Gopal Prasad and Andrei S. Rapinchuk, *Irreducible tori in semisimple groups*, Internat. Math. Res. Notices **23** (2001), 1229–1242.
- [34] Xu Shen, Chia-Fu Yu, and Chao Zhang, EKOR strata for Shimura varieties with parahoric level structure, arXiv e-prints (2019), arXiv:1910.07785, available at 1910.07785.
- [35] Xu Shen and Chao Zhang, Stratifications in good reductions of Shimura varieties of abelian type, arXiv e-prints (July 2017), arXiv:1707.00439, available at 1707.00439.
- [36] J. Tits, Reductive groups over local fields, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, 1979, pp. 29–69. MR546588
- [37] Liang Xiao and Xinwen Zhu, Cycles on shimura varieties via geometric satake, arXiv preprint arXiv:1707.05700 (2017).
- [38] Luciena Xiao Xiao, On The Hecke Orbit Conjecture for PEL Type Shimura Varieties, arXiv e-prints (June 2020), available at 2006.06859.
- [39] Yujie Xu, Normalization in integral models of shimura varieties of hodge type, 2020.
- [40] Rong Zhou, Mod-p isogeny classes on Shimura varieties with parahoric level structure, arXiv e-prints (2017), arXiv:1707.09685, available at 1707.09685.
- [41] Rong Zhou and Yihang Zhu, Twisted orbital integrals and irreducible components of affine Deligne-Lusztig varieties, arXiv e-prints (November 2018), available at 1811.11159.
- [42] Ying Zong, On hypersymmetric abelian varieties, Ph.D. thesis, University of Pennsylvania (2008). https://www.math.upenn.edu/grad/dissertations/YingZongThesis.pdf.