PLAN (1) Construction of $(w, \theta) \longmapsto R_w^\theta$

(2) Show that if $w \rightsquigarrow w'$ $\theta = 1$ then $R_w^1 = R_{w'}^1$

(3) Give alternate construction of $R_w^\theta = R_{T\subset B}^\theta$

$G/\mathbb{F}_q$ connected reductive group $T \subset B \subset G$

F-stable torus and Borel.

$W = \dfrac{N_G(T)}{T}$ Weyl group (has an action of F)

$X = G/B$ $\quad X \times X = \bigsqcup_{w \in W} \mathcal{O}(w)$ ← G-orbits smooth etc.

$(hB, gB) \in \mathcal{O}(w) \iff h^{-1}g \in BwB$

Fact : If $w = w_1 w_2$ $\quad l(w) = l(w_1) + l(w_2)$

and B, B'' Borels with $\mathrm{Rel}(B, B'') = w$

Then $\exists!$ B' s.t. $\quad \mathrm{Rel}(B, B') = w_1$

$\mathrm{Rel}(B', B'') = w_2$

$$\mathcal{O}(w) \xrightarrow{\sim} \mathcal{O}(w_1) \times \mathcal{O}(w_2)$$

(B, B'') ↑ morphism of alg. var (B, B', B'')

Recall $G/B \supset X(w) = \{ g \in G \mid g^{-1}F(g) \in BwB \}/B$

$$\begin{array}{ccc} G/B \supset X(w) \\ \text{T-torsor} \uparrow \quad \uparrow \\ G/U \supset Y(w) = \{ g \in G \mid g^{-1}F(g) \in UwU \}/U \end{array}$$

where $B = T.U$ is the Levi decomposition

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("$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$")

$g \in G^F$ then $g^{-1}F(g) = e$

Observations : $G^F \overset{\text{clearly}}{\curvearrowright} Y(w), X(w)$

and $Y(w) \longrightarrow X(w)$ is equivariant.

Claim: the map $Y(w) \longrightarrow X(w)$ is a Galois cover

with covering group $T(w)^F = \{ t \in T \mid t = wF(t)w^{-1} \}$

This means ① $T(w)^F$ acts freely on $Y(w)$, trivially on $X(w)$

② fibers of $Y(w) \longrightarrow X(w)$ are acted on

simple transitively by $T(w)^F$

③ (étale locally trivial)

in étale topology locally trivial

$$\overset{G^F \times T(w)^F}{\circlearrowleft} \qquad\qquad\qquad \overset{G^F}{\circlearrowright}$$

COR: $$H_c^i(Y(w), \overline{\mathbb{Q}}_\ell) = \bigoplus_{\theta \in \widehat{T(w)}^F} H_c^i(Y(w), \overline{\mathbb{Q}}_\ell)_\theta$$

and $$H_c^i(Y(w), \overline{\mathbb{Q}}_\ell)_1 = H_c^i(X(w), \overline{\mathbb{Q}}_\ell)$$

Def: $$\{(w, \theta \in \widehat{T(w)}^F)\} \mapsto R_w^\theta = \sum_i (-1)^i H_c^i(Y(w), \overline{\mathbb{Q}}_\ell)_\theta$$
$$\in K_0(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G^F))$$
$$\parallel$$
"virtual reps." $=$ { integer linear combinations of irred reps }

Example: $w = e$, then
$X(w) = G^F/B^F$
$Y(w) = G^F/U^F$
$T(w)^F = T^F$

$$H_c^0(Y(w), \overline{\mathbb{Q}}_\ell)_\theta = \mathrm{Ind}_{B^F}^{G^F} \theta$$

$H_c^0(Y(w), \overline{\mathbb{Q}}_\ell) =$ functions $G^F/U^F \to \overline{\mathbb{Q}}_\ell$
$= \mathrm{Ind}_{U^F}^{G^F} 1 \quad \circlearrowleft T^F$

$H_c^0(X(w), \overline{\mathbb{Q}}_\ell) =$ { functions $G^F/B^F \longrightarrow \overline{\mathbb{Q}}_\ell$ }
$=$ { functions $G^F \longrightarrow \overline{\mathbb{Q}}_\ell$ that are

B^F invariant $\}$ $= \operatorname{Ind}_{B^F}^{G^F} 1$

Rmk: DL induction recovers parabolic induction in the degenerate case $w = e$.

Def: $w \xrightarrow{F} w'$ if $w' = w_1 w F(w_1)$ for some $w_1 \in W$

Thm (1.6 in DL) If $w \xrightarrow{F} w'$ then

$$R_w^1 = R_{w'}^1 \qquad [X(w) \neq X(w')]$$

pf. suffices to treat case $w_1 = s$, with $\ell(s) = 1$

a) $\ell(w) = \ell(w')$

b) $\ell(w) = \ell(w') + 2$

Case a) $w = s w_2$ $\qquad w' = w s F(s)$

given $B \in X(w)$

$$\operatorname{Rel}(B, F(B)) = w$$

$\exists!$ B' s.t.

$$\operatorname{Rel}(B, B') = s$$

$$\operatorname{Rel}(B', F(B)) = w_2$$

$$\begin{tikzcd}
B \arrow[r, dashed, "sw"] \arrow[d, dashed, "s"'] & F(B) \arrow[d, dashed, "F(s)"] \\
\sigma(B) \arrow[ur, dashed, "w_2"] \arrow[r, dashed, "w_2 F(s)"'] & F(\sigma(B))
\end{tikzcd}$$

$$\begin{tikzcd}
X(w) \arrow[r, "\sigma"] \arrow[d, "F"'] & X(w') \arrow[d, "F"] \arrow[dl, "B \mapsto \sigma(B)"'] \\
X(Fw) \arrow[r, "\sigma^{(p)}"'] & X(Fw')
\end{tikzcd}$$

Point is F is a universal homomorphism, so induces

+ some diagram chase

gives

$$E_c(X(w)) = E_c(X(w')).$$

b) $w' = swF(s)$, $\ell(w') = \ell(w) + 2$

Find γB, δB s.t. $(B \subset X(w'))$

$\mathrm{Rel}(B, \gamma B) = S$ $\qquad$ $\mathrm{Rel}(\delta B, F B) = F(S)$

$\mathrm{Rel}(\gamma B, \delta B)\ w$

Rmk: $\mathrm{Rel}(\delta B, F(\gamma(B))) \leq S$

$\uparrow$ equal or rel position s.

$X(w') = X_1 \cup X_2$

where $X_1 = (B$ s.t. $\delta B = F(\gamma(B)))$

$X_2 = X(w') \setminus X_1$

$X_1 \longrightarrow X(w)$

$B \longmapsto \gamma(B)$

Fiber of this map over $B' \in X(w)$ consists of B s.t.

$\mathrm{Rel}(B, B') = S$

Fact: $X_1 \longrightarrow X(w)$ is a line bundle so

(fibers are isomorphic to $\mathbb{A}^1 = k$)

Then: $H_c^i(X_1) = H_c^{i-2}(X(w))$

$$H_c^i(\mathbb{A}^1, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & \text{if } i = 2 \\ 0 & \text{o/w} \end{cases}$$

$$\Rightarrow \quad E_c(X_1) = E_c(X(\omega))$$

Miracle $\quad E_c(X_2) = 0 \quad$ and $\quad E_c(X(\omega')) = E_c(X_1) + E_c(X_2)$

$$X_1 \xrightarrow[\text{bundle}]{\mathbb{A}^1 \setminus \{0\}} X(s\omega)$$

$$E_c(X_2) = E_c^{-2}(X(s\omega)) + E_c^{-1}(X(s\omega))$$

$$= E_c(X(s\omega)) - E_c(X(s\omega)).$$