Notation :

- G = reductive group / $\overline{\mathbb{F}}_p$ $\qquad G = G_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_p$

- $T \subset B$ $\qquad$ Borel + max torus

  T is F-stable

- $X_{T\subset B} := X(w)$ $\qquad w = \mathrm{inv}(B, FB)$

  $B \overset{w}{\dashrightarrow} FB$

- $$\begin{tikzcd} \tilde{X}_{T\subset B} \arrow[r] \arrow[loop above, "G^F \times T^F"] & X_{T\subset B} \arrow[loop above, "G^F"] \end{tikzcd}$$ $\qquad T^F$ - torsor

  U unipotent radical of B

  $X_{T\subset B} = \{ g \in G \mid g^{-1} Fg \in FU \} / T^F (U \cap FU)$

  $\tilde{X}_{T\subset B} = \{ \cdots \} / U \cap FU$

  $gBg^{-1} \longleftrightarrow g$

$\theta : T^F \to \overline{\mathbb{Q}}_\ell^\times$

- $R^\theta_{T\subset B} = \sum (-1)^i H^i_c(\tilde{X}_{T\subset B}, \overline{\mathbb{Q}}_\ell)_\theta$

virtual representation

Goal : calculate character of $R^\theta_{T\subset B}$

§1 Fixed point formula :

$X/\mathbb{F}_q$ variety

Idea: If $\sigma : X \to X$ is finite order automorphism

$\sigma = su$ $\quad s =$ prime-to-p order

$\quad u =$ power of p order

then $tr(\sigma^*, H_c^*(X)) = tr(u^*, H_c^*(X^s))$

$\tilde{X} \xrightarrow{\pi} X$ $\quad T^F$-torsor $\quad G^F$-equivariant.

$g \in G^F \quad g = su$

$X^s = \bigsqcup X_i^s$ where $s.x = t(s,i).x$ $\quad t(s,i) \in T^F$

$\forall x \in \pi^{-1}(X_i^s)$

Take $y \in X^s$ $\quad \pi^{-1}(y)$ principal homogeneous space under T^F $\ni x$

$\exists !\, t \in T^F$ s.t. $s.x = t.x$

Prop: we have

$$tr(g, H_c^*(\tilde{X}))_\theta = \sum_{i \in I} \theta(t(s,i)^{-1})\, tr(u, H_c^*(X_i^s))$$

§2 The representation R_{TCB}^θ

Need to calculate $X_{TCB}^s = \bigsqcup_{i \in I} X_{TCB,i}^s$

Throughout let $Z^\circ(s)$ connected centralizer of s in G
s semisimple

let $\mathcal{F}\ell_G = G/B$ denote the flag variety

Idea: construct a map $\tau : \mathcal{F}\ell^s \to Z^\circ(s)\backslash G$
and define $\mathcal{F}\ell^s = \bigsqcup \tau^{-1}(Z^\circ(s)g)$

$$G/U \xrightarrow{\pi} G/B = \mathcal{F}\ell$$

Prop: for any $y \in \mathcal{F}\ell^s$ we can write $y = gB$ such that

$$sg = gt \qquad t \in T \qquad (s \text{ semisimple})$$

(conjugate with elements in U to get element in t)

(action of T as distinct from action of G)

1) if $\pi(x) = y$ then $s \cdot x = t \cdot x$

2) $sg_1 = g_1 t_1 \qquad sg_2 = g_2 t_2$

$$t_1 = t_2 \iff Z^\circ(s) g_1 = Z^\circ(s) g_2$$

3) The map $\tau_{TCB}(y) := Z^\circ(s) g$ is well defined,
$\mathcal{F}\ell^s \to Z^\circ(s)\backslash G$

with fibres $\mathcal{F}l^s(g)$

$\left(\begin{matrix} g \in G \\ gTg^{-1} \subset Z^\circ(s) \end{matrix} \right)$ $\qquad$ $\begin{matrix} \mathcal{F}l^s(g) \xrightarrow{\sim} \mathcal{F}l_{Z^\circ(s)} \\ B' \longmapsto B' \cap Z^\circ(s) \end{matrix}$

and s acts on $\pi^{-1}(\mathcal{F}l^s(g))$ as $g^{-1}sg = t(s,\cdot) \in T$.

$g \longleftrightarrow B'$ $\quad$ g unique up to $B' \cap Z^\circ(s)$

Prop: $\quad$ Take $B' \in \mathcal{F}l^s$ $\qquad$ $s \in G^F$ semisimple

if $\quad \mathrm{inv}(B, FB) = \mathrm{inv}(B', FB')$

then $\quad F\,\tau_{T\subset B}(B') = \tau_{T\subset B}(B')$ $\quad$ so $\exists$ rational pt.

$\qquad \tau_{T\subset FB}(FB') \quad \cancel{=}$

In particular, get map

$$\tau_{T\subset B} : X^s_{T\subset B} \longrightarrow {}_{Z^\circ(s)^F}\backslash G^F$$

fibres : $\quad X^s_{T\subset B}(g) \xrightarrow{\sim} X_{gTg^{-1} \subset gBg^{-1} \cap Z^\circ(s)}$

and s acts on $\pi^{-1}(X^s_{T\subset B}(g))$ as $g^{-1}sg \in T^F$

Decomp $\quad X^s_{T\subset B} = \bigsqcup_{\substack{g \in Z^\circ(s)^F \backslash G^F \\ gTg^{-1} \subset Z^\circ(s)}} X^s_{T\subset B}(g)$ $\qquad$??

Thm : Let $x = su$ Jordan decomp. of $x \in G^F$. Then

$$tr(x, R^{\theta}_{T \subset B}) = \frac{1}{|Z^{\circ}(s)^F|} \sum_{\substack{g \in G^F \\ gTg^{-1} \subset Z^{\circ}(s)}} Q_{gTg^{-1}, Z^{\circ}(s)}(u) \cdot \theta(g^{-1}sg)$$

where $Q_{gTg^{-1}, Z^{\circ}(s)}(u) = \underbrace{tr(u, R^{1}_{gTg^{-1} \subset gBg^{-1} \cap Z^{\circ}(s)})}_{\text{indep. of } B}$

Rem :

1) $Q(u)$ is an integer independent of ℓ.
(Weil conjectures)

2) $R^{\theta}_{T \subset B}$ is indep of B