Math 245B Notes

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These are course notes for Math245B, Topics in Algebraic Geometry, Winter 2023, taught by Pol van Hoften.

Contents

1 Lecture 1: 1/9/23

The course will be on Deligne–Lusztig Theory. In short, we will attempt to understand the $\mathbf{C}$-representation theory of finite groups of Lie types; e.g. $\mathrm{GL}_3(\mathbf{F}_8), \mathrm{SP}_8(\mathbf{F}_{27}), \mathrm{SO}_5(\mathbf{F}_3)$. Goal: "construct" all the representations of these groups.

Example 1.1. *Consider* $\mathrm{SL}_2(\mathbf{F}_p)$ *for* $p > 2$ *(or* $\mathbf{F}_q$ *for* $q = p^r$*). Inside this group we have* $B(\mathbf{F}_p)$*, the* Borel *subgroup of upper triangular groups, and inside* B *we have* $T(\mathbf{F}_p)$*, the (abelian) subgroup of diagonal matrices. Given a character* $\theta : T \to \mathbf{C}^\times$*, we consider it as a representation of* $B(\mathbf{F}_p)$ *via the quotient* $B(\mathbf{F}_p) \to T(\mathbf{F}_p)$ *(mod out by the normal subgroup that is upper triangular matrices with 1's on the diagonal), and then* $Ind_{B(\mathbf{F}_p)}^{\mathrm{SL}_2(\mathbf{F}_p)}\theta$ *is a* SL_2*-representation.*

If θ *is the trivial representation 1, then* $Ind_{B(\mathbf{F}_p)}^{\mathrm{SL}_2(\mathbf{F}_p)}\theta$ *is just the functions from* $\mathbf{P}^1(\mathbf{F}_p) \cong \mathrm{SL}_2(\mathbf{F}_p)/B(\mathbf{F}_p)$ *to* $\mathbf{C}$*.*

There is a short exact sequence

$$0 \to \mathbf{C} \xrightarrow{cst} Ind_{B(\mathbf{F}_p)}^{\mathrm{SL}_2(\mathbf{F}_p)} 1 \to st \to 1,$$

where st *is the* Steinberg *representation. Exercise: prove it is irreducible.*

Next time: If $\theta^2 \neq 1$, then $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{SL}_2(\mathbf{F}_p)}\theta \cong \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{SL}_2(\mathbf{F}_p)}\theta^{-1}$.

Fact: If $p > 2$ and $q = p^r$, then we get $\frac{q+5}{2}$ irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$ from inducing up from B.

Exercise: $\mathrm{SL}_2(\mathbf{F}_q)$ has $q + 4$ conjugacy classes. In lieu of doing the computation (using Jordan canonical forms and taking care that we only conjugate by elements in SL_2, not GL_2), we just give the following classification:

Representative	# of Elements	# of Classes
$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	1
$-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	1	1
$J_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\frac{q^2-1}{2}$	1
$J_2 = \begin{bmatrix} 1 & \epsilon \\ 0 & 1 \end{bmatrix}$	$\frac{q^2-1}{2}$	1
$J_3 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$	$\frac{q^2-1}{2}$	1
$J_4 = \begin{bmatrix} -1 & \epsilon \\ 0 & -1 \end{bmatrix}$	$\frac{q^2-1}{2}$	1
$c_x = \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}, x \neq \pm 1$	$q^2 + q$	$\frac{q-3}{2}$
$d_{x,y} = \begin{bmatrix} x & y \\ \epsilon y & x \end{bmatrix}, x \neq \pm 1, x^2 - \epsilon y^2 = 1$	$q^2 - q$	$\frac{q-1}{2}$

,

where ϵ is a generator of $\mathbf{F}_q^\times$. Note that c_x and $c_{x^{-1}}$ determine the same conjugacy class (hence exactly $\frac{q-3}{2}$ classes labeled c), and so do $d_{x,y}$ and $d_{x,-y}$. We can identify $d_{x,y}$ with $\zeta = x + y\sqrt{\epsilon}$ in the cyclic subgroup $\{\zeta \in (\mathbf{F}_{q^2})^\times : \zeta^{q+1} = 1\}$, where we skip over the elements ± 1, and ζ, ζ^{-1} determine the same class (hence exactly $\frac{q+1-2}{2} = \frac{q-3}{2}$ classes labeled d). Moreover, $|\mathrm{SL}_2(\mathbf{F}_q)| = q(q-1)(q+1)$, so the above list accounts for all elements. There are indeed a total of $6 + \frac{q-3}{2} + \frac{q-1}{2} = q + 4$ conjugacy classes.

Hence the other half of the representations must come from a different construction. MacDonald's conjecture says that these are related to characters to $T'(\mathbf{F}_q) \subseteq SL_2(\mathbf{F}_q)$. Idea: take another maximal torus $\mathbf{F}_{q^2}^\times \subseteq \mathrm{GL}_2(\mathbf{F}_q)$ (think of $\mathbf{F}_{q^2}$ as a 2-dimensional vector space, and its elements acting on this space), and $\mathbf{F}_{q^2}^\times \cap \mathrm{SL}_2 =: \mu_{q+1}$. Want to induct up. The issue is that there is no analogue of $B(\mathbf{F}_q)$ containing $T'(\mathbf{F}_q)$.

Drinfeld: look at the curve C: $xy^q - yx^1 = 1$ inside $\mathbf{A}^2_{\mathbf{F}_q}$. This has commuting actions of $\mathrm{SL}_2(\mathbf{F}_q)$ and μ_{q+1}, given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot (x, y) = (ax + by, cx + dy)$ and $\zeta \cdot (x, y) = (\zeta x, \zeta y)$. Then for $\theta : \mu_{q+1} \to \mathbf{C}^\times \cong \overline{\mathbf{Q}_\ell}$, look at $H^1_{et}(C_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}_\ell})[\theta]$. Drinfeld says these should explain the rest of the irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$.

Remark: C is a μ_{q+1}-cover of $\mathbf{P}^1_{\mathbf{F}_q} - \mathbf{P}^1(\mathbf{F}_q)$, see [1, Sec. 2.2.3]

In the first three weeks of this course we will prove this result, mostly following [1].

1.1 Representation Theory Preliminaries

Definition 1.2. *Let G be a finite group, and k a field. Then a k-linear* representation *of G is a pair (V, π) where V is a finite dimensional k-vector space, and $\pi : G \times V \to V$ is an action of G that is k-linear. Can also think of it as a homomorphism $\rho : G \to \mathrm{GL}(V)$.*

A *morphism* of representations (V, π) and (V', π') is a map $f : V \to V'$ such that

$$\begin{tikzcd} G \times V \arrow[r, "1\times f"] \arrow[d, "\pi"] & G \times V' \arrow[d, "\pi'"] \\ V \arrow[r, "f"] & V' \end{tikzcd}$$

We will write $Rep_k(G)$ for the category of k-linear representations of G.

Fact: This category is abelian, and the forgetful functor $Rep_k(G) \to Vect_k$ commutes with limits and colimits. In other words, the categorical kernels and quotients are just given by the kernels and quotients of the maps on vector spaces, equipped with the natural action of G.

Fact: there is a tensor product of representations, which is the usual thing on underlying vector spaces.

Theorem 1.3 (Maschke)**.** *If $|G|$ is invertible in k, then $Rep_k(G)$ is semisimple; i.e. all representations are a direct sum of irreducible representations.*

Definition 1.4. *Given (V, π, ρ) a representation, there is a function $\chi_V = \chi_\pi = \chi_\rho : G \to k$, the* character*, which is given by $g \mapsto tr(\rho(g))$.*

Observe that χ_V is conjugation-invariant (as trace is).

Theorem 1.5 (Schur Orthogonality)**.** *If $|G| \in k^\times$, and V, V' are representations, then $\langle \chi_V, \chi_{V'} \rangle \coloneqq \frac{1}{|G|} \sum_{g\in G} \chi_V(g)\chi_{V'}(g^{-1}) = \dim Hom_G(V, V')$ holds* in k *(consider the right hand side inside k).*

Note that in $\mathbf{C}$, this is $\frac{1}{|G|} \sum_{g\in G} \chi_V(g)\overline{\chi_{V'}(g)}$ (complex eigenvalues lie on the unit circle, where inverse equals conjugate).

Proof. Sketch: The LHS is

$$\frac{1}{|G|} \sum_{g\in G} \chi_V(g)\chi_{V'}(g^{-1}) = \frac{1}{|G|} \sum_{g\in G} \mathrm{tr}(g|\mathrm{Hom}(V, V')),$$

(the right hand side means the trace of g acting on the representation $\mathrm{Hom}(V, V')$), and for any representation W, it is not hard to see that $\frac{1}{|G|} \sum_{g\in G} \mathrm{tr}(g|W) = \dim W^G$, the space of G-fixed elements of W. □

Fact: If $|G|$ is invertible in k and k algebraically closed, consider the k-vector space of conjugation-invariant maps $G \to k$. Then the inclusion of the set of irreducible characters into this space is an equality.

2 Lecture 2: 1/11/23

Today: Mackey theory and $\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)}\theta$.

Fix a finite group G and an algebraically closed field k such that $|G| \in k^\times$ (can take $k = \mathbf{C}$, but for now we can work with general $k = \overline{k}$).

If $H \subseteq G$ is a subgroup, there is a functor $Rep_k(G) \to Rep_k(H)$ given by restriction. It has both a left and right adjoint (the same), which we will describe. It is the *induced representation* $\mathrm{Ind}_H^G : Rep_k(H) \to Rep_k(G)$, given by $V \mapsto \{f : G \to V : f(hg) = \rho_V(h)f(g)\ \forall h \in H, g \in G\}$. The adjointness was proven by Frobenius (Frobenius reciprocity), in terms of characters.

Remark 2.1. *Dimension of $Ind_H^G V$ is $\dim V \cdot [G : H]$.*

Notation: For a representation V of H we will use χ_V^G to denote the character of $Ind_H^G V$. For a representation W of G we will use $(\chi_W)_H$ for the character of the restriction of W to H.

Theorem 2.2 (Frobenius reciprocity). *$\langle \chi_V^G, \chi_V^G \rangle_G = \langle \chi_V, (\chi_V^G)_H \rangle_H$. More generally, for χ a character of G and θ a character of H, we have $\langle \theta^G, \chi \rangle_G = \langle \theta, \chi_H \rangle_H$ (χ_H being the restriction of χ to H).*

Recall that $\mathrm{Ind}_H^G V$ is irreducible if and only if $\langle \chi_V^G, \chi_V^G \rangle_G = 1$.

Definition 2.3. *For $g \in G$, write H^g for gHg^{-1}. For $\rho : H \to \mathrm{GL}(V)$, write $\rho^g = gHg^{-1} \to GL(V)$ given by $ghg^{-1} \mapsto \rho(h)$.*

Then:

Theorem 2.4 (Mackey decomposition). *$Res_H^G Ind_H^G \rho = \bigoplus_{[g] \in H\backslash G/H} Ind_{H\cap H^g}^H Res_{H\cap H^g}^{H^g} \rho^g$.*

Corollary 2.5. *$Ind_H^G V$ is irreducible if and only if V is irreducible (with representation ρ and character χ) and $Res_{H\cap H^g}^{H^g} \rho^g$ and $Res_{H\cap H^g}^{H^g} \rho$ share no common irreducible factors (other than when $g = 1$).*

Proof. Using Frobenius reciprocity, we have

$$\begin{aligned}
\langle \chi_V^G, \chi_V^G \rangle_G &= \langle \chi_V, (\chi_V^G)_H \rangle_H \\
&= \Big\langle \chi_V, \sum_{[g] \in H\backslash G/H} \chi_{\mathrm{Ind}_{H\cap H^g}^H \mathrm{Res}_{H\cap H^g}^{H^g} \rho^g} \Big\rangle \\
&= \sum_{[g] \in H\backslash G/H} \langle \chi_V, \chi_{\mathrm{Ind}_{H\cap H^g}^H \mathrm{Res}_{H\cap H^g}^{H^g} \rho^g} \rangle \\
&= \sum_{[g] \in H\backslash G/H} \langle \mathrm{Res}_{H\cap H^g}^{H^g} \chi, \mathrm{Res}_{H\cap H^g}^{H^g} \chi^g \rangle.
\end{aligned}$$

We want to know when this is 1. It is certainly at least 1 (coming from when $g = 1$), so is equal to 1 precisely when the summands for $[g] \neq 1$ vanish. $\square$

Let's apply this to $G = \mathrm{SL}_2(\mathbf{F}_q)$, $H = B(\mathbf{F}_q)$. Let $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}_2(\mathbf{F}_q)$, and the conjugation of a generic matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{F}_q)$ is $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$. So conjugation by S preserves $T(\mathbf{F}_q)$ (diagonal matrices), and acts as -1 on it (it swaps a and d, which must be inverses as we're in $\mathrm{SL}_2(\mathbf{F}_q)$). Also, this conjugation makes an upper triangular matrix lower triangular, and vice versa. Hence $B(\mathbf{F}_q) \cap SB(\mathbf{F}_q)S^{-1} = T(\mathbf{F}_q)$.

Lemma 2.6 (Bruhat decomposition)**.** $\mathrm{SL}_2(\mathbf{F}_q) = B(\mathbf{F}_q) \cup B(\mathbf{F}_q)SB(\mathbf{F}_q)$.

Proof. Suppose $g \coloneqq \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{F}_q)$ is not in $B(\mathbf{F}_q)$. Then $c \neq 0$, and one checks that $S^{-1} \begin{bmatrix} 1 & -\frac{a}{c} \\ 0 & 1 \end{bmatrix} g \in B(\mathbf{F}_q)$. Rearrange terms to get the answer. $\square$

The upshot is that if we start with $\theta_1, \theta_2 : T(\mathbf{F}_q) \to \mathbf{C}^\times$, and consider them as representations of the quotient map $B(\mathbf{F}_q) \to T(\mathbf{F}_q)$ (see Lecture 1), then we have

$$\langle \mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)} \theta_1, \mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)} \theta_2 \rangle_{\mathrm{SL}_2} = \langle \theta_1, \theta_2 \rangle_T + \langle \theta_1, \theta_2^{-1} \rangle_T.$$

Proof. By Frobenius reciprocity and Mackey's formula, the LHS is equal to

$$\langle \theta_1, \mathrm{Res}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)} \mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)} \theta_2 \rangle_B = \langle \theta_1, \theta_2 \rangle_T + \langle \theta_1, \theta_2^S \rangle_T = \langle \theta_1, \theta_2 \rangle_T + \langle \theta_1, \theta_2^{-1} \rangle_T.$$

In particular, note that the double cosets of B in SL_2 are represented by either 1 or S, so that $B \cap B^g$ (in Mackey's formula) for $g \in \{1, S\}$ is either $T(\mathbf{F}_q)$ or B. $\square$

Corollary: if $\theta_1 = \theta_2 = \theta$, we find that $\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)} \theta$ is irreducible if $\theta \neq \theta^{-1}$. If θ_1 is in $\{\theta_2, \theta_2^{-1}\}$, then $\mathrm{Ind}\theta_1, \mathrm{Ind}\theta_2$ share no common factors.

So now, if $p > 2$ and q is a power of p, then there are $q - 3$ characters θ with $\theta \neq \theta^{-1}$, so $\frac{q-3}{2}$ irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$. We have $\mathrm{Ind}1 = 1 + st$ (the "Steinberg representation"), and Ind_α (for $\alpha \neq 1$, $\alpha^2 = 1$) splits into irreducibles as $R(\alpha)_+ + R(\alpha)_-$. Fact: $R(\alpha)_+ \ncong R(\alpha)_-$, but they have the same dimension. This gives $\frac{q-3}{2} + 4 = \frac{q+5}{2}$ irreducible representations.

Definition 2.7. *A representation of* $\mathrm{SL}_2(\mathbf{F}_q)$ *that does* not *contain any of the previous* $\frac{q+5}{2}$ *irreducible representations as a summand is called* cuspidal*.*

Exercise (if you know what the words mean): Consider the natural quotient map $\mathrm{SL}_2(\mathbf{Z}_p) \to \mathrm{SL}_2(\mathbf{F}_p)$ inside $\mathrm{SL}_2(\mathbf{Q}_p)$. Let $\mathrm{SL}_2(\mathbf{Z}_p)$ act on V via a cuspidal representation. Then $c\text{−}\mathrm{Ind}_{\mathrm{SL}_2(\mathbf{Z}_p)}^{\mathrm{SL}_2(\mathbf{Q}_p)} V$ is cuspidal.

3 Lecture 3: 1/13/23

Today: introduction to ℓ-adic etale cohomology.

Let X be a smooth projective variety over $\mathbf{F}_p$. Define

$$Z_X := \exp\left(\sum_{n\geq 1}|X(\mathbf{F}_{q^n})|\frac{T^n}{n}\right) \in \mathbf{Q}[[T]].$$

Example: if $X = \mathrm{Spec}(\mathbf{F}_q)$, then $Z_X = \exp\left(\sum_{n\geq 1}\frac{T^n}{n}\right) = \exp(-\log(1-T)) = \frac{1}{1-T}$ (ignoring issues like exp of log is 1, but this actually works).

Another example: if $X = \mathbf{P}^1_{\mathbf{F}_q}$, then $Z_X = \frac{1}{(1-T)(1-qT)}$.

Another example: if X is an elliptic curve, there are $\alpha, \beta \in \overline{\mathbf{Q}}$ such that $Z_X = \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-qT)}$.

One of the Weil conjectures: Z_X is always a rational function (proved by Dwork in the 1960s).

Proof. Idea: we are counting fixed points of Frob^n on $X_{\overline{\mathbf{F}_q}}$. Here is a useful fact (Lefschetz fixed-point theorem): if M is a compact oriented manifold, and $\psi : M \to M$ continuous with isolated fixed points, then the number of fixed points of ψ is $\sum_i (-1)^i \mathrm{tr}(\psi_*, H^i_{sing}(M, \mathbf{R}))$. Exponential generating function of $\mathrm{Fix}(\psi^m)$ is nice. □

The following linear algebra lemma was stated orally in class, but never written down.

Lemma 3.1. *Let V be a finite dimensional vector space over a field K and let $f : V \to V$ be an endomorphism. Then*

$$\exp\left(\sum_{n\geq 1}\mathrm{Tr}(f^n, V)\frac{T^n}{n}\right) \in K[[T]] \tag{1}$$

is given by the characteristic polynomial of f.

Proof. exercise □

Question: is there an "algebraic definition" of singular cohomology of nice X over $\mathbf{C}$? We know $H^0_{sing}(X(\mathbf{C}), \mathbf{Q})$ is $\mathbf{Q}$[connected components] (vector space with basis corresponding to the connected components). Also, $H^1_{sing}(X(\mathbf{C}), \mathbf{Z}) = \pi_1(X(\mathbf{C}))_{ab}$, and $C^\times$ has a $\mathbf{Z}$-cover given by the exponential map, which is not algebraic. However, Riemann existence theorem proves that all *finite* covering spaces are algebraic, so that $H^1_{sing}(X(\mathbf{C}), \mathbf{Z}/n\mathbf{Z})$ "has an algebraic definition".

Serre has a simple argument that shows that there cannot exist a cohomology theory for nice varieties over $\overline{\mathbf{F}}_p$ in $\mathbf{Q}$-vector spaces such that H^1(Elliptic curve) is 2-dimensional. The issue is that there is an elliptic curve E over $\overline{\mathbf{F}}_p$ such that $\hom(E, E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a quaternion division algebra over $\mathbb{Q}$, and such algebras cannot act nontrivially on a two-dimensional $\mathbb{Q}$-vector space. However one can show that the n-th power map $[n] \in \hom(E, E)$ has to induce multiplication by $[n]$ on cohomology.

So we could hope to define a cohomology theory with values in $\mathbf{Z}/\ell^n\mathbf{Z}$ for $\ell \neq p$. Should hope to get a theory with $\varprojlim \mathbf{Z}/\ell^n\mathbf{Z} = \mathbf{Z}_\ell$ or in $\mathbf{Z}_\ell[1/\ell] = \mathbf{Q}_\ell$. This is possible (Grothendieck-Deligne-Artin).

Facts: there is a contravariant functor (for all $\ell \geq 0$, $\ell \neq p$) $H^i_{et}(-, \mathbf{Q}_\ell)$ from the category of nice varieties over $\overline{\mathbf{F}}_p$ to the category of finite-dimensional $\mathbf{Q}_\ell$ vector spaces. Here are its properties:

1. It is 0 unless $0 \leq i \leq 2\dim(X)$ (since turning a complex manifold into a real manifold doubles the dimension).

2. $H^0_{et}(X, \mathbf{Q}_\ell)$ is just the vector space over $\mathbf{Q}_\ell$ with basis corresponding to the connected components.

3. If X lifts to $\widetilde{X}$ over $\mathbf{C}$, then
$$H^i_{sing}(\widetilde{X}(\mathbf{C}), \mathbf{Q}_\ell) = H^i_{sing}(\widetilde{X}(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \cong H^i_{et}(X, \mathbf{Q}_\ell).$$

4. Poincare duality: assuming X equidimensional of dimension d, then $H^i_{et}(X) \cong H^{2d-i}_{et}(X)^\vee$.

5. If X is defined over $\mathbf{F}_q$, $|X(\mathbf{F}_{q^m})| = \sum_i (-1)^i \mathrm{tr}(\mathrm{Frob}_{q^m}, H^i_{et}(X_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell))$.

6. If $\psi : X \to X$ has isolated fixed points, then $|\mathrm{Fix}(\psi)| = \sum_i (-1)^i \mathrm{tr}(\psi_*, H^i_{et}(M, \mathbf{Q}_\ell))$.

There is an extension: a contravariant functor $H^i_c(\_, \mathbf{Q}_\ell)$ ("c" stands for "compact") from the category of (all) varieties over $\overline{\mathbf{F}}_p$ with proper maps, to the category of finite-dimensional $\mathbf{Q}_\ell$ vector spaces, such that:

1. $H^i_c(X, \mathbf{Q}_\ell) = H^i(X, \mathbf{Q}_\ell)$ if X is proper/projective.

2. Vanishes unless $i \in [0, 2\dim(X)]$.

3. If X is smooth and affine, then (Artin vanishing) $H^i_c(X, \mathbf{Q}_\ell) = 0$ for $0 \leq i \leq \dim(X)$.

4. if $Z \subseteq X$ is closed and $U = X - Z$, then there is a long exact sequence
$$\ldots \to H^i_c(U, \mathbf{Q}_\ell) \xrightarrow{\text{extension by 0}} H^i_c(X, \mathbf{Q}_\ell) \xrightarrow{\text{restriction}} H^i_c(Z, \mathbf{Q}_\ell) \to H^{i+1}_c(U, \mathbf{Q}_\ell) \to \ldots$$

5. Same as previous (5).

6. Same as previous (6).

Let C be the Drinfeld curve $(xy^q - x^q y = 1)$ over F_q equipped with actions of $\mathrm{SL}_2(\mathbf{F}_q)$ and μ_{q+1}. Let θ be a character of μ_{q+1} with values in $\mathbf{Q}_\ell$. Then:

Definition 3.2 (Deligne-Lusztig induction)**.** *Write $R(\theta)$ for $H^0_c(C_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell)[\theta] - H^1_c(C_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell)[\theta] + H^2_c(C_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell)[\theta]$, where $[\theta]$ denotes $Hom_{\mu_{q+1}}(\theta, \_)$.*

No lecture Monday 1/16/23 (MLK day).

4 Lecture 4: 1/18/23

Recall the *Drinfeld curve C* given by $xy^q - x^q y = 1$ inside $\mathbf{A}^2_{\mathbf{F}_q}$, where $q = p^r$. This has commuting actions of $\mathrm{SL}_2(\mathbf{F}_q)$ and μ_{q+1}, given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot (x, y) = (ax + by, cx + dy)$ and $\zeta \cdot (x, y) = (\zeta x, \zeta y)$ (see Lecture 1).

Observation: $C(\mathbf{F}_q) = \varnothing$, since $x = x^q$ and $y = y^q$.

We define, for a character $\theta_{q+1} \to \overline{\mathbf{Q}_l}^\times$ (or $\mathbf{C}^\times$), the *virtual representation* $R'(\theta) := H^2_c(C_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}_l})[\theta] - H^1_c(C_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}_l})[\theta]$. Here, for $W \in \mathrm{Rep}(\mu_{q+1})$, we write $W[\theta] = \{w \in W : \zeta w = \theta(\zeta) w\}$.

We start by computing $R'(1)$. We have

$$H^i_c(C_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}_l}) = H^i_c((C_{\overline{\mathbf{F}_q}}/\mu_{q+1}, \overline{\mathbf{Q}_l})$$

as $\mathrm{SL}_2(\mathbf{F}_q)$-representations (using that the actions commute).

Lemma 4.1. *The map $C \to \mathbf{P}^1_{\overline{\mathbf{F}_q}} - \mathbf{P}^1(\mathbf{F}_q)$ realizes the target as the quotient C/μ_{q+1}.*

Proof. We need that:

1. $[\zeta x, \zeta y] = [x, y]$ in $\mathbf{P}^1_{\mathbf{F}_q}$ (clear).
2. The map is surjective on $\overline{\mathbf{F}_q}$-points.
3. If $(\lambda, \lambda x)$ and $(\lambda', \lambda' x)$ are points of $C_{\overline{\mathbf{F}_q}}$, then $\lambda = \zeta \lambda'$ for some $\zeta \in \mu_{q+1}$.

For (2), note that given $[1, x] \in \mathbf{P}^1_{\overline{\mathbf{F}_q}} - \mathbf{P}^1(\mathbf{F}_q)$, then $x^q \neq x$. We want to find $\lambda \in \overline{\mathbf{F}_q}^\times$ such that with $y = \lambda$, we have $[\lambda, \lambda x]$ implying $\lambda^{q+1} x^q - \lambda^{q+1} x = 1$. We can solve this using linear algebra ($q + 1$'st roots exist in $\overline{\mathbf{F}_q}^\times$).

For (3), note that $\lambda^{q+1} x^q - \lambda^{q+1} x = (\lambda')^{q+1} x^q - (\lambda')^{q+1} x$, so that $\lambda^{q+1} = (\lambda')^{q+1}$.

So we conclude that $C(\overline{\mathbf{F}_q})/\mu_{q+1} = (\mathbf{P}^1_{\overline{\mathbf{F}_q}} - \mathbf{P}^1(\mathbf{F}_q))(\overline{\mathbf{F}_q})$. □

Now, we compute $H^1_c(U_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}_l})$ (with $U = \mathbf{P}^1_{\overline{\mathbf{F}_q}} - \mathbf{P}^1(\mathbf{F}_q)$) via the long exact sequence (see Lecture 3) for $X = \mathbf{P}^1_{\overline{\mathbf{F}_q}}$, $Z = \mathbf{P}^1(\mathbf{F}_q)$, $U = X - Z$. Note that $H^0_c(Z_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}_l}) = \mathrm{Fun}(\mathbf{P}^1(\mathbf{F}_q), \overline{\mathbf{Q}_l}) = 1 \oplus st$.

Another fact: $H^i_c(\mathbf{P}^1_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}_l}) = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \\ 1 & i = 2 \end{cases}$ as $\mathrm{SL}_2(\mathbf{F}_q)$-representations.

Going back to the long exact sequence, we get

$$0 \to H^0_c(\mathbf{P}^1) \to H^0_c(Z) \to H^1_c(U) \to H^1_c(\mathbf{P}^1) \to 0 \to H^2_c(\mathbf{P}^1) \xrightarrow{\simeq} H^2_c(U) \to 0.$$

Note that $H^0_c(\mathbf{P}^1)$ is 1, $H^0_c(Z)$ is $1 + st$, and the map is injective and SL_2-equivariant. Hence we conclude it maps 1 to 1. The upshot is that $R(1) = 1 - st$ and $R'(1) = st - 1$.

Observation: for $\zeta \in \mu_{q+1}, \zeta \neq 1$, we have $C^{\zeta}_{\overline{\mathbf{F}_q}} = \varnothing$ (clear).

The trace formula tells us that

$$\mathrm{tr}(\zeta, H^2_c(C)) - \mathrm{tr}(\zeta, H^1_c(C)) = 0.$$

This also characterises the regular representation of μ_{q+1}. So the character of the (virtual) representation $H^1_c(C) - H^2_c(C)$ is a multiple of the regular representation of μ_{q+1}. On applying $[\theta]$, for $\theta \neq 1$, we get an actual character. Upshot: the "degree" of $H^1_c(C)[\theta]$ equals the "degree" of $H^1_c(C)[1] - H^2_c(C)[1]$. So $H^1_c(C)[\theta]$ has dimension $q - 1$.

Theorem 4.2. *If $\theta \neq 1$, then $H^1_c(C_{\overline{\mathbf{F}_q}}, \overline{\mathbf{Q}_l})[\theta]$ is cuspidal (recall Definition 2.7).*

Proof. Let U be the subgroup of matrices of the form $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ inside $\mathrm{SL}_2(\mathbf{F}_q)$. Consider the functors

$$Rep_{\overline{\mathbf{Q}_l}} T \to Rep_{\overline{\mathbf{Q}_l}} B \xrightarrow{\text{induce}} Rep_{\overline{\mathbf{Q}_l}} \mathrm{SL}_2(\mathbf{F}_q).$$

To go backwards we restrict to B, and take U-coinvariants (quotient by submodule generated by all $x - ux$ for $u \in U$).

For us, it suffices to show that $(H^1_c(C)[\theta])_U = (H^1_c(C)[\theta])^U = 0$. So we need to understand $H^1_c(C_{\overline{\mathbf{F}_q}}/U, \overline{\mathbf{Q}_l})$ with its action of μ_{q+1}. What is the quotient by U? Looking back at the action of $\mathrm{SL}_2(\mathbf{F}_q)$ on C, a good guess for the quotient is $C \to \mathbf{A}^1 - \{0\}$ given by $(x, y) \mapsto y$ (which is actually correct, as we will show next time). $\square$

5 Lecture 5: 1/20/23

Recall the Drinfeld curve C

$$\{xy^q - yx^q = 1\} \subset \mathbf{A}^2_{\mathbf{F}_q}$$

equipped with actions of $\mathrm{SL}_2(\mathbf{F}_q)$ and μ_{q+1}. Last time we proved that $H^2_C(C, \overline{\mathbf{Q}_l}) = 1$ as a representation of $\mathrm{SL}_2(\mathbf{F}_q) \times \mu_{q+1}$. For $\theta : \mu_{q+1} \to \overline{\mathbf{Q}_l}^\times$ non-trivial, we have $H^1_C(C)[\theta]$ is a $q-1$ dimensional representation on $\mathrm{SL}_2(\mathbf{F}_q)$. Our goal was to show this representation is cuspital. It turns out that this is equivalent to proving that

$$(H^1_C(C)[\theta])_U = (H^1_C(C)[\theta])^U = (H^1_C(C))^U[\theta] = 0$$

So now we compute $H^1_C(C/U)$ as a representation of μ_{q+1}.

Lemma 5.1. *The map $C \to \mathbf{A}^2 - \{0\}$ induces an isomorphism $C/U \xrightarrow{\sim} \mathbf{A}^2 - \{0\}$.*

Proof. The map is U-invariant since U acts as $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}(x, y) = (x + by, y)$. For surjectivity, given $y \neq 0$ we can always solve the equation

$$xy^q - yx^q - 1 = 0$$

in $\overline{\mathbf{F}_q}$ since it's algebraically closed. The final step is to show that any two solutions (x_1, y) and (x_2, y) are related by the action by U, i.e. we want to find $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ such that $x_2 = x_1 + by$. We see that b must be $\frac{x_2 - x_1}{y} \in \overline{\mathbf{F}_q}$, and we need to show it's in $\mathbf{F}_q$. Subtracting the equations and dividing by y^{q+1}, we get

$$\left(\frac{x_2}{y}\right)^q - \frac{x_2}{y} = \left(\frac{x_1}{y}\right)^q - \frac{x_1}{y}$$

so indeed

$$\left(\frac{x_2 - x_1}{y}\right)^q = \frac{x_2 - x_1}{y}.$$

This shows $C/U \xrightarrow{\sim} \mathbf{A}^2 - \{0\}$ is an isomorphism. □

Recall that $H^i_C(\mathbf{A}^n) = 1$ if $i = 2n$ and 0 otherwise, and the map $\{0\} \hookrightarrow \mathbf{A}^1$ is μ_{q+1}-equivariant. Thus by considering the long exact sequence (of the pair $\{0\} \subset \mathbf{A}^1$), $H^1(\mathbf{A}^1 - \{0\})$ is the trivial representation of μ_{q+1}.

Corollary 5.2. *If $\theta \neq 1$, then $H^1_C(C)[\theta]$ is cuspidal.*

As this point, we have already found $\frac{q+5}{2}$ irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$ (there are $q+4$ in total, assuming $p>2$). We are missing $\frac{q+3}{2}$ of them. Notice that $\frac{q+3}{2}=\frac{q-1}{2}+2$. We might hope that the 2 is contributed by the unique representation of order 2 of μ_{q+1}, and the remaining $q-1$ nontrivial representations of μ_{q+1} ones pair up. [The trivial representation of μ_{q+1} does not produce any new characters, as we have seen.]

Observe that the map $F: C\to C$ given by $(x,y)\mapsto(x^q,y^q)$ is $\mathrm{SL}_2(\mathbf{F}_q)$-equivariant, but it is not μ_{q+1}-equivariant because $F(\zeta x,\zeta y)=\zeta^q F(x,y)$, and in μ_{q+1} we know $\zeta^q=\zeta^{-1}$. So F induces a $\mathrm{SL}_2(\mathbf{F}_q)$-equivariant isomorphism $H^1_C(C)\to H^1_C(C)$ taking $H^2_C(C)[\theta]$ isomorphically to $H^2_C(C)[\theta^{-1}]$. (It is a fact that a Frobenius map induces isomorphism on etale cohomology.) The situation is now analogous to the induction procedure we went through with $B(\mathbf{F}_q)$ and $T(\mathbf{F}_q)$:

Theorem 5.3 (Geometric Mackey Formula)**.** *Let θ_1,θ_2 be non-trivial representations. Then*

$$\langle H^1_C(C)[\theta_1], H^1_C(C)[\theta_2]\rangle_{\mathrm{SL}_2(\mathbf{F}_q)}=\langle\theta_1,\theta_2\rangle_{\mu_{q+1}}+\langle\theta_1,\theta_2^{-1}\rangle_{\mu_{q+1}}$$

Fact: $H^1_C(C)[\theta]^\vee=H^1_C(C)[\theta]=H^1_C(C)[\theta^{-1}]$ where the dual is as $\mathrm{SL}_2(\mathbf{F}_q)$-vector spaces.
We compute

$$\begin{aligned}\langle R'(\theta_1),R'(\theta_2)\rangle&=\langle 1,R'(\theta_1)\otimes R'(\theta_2)\rangle\\&=\dim(H^1_C(C)[\theta_1]\otimes H^1_C(C)[\theta_2])^{\mathrm{SL}_2(\mathbf{F}_q)}.\end{aligned}$$

The idea then is to use the Kunneth formula. From now on we write $H^*_C(X)$ for $\sum_{i=1}^\infty(-1)^iH^i_C(X)$. So we want to understand the dimension of the virtual character (of $\mu_{q+1}\times\mu_{q+1}$)

$$H^*_C(C\times C)[\theta_1\times\theta_2]^{\mathrm{SL}_2(\mathbf{F}_q)}.$$

It amounts to understanding the virtual character $H^1_C(\frac{C\times C}{\mathrm{SL}_2(\mathbf{F}_q)})$.

Write $Z=C\times C\subset\mathbf{A}^4_{\mathbf{F}_q}$ (in variables x,y,z,w), and write $Z=Z_0\cup Z_{\neq0}$ where $Z_0=\{xw-yz=0\}$ and $Z_{\neq0}$ is its complement.

Lemma 5.4. *$Z_0\subset C\times C$ is $\mu_{q+1}\times\mu_{q+1}\times\mathrm{SL}_2(\mathbf{F}_q)$ stable.*

Proof. Only the stability under the $\mathrm{SL}_2(\mathbf{F}_q)$ action needs a proof. We have

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}(x,y,z,w)=(ax+by,cx+dy,az+bw,cz+dw),$$

and it is easy to observe that

$$(ax+by)(cz+dy)-(cx+dy)(az+bw)=(bc-ad)(yz-wx).$$

∎

6 Lecture 6: 1/23/23

Recall the Drinfeld curve C with actions of $\mathrm{SL}_2(\mathbf{F}_q)$ and μ_{q+1}. Moreover, we have seen that for $\theta : \mu_{q+1} \to \overline{\mathbf{Q}_l}^\times$ then $H^1_c(C)[\theta]$ is cuspidal of dimension $q-1$. Recall that we also have:

Theorem 6.1 (Mackey formula)**.**

$$\langle H^1_c(C)[\theta_1], H^1_c(C)[\theta_2]\rangle = \langle \theta_1, \theta_2\rangle + \langle \theta_1, \theta_2^{-1}\rangle.$$

From Lecture 5, we also saw that this is closely related to computing $H^*_c\left(C \times C/\,\mathrm{SL}_2(\mathbf{F}_q)\right)$ as a representation of $\mu_{q+1} \times \mu_{q+1}$.

Let's start by computing $C/\,\mathrm{SL}_2(\mathbf{F}_q)$.

Lemma 6.2. *The map $\varphi : C \to \mathbf{A}^1_{\mathbf{F}_q}$ sending (x, y) to $(xy^{q^2} - yx^{q^2})$ identifies $C/\,\mathrm{SL}_2(\mathbf{F}_q) \simeq \mathbf{A}^1_{\mathbf{F}_q}$.*

Proof. First show this map is G-invariant. Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{F}_q)$. We have

$$\begin{aligned}
&(ax + by)(cx^{q^2} + dy^{q^2}) - (cx + dy)(ax^{q^2} + by^{q^2}) \\
&= (adxy^{q^2} + bcyx^{q^2}) - (cbxy^{q^2} + adyx^{q^2}) \\
&= (ad - bc)xy^{q^2} + (bc - ad)yx^{q^2} \\
&= xy^{q^2} - yx^{q^2}.
\end{aligned}$$

We next want to show that the action of $\mathrm{SL}_2(\mathbf{F}_q)$ on C is free. If $g \in \mathrm{SL}_2(\mathbf{F}_q)$ has a fixed point, then it has an eigenvalue equal to 1. Without loss of generality, we can say $g = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$, and then $g \cdot (x, y) = (x + by, dy)$, implying $b = 0, d = 1$, as needed.

Next, we need to show that for any $a \in \mathbf{A}^1(\mathbf{F}_q)$, that $\varphi^{-1}(a)$ contains $|\mathrm{SL}_2(\mathbf{F}_q)| = q(q^2-1)$ elements. Write $(z, t) = (x, y/x)$, then we are solving $(t^{q^2} - t)(z^{q^2+1}) = a$ and $(t^q - 1)(z^{q+1}) = 1$. Use $t^{q^2} - t = (t^q - t)^q + (t^q - t)$ (we're in characteristic p), so that these two equations are satisfied if and only if $(t^q - t)(z^{q+1}) = 1$ and $\frac{1}{z^{q+1}} + \frac{1}{z^{q^2+q}} = \frac{a}{z^{q^2+1}}$. This happens if and only if $t^q - t = \frac{1}{z^{q+1}}$ and $z^{q^2-1} - az^{q+1} + 1 = 0$.

So we are done if these polynomials always have distinct roots (there are $q^2 - 1$ roots for z, and then q roots for t for each z). Can check this by taking the derivative.

Here is a fact that completes the proof: the map φ is etale (hence smooth). Thus $C/\,\mathrm{SL}_2(\mathbf{F}_q) \simeq \mathbf{A}^1$. $\square$

We now turn to discussing $C \times C$. Recall this sits inside $\mathbf{A}^2_{\mathbf{F}_q}$ (with coordinates (x, y, z, t) with diagonal $\mathrm{SL}_2(\mathbf{F}_q)$ action and $\mu_{q+1} \times \mu_{q+1}$-action $(\zeta_1, \zeta_2) \cdot (x, y, z, t) = (\zeta_1 x, \zeta_1 y, \zeta_2 z, \zeta_2 t)$.

In Lecture 5, we introduced $C \times C = Z_{\neq 0} \cup Z_0$ where Z_0 is cut out by $\{xt - yz = 0\}$ (and similarly for $Z_{\neq 0}$). We checked that is stable under all the above actions.

Define now $V \subseteq (\mathbf{A}^1 - \{0\}) \times \mathbf{A}^2$ by $u^{q+1} - ab = 1$ for coordinates (u, a, b).

Lemma 6.3. *The map $Z_{\neq 0} \to V$ given by $(x, y, z, t) \mapsto (xt - zy, xt^q - yz^q, x^q t - y^q z)$ induces an isomorphism $Z_{\neq 0}/G \simeq V$.*

Proof. See [1] (may have some computational errors below). Here is an outline:

1. Check that φ actually lands in V. We have

$$\begin{aligned}(xt^q - yz^q)(x^q t - y^q z) &= x^{q+1}t^{q+1} + y^{q+1}z^{q+1} - xt^q y^q z - yz^q x^q t \\ &= (x^q t^q - y^q z^q)(xt - yz) - xt^q y^q z - yz^q x^q t + x^q t^q yz + y^q z^q xt \\ &= (xt - yz)^{q+1} - (xy^q - yx^q)(zt^q - tz^q)\end{aligned}$$

   It is true that $(xy^q - yx^q)(zt^q - tz^q) = -1$. I am not sure...

2. Check that the map is $\mathrm{SL}_2(\mathbf{F}_q)$-invariant. For the first coordinate, we have

$$(ax + by)(cz + dt) - (cx + dy)(za + dt) = (ad - bc)(xt - yz) = xt - yz.$$

   For the second coordinate, we have

$$(ax + by)(cz^q + dt^q) - (cx + dy)(az^q + dt^q) = (ad - bc)(xt^q - yz^q) = xt^q - yz^q.$$

3. Fix $(u, a, b) = V(\mathbf{F}_q)$. Then $\varphi^{-1}(u, a, b)$ consists of those (x, y, z, t) satisfying the 5 equations:

   (a) $xy^q - yx^q = 1$

   (b) $zt^q - tz^q = 1$

   (c) $xt - yz = u$

   (d) $xt^q - yz^q = a$

   (e) $x^q t - y^q z = b$.

   We rewrite (c) and (e) as

$$\begin{bmatrix} x & -y \\ x^q & -y^q \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} u \\ b \end{bmatrix},$$

   and invert the matrix to get equations for u, b in terms of t, z. In particular, we have $z = ux^q - bx$ and $t = uy^q - by$.

   The upshot is that the system reduces to i $xy^q - yx^q = 1$

ii $z = ux^q - bx$

iii $xy^{q^2} - yx^{q^2} = \frac{a+b^q}{u^q}$

iv $t = uy^q - by$.

From before, there are exactly $(q^2-1)q$ pairs of (x, y) satisfying (i) and (iii), and z and t are determined by u, x, y.

4. Finally, a tangent space computation shows that the map is etale. Conclude as in Lemma 6.2.

□

Moreover, V has an action of $\mu_{q+1} \times \mu_{q+1}$ by

$$(\zeta_1, \zeta_2) \cdot (u, a, b) = (\zeta_1\zeta_2 u, \zeta_1\zeta_2^{-1} a, \zeta_1^{-1}\zeta_2 b),$$

which commutes with the $G_m = \overline{\mathbf{F}_q}^\times$ action given by $\lambda(u, a, b) = (u, \lambda a, \lambda^{-1} b)$.

We end with the following fact (Torus-equivariant localization):

$$H_c^*(V) = H_c^*(V^{G_m})[z].$$

We will discuss this more next time (some disagreement in class).

7 Lecture 7: 1/25/23

Last time we were trying to the geometric Mackey formula for $H^1_C(C)[\theta]$ for $\theta : \mu_{q+1} \to \overline{\mathbf{Q}_l}^\times$. Geometrically, this amounts to understanding

$$H^*_C\left(\frac{C \times C}{\mathrm{SL}_2(\mathbf{F}_q)}\right)$$

as a virtual representation of $\mu_{q+1} \times \mu_{q+1}$. We decomposed $C \times C = Z_0 \cup Z_{\neq 0}$ into $\mathrm{SL}_2(\mathbf{F}_q) \times \mu_{q+1} \times \mu_{q+1}$ stable parts, and showed that

$$Z_{\neq 0}/G \xrightarrow{\sim} V = \{u^{q+1} - ab = 1\} \subset (\mathbf{A}^q - \{0\}) \times \mathbf{A}^2$$

with $(\zeta_1, \zeta_2)(u, a, b) = (\zeta_1\zeta_2 u, \zeta_1\zeta_2^{-1} a, \zeta_1^{-1}\zeta_2 b)$. The question was how to compute $H^*_C(V)$ as a virtual representation of $\mu_{q+1} \times \mu_{q+1}$. This is equivalent to computing the trace $\mathrm{tr}\left((\zeta_1, \zeta_2), H^*_C(V)\right)$ for all $(\zeta_1, \zeta_2) \in \mu_{q+1} \times \mu_{q+1}$. The idea is to use the G_m-action $\lambda(u, a, b) = (u, \lambda^{-1} a, \lambda b)$ and compare to $\mathrm{tr}(-, H^*_C(V^{G_m}))$. What is the $-$ suppose to be? Here are some facts:

1. Since V is affine, there exists $t \in \overline{\mathbf{F}_q}^\times = G_m(\overline{\mathbf{F}_q})$ such that $V^{G_m} = V^t$.

2. Suppose γ is a finite order automorphism of a variety of V, $\gamma = su$ such that u has p-power order, s has prime-to-p-power order, $us = vs$. (not sure what is going on here) Then
$$\mathrm{tr}(\gamma, H^*_C(V)) = \mathrm{tr}(u, H^*_C(V^s))$$

Lemma 7.1. *Let Γ be a finite group. Suppose that $\Gamma \times G_m$ acts on an affine variety V, then $tr(\gamma, H^*_C(V)) = tr(\gamma, H^*_C(V^{G_m}))$. When Γ is the trivial group, this is due to B-B (how to spell?).*

Proof. Choose $t \in G_m(\overline{\mathbf{F}_q})$ such that $v^t = V^{G_m}$. Let $\gamma = su$ as before. Then

$$\mathrm{tr}(\gamma, H^*_C(V^{G_m})) = \mathrm{tr}(\gamma, H^*_C(V^t)) = \mathrm{tr}(u, (V^t)^s) = \mathrm{tr}(u, (V^s)^t)$$

Now using fact 2 again for $\gamma = ut$, this is equal to

$$\mathrm{tr}(ut, V^s) = \mathrm{tr}(s, V^u) = \mathrm{tr}(su, V) = \mathrm{tr}(\gamma, V)$$

□

Let's go back to the previous situation: $V = \{u^{q+1} - ab = 1\} \subset (\mathbf{A}^q - \{0\}) \times \mathbf{A}^2$, G_m acting on V by $\lambda(u, a, b) = (u, \lambda a, \lambda^{-1} b)$. So the fixed points are

$$V^{G_m} = \mu_{q+1} \times \{0\} \times \{0\}$$

with $\mu_{q+1} \times \mu_{q+1}$ acting by $(\zeta_1, \zeta_2)(\zeta) = \zeta_1\zeta_2\zeta$.

To compute $Z_0 \subset C \times C \subset \mathbf{A}^4$ cut out by (some equations), we need

Lemma 7.2. *The map $\phi : \mu_{q+1} \times C \to Z_0$ given by $(\zeta, x, y) \mapsto (x, y, \zeta x, \zeta y)$ is a $\mathrm{SL}_2(\mathbf{F}_q)$-equivariant isomorphism.*

Proof. We can easily see the action lands in Z_0 by checking the defining equations. Given $(x, y) \in C(\overline{\mathbf{F}_q})$, we want to show that there are at most $q+1$ options for (z, t) such that $(x, y, z, t) \in Z_0(\overline{\mathbf{F}_q})$. Such (x, y) must satisfy $t = \frac{yz}{x}$ and z must satisfy $z^{q+1}\left(\frac{y}{x}\right)^q - z^{q+1}\left(\frac{y}{x}\right) = 1$. This shows that ϕ is a bijection on $\overline{\mathbf{F}_q}$-points. □

Corollary 7.3. $Z_0/G \cong \mu_{q+1} \times \mathbf{A}^1$ *with* $\mu_{q+1} \times \mu_{q+1}$ *acting by* $(\zeta_1, \zeta_2)(\zeta, z) = (\zeta_1^{-1}\zeta_2\zeta, \zeta_1^2 z)$.

We can now prove the geometric Mackey formula.

Theorem 7.4 (Mackey)**.** *Let θ_1, θ_2 be non-trivial characters of μ_{q+1}. Then*

$$\langle H^1_C(C)[\theta_1], H^1_C(C)[\theta_2]\rangle_{\mathrm{SL}_2(\mathbf{F}_q)} = \langle \theta_1, \theta_2\rangle_{\mu_{q+1}} + \langle \theta_1, \theta_2^{-1}\rangle_{\mu_{q+1}}.$$

Proof. We have seen that

$$\langle H^1_C(C)[\theta_1], H^1_C(C)[\theta_2]\rangle_{\mathrm{SL}_2(F_q)} = \dim H^*_C((C \times C)^{\mathrm{SL}_2(\mathbf{F}_q)})[\theta_1 \times \theta_2].$$

The right side decomposes as

$$\begin{aligned}
&\dim H^*_C((Z_0)^{\mathrm{SL}_2(\mathbf{F}_q)})[\theta_1 \times \theta_2] + \dim H^*_C((Z_{\neq 0})^{\mathrm{SL}_2(\mathbf{F}_q)})[\theta_1 \times \theta_2] \\
&= \dim H^*_C(Z_0/\mathrm{SL}_2(\mathbf{F}_q))[\theta_1 \times \theta_2] + \dim H^*_C(Z_{\neq 0}/\mathrm{SL}_2(\mathbf{F}_q))[\theta_1 \times \theta_2] \\
&= \dim \mathrm{Ind}_{\mu^1_{q+1}}^{\mu_{q+1} \times \mu_{q+1}}[\theta_1 \times \theta_2]1 + \dim \mathrm{Ind}_{\mu^2_{q+1}}^{\mu_{q+1} \times \mu_{q+1}}[\theta_1 \times \theta_2]1 \\
&= \langle 1, \theta_1 \otimes \theta_2\rangle_{\mu_{q+1}} + \langle 1, \theta_1 \otimes \theta_2^{-1}\rangle_{\mu_{q+1}}.
\end{aligned}$$

□

Corollary 7.5. *1. $H^1_C(C)[\theta]$ is irreducible of dimension $q-1$ if $\theta^2 \neq 1$.*

2. $H^1_C(C)[\theta_0]$ has two irreducible non-trivial summands $?_+$ and $?_-$.

By counting, we have now found all irreducible representations of $\mathrm{SL}_2(\mathbf{F}_q)$ $(p > 2)$. One can compute that both $?_+$ and $?_-$ have dimension $\frac{q-1}{2}$.

Remark 7.6. *One should think of the usual induction $\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{SL}_2(\mathbf{F}_q)}\alpha$ as a special case of DL induction with a 0-dimensional DL variety $H^*_C\left(\frac{\mathrm{SL}_2(\mathbf{F}_q)}{U(\mathbf{F}_q)}\right)[\alpha]$.*

8 Lecture 8: 1/27/23

Today: how to we generalize the Drinfeld curve? A very rough sketch (no proofs today).

For SL_2, there are 2 varieties with a $\mathrm{SL}_2(\mathbf{F}_q)$ action and also an action of $T'(\mathbf{F}_q)$, $T(\mathbf{F}_q)$. Recall that C is:

1. A $\mathrm{SL}_2(\mathbf{F}_q)$-cover of $\mathbf{A}^1$.
2. A $U(\mathbf{F}_q)$-cover of $\mathbf{A}^1 - \{0\}$.
3. $\mu_{q+1} = T'(\mathbf{F}_q)$-cover of $\mathbf{P}^1_{\overline{\mathbf{F}_q}} - \mathbf{P}^1(\mathbf{F}_q)$.

We will seek to generalize (3).

How do we generalize the action of SL_2 on $\mathbf{P}^1$? Say for GL_3, we could take $\mathbf{P}^2$. We know that $\mathbf{P}^2(\overline{\mathbf{F}_q})$ is the set of lines $L \subseteq \overline{\mathbf{F}_q}^{\oplus 3}$ (with an obvious action by $\mathrm{GL}_2(\overline{\mathbf{F}_q})$). But we also have the set of flags:

$$FL(\overline{\mathbf{F}_q}) \coloneqq \{0 \subsetneq L_1 \subsetneq L_2 \subsetneq \overline{\mathbf{F}_q}^{\oplus 3}\}.$$

$\mathrm{GL}_3(\overline{\mathbf{F}_q})$ acts on FL, but since the action extends to GL_3, it is not interesting on cohomology. So our next idea might be to look at $FL - FL(\mathbf{F}_q)$.

Observe that $FL = \mathrm{GL}_3/B$, where B is the subgroup of upper triangular matrices. A matrix in B with columns e_1, e_2, e_3 corresponds to the flag $\{0 \subsetneq e_1 \subsetneq e_1 \oplus e_2 \subsetneq FL(\overline{\mathbf{F}_q})\}$. Hence $FL(\mathbf{F}_q) = \mathrm{GL}_3(\mathbf{F}_q)/B(\mathbf{F}_q)$ and $H^*_c(FL(\mathbf{F}_q)) = \mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_3(\mathbf{F}_q)} 1$.

A good theory of "relative positions of flags" can be done in two ways: either $L_1 \not\subset L_2$ or $L_1' \not\subset L_2'$. So we want a "relative map" $FL(\overline{\mathbf{F}_q}) \times FL(\overline{\mathbf{F}_q})$ to somewhere (TBD). First, we note that there is *no* algebraic map $\mathbf{P}^1(\overline{\mathbf{F}_q}) \times \mathbf{P}^1(\overline{\mathbf{F}_q}) \to \{0, 1\}$.

Better way to see this: use the Bruhat decomposition $\mathrm{GL}_2 = B \cup BSB$. Hence there are two left B-orbits on G/B. Moreover, there are 2 left G-orbits on $\mathbf{P}^1 \times \mathbf{P}^1$. $G - (G/B \times G/B) = B \backslash G/B = 1 \cup S$. This isn't right/missing many details, I got lost around here. In general, for GL_3, we want to look at $\mathrm{GL}_3 \backslash FL \times FL$.

Fact/exercise: Let S_n be the symmetric group, and conflate $\sigma \in S_n$ with the corresponding permutation matrix in GL_n. Let B be the set of upper triangular matrices in GL_n Then $\mathrm{GL}_n = \bigcup_{\sigma \in S_n} B\sigma B$.

Geometrically, this translates into $\mathrm{GL}_n/B \times \mathrm{GL}_n/B = \bigcup_{\sigma \in S_n} O(\sigma)$, where $O(\sigma)$ is the GL_n-orbit of $(1, \sigma)$. We have that the dimension of $O(\sigma)$ is $l(\sigma) + \dim \mathrm{GL}_n/B$, l denoting the length of the permutation σ.

So we get two decompositions of GL_n/B:

1. $\mathrm{GL}_n/B \to \mathrm{GL}_n/B \times \mathrm{GL}_n/B$ via $x \mapsto (x, 1)$ and take inverse image of $O(\sigma) \rightsquigarrow (\mathrm{GL}_n/B)^\sigma$.
2. $\mathrm{GL}_n/B \to \mathrm{GL}_n/B \times \mathrm{GL}_n/B \rightsquigarrow (\mathrm{GL}_n/B)(\sigma)$ via Frobenius and pull back $\mathcal{O}(s)$. This leads to Deligne-Lusztig varieties.

Observe that $X(\omega) = (\mathrm{GL}_n / B)(\omega)$ are $\mathrm{GL}_n(\mathbf{F}_q)$-stable. We are almost "in business", but we need to construct $Y(\omega) \to X(\omega)$, where $Y(\omega)$ has an action by $T_\omega(\mathbf{F}_q) \times \mathrm{GL}_n(\mathbf{F}_q)$.

Facts:

1. All the $X(\omega)$ are smooth of pure dimension $l(\sigma)$ (like the X^σ).
2. Their closures have the same singularities as the closures of the X^σ.
3. $X(\sigma)$ is usually not connected.

No class next week (Pol is gone).

9 Lecture 9: 2/6/23

This week: Algebraic groups. We want to define/motivate "finite groups of Lie type".

Let k be a perfect field (probably algebraically closed in a bit).

Definition 9.1. *A variety is a separated, geometrically reduced over k of finite type.*

Definition 9.2. *A* group scheme *is a quadruple (G,m,e,i) where G is a k-scheme locally of finite type, $m : G \times_k G \to G$, $i : G \to G$, and $e : Spec(k) \to G$ are k-morphisms such that the "group axioms hold".*

This is the same thing as saying that for all k-schemes T, the operations m, e, i make $G(T)$ into a group with inverse operation i_T and ideneity e_T.

Basic group theory tells us that e and i are uniquely determined by m.

Definition 9.3. *G as above is an* algebraic group *if G is a variety over k.*

Here are some examples:

1. $\mathbb{G}_m \coloneqq \mathrm{Spec} k[x, x^{-1}]$, which represents the functor $R \mapsto (R^\times, \cdot)$ for k-algebras R.
2. $\mathbb{G}_a \coloneqq \mathrm{Spec} k[x]$, which which represents the functor $R \mapsto (R^+, +)$.
3. Elliptic curves.
4. If Γ is a finite group, then we can take $G = \coprod_{\gamma \in \Gamma} \mathrm{Spec}(k)$ as a group scheme. It is an algebraic group denoted $\underline{\Gamma}$.
5. If Γ is a finite abelian group of order invertible in k, then $\mathrm{Spec}(k[\Gamma])$ is an algebraic group. Need the order to be invertible so that the scheme is reduced (for instance, $\mathbf{F}_p[\mathbf{Z}/p\mathbf{Z}] = \mathbf{F}_p[x]/(x^p - 1)$ gives μ_p, nonreduced in characteristic p.

6. $G = \mathrm{GL}_n = \mathrm{Spec}(k[X_{ij} : 1 \leq i, j \leq n][1/D])$, where $D = \det(X_{ij})$. This represents the functor $R \mapsto \mathrm{GL}_n(R)$.

We will not be interested in projective (proper) algebraic groups (boring representation theory, because they are commutative). We also have to exclude disconnected algebraic groups, like $\underline{\Gamma}$.

More subtly, the representation theory of $\mathbb{G}_a$ is also not so interesting.

Note: we mean the representation theory of $\mathbb{G}_a(\mathbf{F}_q)$ when $k = \mathbf{F}_q$.

Conrad's addendum: we should really mean k-homomorphisms $G \to \mathrm{GL}_n$ by "representation theory".

Conrad's addendum 2: It turns out any k-homomorphism $G \to \mathrm{GL}_n$ can be conjugated inside $\mathrm{GL}_n(k)$ to land inside U_n (closed subgroup of upper-triangular unipotent matrices), so is rarely "completely reducible". That failure is one reason why the representation theory of $\mathscr{G}_a$ as a k-group is bad.

Fact: all affine algebraic groups can be realized as closed subgroups of GL_n over k. For a finite group this is obvious by Cayley's theorem. But for a group like $\mathrm{PGL}_2 = \mathrm{GL}_2/\mathbb{G}_m$, it is not so obvious how this is done. Proof idea: G "acts" on its own coordinate ring ($G = \mathrm{Spec}(R)$), and this will be an increasing union of G-stable finite-dimensional k-vector spaces. So should get some finite piece.

Such algebraic groups are called "linear algebraic groups".

Warning: non-reduced group schemes appear in nature, and are useful. For example, $Z_{\mathrm{SL}_2} = \mu_2 \subsetneq \mathrm{SL}_2$ in characteristic 2 (Z is the center).

Fact: quotients of an algebraic group G by a closed subgroup $H \subset G$ "exist". They are again groups if H is normal. In other words, $G \to G/H$ is H-invariant and universal for H-invariant maps of schemes. Note that although $G(\overline{k})/H(\overline{k}) = (G/H)(\overline{k})$ if $\overline{k} \supset k$ is algebraically closed, but *not in general true* for all k-points.

Fancy language: $G \to G/H$ is an H-torsor in the etale topology.

Example: $\mathrm{GL}_n/\mathbb{G}_m = \mathrm{PGL}_n = \mathrm{SL}_n/\mu_n$. But it's not true that $\mathrm{SL}_2(\mathbf{Q})/\mu_2(\mathbf{Q}) \neq \mathrm{PGL}_2(\mathbf{Q})$, because the RHS has "determinant in $\mathbf{Q}^*/(\mathbf{Q}^*)^2$", but the LH has "determinant 1". In fact, there is a long exact sequence

$$1 \to \mu_2(\mathbf{Q}) \to \mathrm{SL}_2(\mathbf{Q}) \to \mathrm{PGL}_2(\mathbf{Q}) \to \mathbf{Q}^*/(\mathbf{Q}^*)^2 \to 1.$$

Another example: $\mathrm{GL}_2/B_2 \cong \mathbf{P}^1$, where B_2 is the subgroup of upper triangular matrices in GL_2. But here $\mathrm{GL}_2(\mathbf{Q})/B_2(\mathbf{Q})$ really is $\mathbf{P}^1(\mathbf{Q})$.

An algebraic group G over k has:

1. A maximal quotient G^{ab}, fitting inside a short exact sequence

$$1 \to G^{der} \to G \to G^{ab} \to 1,$$

where G^{der} is the "derived group" of G. For instance,

$$1 \to \mathrm{SL}_n \to \mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_{\gtrdot} \to 1.$$

Conrad's addendum 3: The right way to think of G^{der} is a smooth closed k-subgroup for which $G^{der}(\overline{k})$ is the commutator subgroup of $G(\overline{k})$. For example, SL_n turns out to be its own derived k-subgroup. Same is true for PGL_n, but the commutator subgroup of $\mathrm{PGL}_n(k)$ is typically a proper subgroup (it's the image of $\mathrm{SL}_n(k)$).

2. The connected component $G^0 \subseteq G$ containing e (the "identity" or "neutral" component) is a *normal* subgroup, and the quotient G/G^0 is finite.

3. Given $H \subseteq G$ a closed subgroup, there is a centralizer $C_G(H) \subseteq G$, which represents

$$R \mapsto \left(\bigcap_{R \to R'} C_{G(R)}(H(R')) \right) \subseteq G(R)$$

.

We now turn to finite groups of Lie type. They are motivatived by classical compact connected real lie groups such as $SO_n(\mathbf{R})$, $SU_n(\mathbf{R})$ by loose analogy.

Definition 9.4. *An algebraic group G is called* reductive *if G does not contain a normal subgroup $H \subseteq G$ isomorphic to $\mathbb{G}_a^{\oplus m}$. for all m.*

If we have one, we should take the quotient (doesn't really change the representation theory).

1. $\mathbb{G}_m$ is reductive.

2. But if we take B_2, this is not reductive since $N_2 = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}$ is isomorphic to $\mathbb{G}_a$. Note that B_2/N_2 is the diagonal matrices, isomorphic to $\mathbb{G}_m^2$.

3. $G = \mathrm{GL}_n$ is reductive (very hard exercise).

Fact: when $k = \mathbf{C}$, the reductive groups G are precisely those such that $Lie(G) = Lie(H) \otimes_{\mathbf{R}} \mathbf{C}$, where H is a compact real Lie group.

We expect reductive groups, or at least connected reductive groups, to have simple representation theory. For instance, maps $G \to \mathrm{GL}_n$ of algebraic groups for $k = \mathbf{C}$ are very simple and have a good classification.

Definition 9.5 (Strictly for this course only!)**.** *A* finite group of Lie type *is a finite group isomorphic to $G(\mathbf{F}_q)$, for some $G/\mathbf{F}_q$ connected reductive group (perhaps not a good definition for the classification of finite simple groups, but better for us).*

10 Lecture 10

Let k be a perfect field (which we often assume to be algebraically closed). Recall that a connected reductive group is an affine, reduced, finite type k-group scheme G such that G contains no normal subgroups isomorphic to $\mathbb{G}_{a,k}^{\oplus m}$ for all m. [For nonperfect fields, the correct condition is that the basechange of G to an algebraic closure of k is reductive.] Examples:

1. $G = \mathbb{G}_{m,k}^{\oplus n}$. Need to check that there are no isomorphisms $\mathbb{G}_{a,k} \to \mathbb{G}_{m,k}$.
2. $\mathrm{GL}_{n,k}$. The subgroup $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is isomorphic to $\mathbb{G}_{a,k}$ but not normal.

Definition 10.1. *A torus T is an algebraic group over k scuh that $T_{\overline{k}} \cong \mathscr{G}_m^{\oplus n}$.*

e.g. $\mathscr{G}_m$

Definition 10.2. *Let G be a connected reductive group. A maximal torus T is a closed subgroup with T a torus, maximal with this property.*

e.g. the diagonal matrices in GL_3.
Fact:

- Maximal tori exist, and if $T \subset G$ is a maximal torus, then so is $T_{\overline{k}} \subset G_{\overline{k}}$.
- When k is algebraically closed, any two maximal tori are $G(k)$-conjugate.
- $C_G(T) = T$ when $T \subset G$ is a maximal torus.

Example: Let L over k be a finite etale k-algebra of rank n (i.e. $L = \prod_{i=1}^m k_i$, where k_i/k is a degree j_i (separable since k is perfect) field extension and $\sum j_i = n$). Then we consider

$$\mathrm{Res}_{L/k}\mathbb{G}_m$$

which has functor of points

$$R \mapsto (L \otimes_k R)^\times.$$

This is a torus, because

$$L \otimes_k \overline{k} \cong \prod_{i=1}^n \overline{k}$$

and so $(\mathrm{Res}_{L/k}\mathbb{G}_m) \otimes_k \overline{k} = \mathbb{G}_{n,\overline{k}}^{\oplus n}$.

If we choose a k-basis of L and we let L act on itself $L = k^{\oplus n}$ by left translation, then there is an induced morphism

$$\mathrm{Res}_{L/k}\mathbb{G}_{m,L} \to \mathrm{GL}_{n,k}.$$

Fact: This procedure produces all maximal tori of $GL_{n,k}$, so the Galois theory of k enters the picture.

Definition 10.3. *A representation of G is a pair (V, ρ) where V is a k-vector space and ρ is a natural transformation (of functors on k-algebras)*

$$(R \mapsto G(R)) \mapsto (R \mapsto \mathrm{Aut}_r(V \otimes_k R)).$$

A morphism $f : (V, \rho) \to (V', \rho')$ is a k-linear map $V \to V'$ such that for all R and $g \in G(R)$, $v \in V \otimes_k R$,

$$f_R(\rho(g).v) = \rho'(g) f_R(v).$$

This gives an abelian category of finite dimensional representations.

Example: The adjoint representation. Let $R[\epsilon] = R[x]/(x^2)$, then we define

$$\mathrm{Lie}\, G(R) = \ker(G(k[\epsilon]) \to G(R))$$

E.g. when $G = GL_n$, we get

$$1 + \epsilon M_{n\times n}(R) \subset \mathrm{GL}_n(k[\epsilon])$$

with the group structure

$$(1 + \epsilon A)(1 + \epsilon B) = 1 + \epsilon(A + B).$$

Fact: $\mathrm{Lie}\, G \subset R$ is canonically an R-module, and $(\mathrm{Lie}\, G)(k) \otimes_k R \cong (\mathrm{Lie}\, G)(k)$. The left hand is also the tangent space to G at $e \in G(k)$. Then the adjoint representation is

$$\begin{aligned}(R \mapsto G(R)) &\mapsto (\mathrm{Aut}_R(LieG(R))) \\ g &\mapsto \text{conjugation by } g \in G(R) \subset G(R[\epsilon])\end{aligned}$$

Often $\mathfrak{g}$ is used to denote $\mathrm{Lie}\, G$, which is a Lie algebra.

Fact: The category of finite dimensional representations of $\mathbb{G}_m$ is semisimple, with simple objects the one-dimensional representations, corresponding to

$$\begin{aligned}\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) &\cong \mathbf{Z} \\ (x \mapsto x^n) &\mapsto n\end{aligned}$$

For a torus T, we write $X^*(T) = \mathrm{Hom}_{\overline{k}}(T_{\overline{k}}, \mathbb{G}_m)$ for the character lattice. Dually, the co-character lattice $X_*(T) = \mathrm{Hom}_{\overline{k}}(\mathbb{G}_m, T_{\overline{k}})$. They are finite free $\mathbf{Z}$-modules equipped with actions of $\mathrm{Gal}(\overline{k}/k)$, which factor through $\mathrm{Gal}(\overline{k}/k) \to \mathrm{Gal}(k'/k)$ for some finite extension k' of k.

There is a pairing (composition)

$$\begin{aligned}X^*(T) \times X_*(T) &\to \mathrm{Hom}_{\overline{k}}(\mathbb{G}_m, \mathbb{G}_m) = \mathbf{Z} \\ (\chi, \mu) &\mapsto \langle \chi, \mu \rangle = \chi \circ \mu\end{aligned}$$

Definition 10.4. *Let $T \subset G$ be a maximal torus, and assume $k = \overline{k}$. Consider* $\operatorname{Lie} G|_T$ *as a T-representation. Then*

$$\operatorname{Lie} G|_T = \operatorname{Lie} T \oplus \bigoplus_{\alpha \in \Phi} (\operatorname{Lie} G)_\alpha$$

$\Phi \subset X^*(T)$ is the subset of those characters α such that $\operatorname{Lie} G_\alpha \neq 0$. *These characters are called the* roots *of G with respect to T.*

Fact: Each $\operatorname{Lie} G_\alpha$ is one-dimensional. Fact: For each $\alpha \in \Phi$, there is a unique $\alpha^\vee \in X_*(T)$ such that

$$\alpha, \alpha^\vee = 2$$

and $S_\alpha(\Phi) = \Phi$ and $S_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$, where

$$S_\alpha : X^*(T) \to X^*(T)$$

is given by $x \mapsto x - \langle x, \alpha^\vee \rangle \cdot \alpha$, and similarly

$$S_{\alpha^\vee} : X_*(T) \to X_*(T)$$

is given by $y \mapsto y - \langle \alpha, y \rangle \cdot \alpha^\vee$.

The upshot is that $(X^*(T), X_*(T), \Phi, \Phi^\vee)$ is a root datum, and connected reductive groups G over $k = \overline{k}$ are classified by root data(?).

A strange observation is that $(X_*(T), X^*(T), \Phi^\vee, \Phi)$ is also a root datum, so we get a dual group $\widehat{G}$ over k with this root datum. For example, under this operation, SL_n goes to PGL_n, SP_{2n} goes to SO_{2n+1}, and GL_n is self-dual. This construction is the heart of all of "Langlands duality". The example of SP_{2n} and SO_{2n+1} is illustrative.

Bruhal Decomposition:

Definition 10.5. *A borel subgroup $B \subset G$ is a subgroup such that G/B is a projective variety, and B is minimal with this property.*

Fact: When $k = \overline{k}$ or $k = \mathbf{F}_q$, Borel subgroups exists and they are all $G(k)$-conjugates. They are maximal solvable connected subgroups of G over $\overline{k}$.

E.g. for GL_n, B is the subgroup of upper triangular matrices. GL_n / B is the variety of full flags.

Fact: Every maximal torus is contained in a Borel subgroup. This is only true for $k = \overline{k}$. It fails already for GL_2 over a finite field (as we have seen for $\operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{G}_m \subset \mathrm{GL}_{2,\mathbb{F}_q}$)

Now we get ($T \subset BG$ maximal torus contained in a Borel subgroup B, and $k = \overline{k}$)

$$\Phi_B^+ \subset \Phi$$

of those $\alpha \in \Phi$ such that $(\operatorname{Lie} B)_\alpha \neq G$.

Fact: If $\alpha \neq \beta \in \Phi^+$, then $\alpha + \beta \in \Phi_B^+$. Every $\alpha \in \Phi$ is in Φ_B^+ or $-\alpha \in \Phi_B^+$. We can define $\Delta_B \subset \Phi_B^+$ as those $\alpha \in \Phi_B^+$ that are not of the form $\alpha_1\alpha_2$ for $\alpha_1 \neq \alpha_2 \in \Phi_B^+$. These are called the simple positive roots.

The Weyl groups. Let $T \subset G$ be a maximal torus. Let $W = N_G(T)/T$ which is a finite group, and W acts on T, hence on $X^*(T)$. It acts faithfully on $X^*(T)$.

Fact: $W \hookrightarrow \mathrm{Aut}(X^*(T))$ is generated by $S_\alpha : X^*(T) \to X^*(T)$ for $\alpha \in \Delta$. This exhibits W as a Coxeter group.

Fact: $G = \coprod_{w \in W} BwB$, where $w \in N_G(T)(k)$ is a representative of $w \in W$.

11 Lecture 11

Today we continue the discussion last time with some examples.

Let $G = \mathrm{SL}_{n+1}$ and T be the diagonal torus. Let B be the upper triangular matrices. $T = T' \cap \mathrm{SL}_{n+1}$ where $T' \subset \mathrm{GL}_{n+1}$ is isomorphic to $\mathbb{G}_m^{\oplus(n+1)}$. We can also think of T as the kernel of

$$T' \to G_m^{\oplus(n+1)}(x_0, \cdots, x_n) \mapsto x_0 \cdots x_n$$

So we can identify $* : X^*(T') = \mathbf{Z}^{n+1} \to X^*(T)$, where if e_i corresponds to the character sending $(t_0, \cdots, t_n)$ to t_i, then the kernel of $*$ is $e_0 + \cdots + e_n$. So $X^* \cong \mathbf{Z}^{n+1}/\mathbf{Z}$, where $\mathbf{Z}$ is embedded diagonally.

$\mathrm{Lie}\,\mathrm{SL}_{n+1} \subset \mathrm{Lie}\,\mathrm{GL}_{n+1}$ (the latter is just $(n+1)$-by-$(n+1)$ matrices), and one can check that T preserves the linear subspaces U_{ij}, which is those matrices with only (i,j)-th entry nonzero. And $(t_0, \cdots, t_n)$ acts on $U_{i,j}$ by the scalar $\frac{t_i}{t_j}$. Hence we see that the roots oare the characters $\alpha_{i,j} = e_i - e_j$ for $i \neq j$.

The positive roots are those that are upper triangular, so $i < j$. Namely, $\Phi^+ = \{\alpha_{ij} \mid i < j\}$. The simple roots are

$$\Delta = \{e_i - e_{i+1}\}_{0 \le i \le n-1}$$

One can check the coroots $\alpha_{i,j}^\vee$ dual to α_{ij} is given by the cocharacter

$$y \mapsto (1, \cdots, 1, y, \cdots, \frac{1}{y}, 1, \cdots 1)$$

where the y appears on the i-th slot and $\frac{1}{y}$ appears on the j-th slot. We see that indeed $\alpha_{ij} \circ \alpha_{ij}^\vee$ sends y to y^2.

For the case of SP_{2n} and SO_{2n}, see Conrad's notes.

From next week on, we will be working with a connected reductive group G_0 over $\mathbf{F}_q$ and $G = G_0 \otimes \overline{\mathbf{F}_q}$. We will describe a procedure that takes as inputs:

1. $T_0 \subset G_0$ a maximal torus
2. $B \supset T (T = T_0 \otimes \overline{\mathbf{F}_q})$
3. $\Theta : T_0(\mathbf{F}_q) \to \overline{\mathbf{Q}_l}^\times$

and outputs a virtual representation $R_{T_0,B}^{\Theta}$ of $G_0(\mathbf{F}_q)$ over $\overline{\mathbf{Q}_l}$. Today, we investigate how many choices of $(T_0, B \subset T)$ there are. (For SL_2 over $\mathbf{F}_q$, there are 2 choices up to conjugation.)

Fix a Borel pair $\mathbb{T} \subset \mathbb{B}_0$ (think of this as a base point). Then $(G_0/B_0 \times G_0/B_0)_{\overline{\mathbf{F}_q}}$ has finitely many G-orbits, indexed by $\underline{W}(\overline{\mathbf{F}_q})$ where $\underline{W} = N_G(\mathbb{T}_0)/\mathbb{T}_0$. This is a base point in the sense that the set of varieties of Borel subgroups can be identified with $G_0/\mathbf{B}_0$ by sending g to $g\mathbf{B}_0 g^{-1}$.

Warm-up:

1. There is only one choice of F-stable Borel pair $\mathbf{T}_0 \subset \mathbf{B}_0$ up to conjugacy.

2. There is only one choice of $\mathbf{B}_0$ up to conjugacy.

Proof. The variety of Borel subgroups can be identified with $G_0/\mathbf{B}_0$ using our base point. Now $(G_0/\mathbf{B}_0)(\mathbf{F}_q) = G_0(\mathbf{F}_q)/\mathbf{B}_0(\mathbf{F}_q)$ because $\mathbf{B}_0$ is connected and we are over a finite field. This shows the second claim since it says all $B_0 \subset G_0$ is $G_0(\mathbf{F}_q)$-conjugate to $\mathbf{B}_0$.

The variety of Borel pairs $(T_0 \subset B_0 \subset G_0)$ is a homogenous space for G_0, and using the base point $(\mathbf{T_0} \subset \mathbf{B_0})$, we can identify it with $G_0/(N_{G_0}(\mathbf{T_0}) \cap N_{G_0}(\mathbf{B}_0))$. It is a fact that the intersection $N_{G_0}(\mathbf{T_0}) \cap N_{G_0}(\mathbf{B}_0)$ is equal to $\mathbf{T}_0$, and so the variety of Borel pairs is identified with $G_0/\mathbf{T}_0$. Using the same argument as above, we see that $G_0/\mathbf{T}_0(\mathbf{F}_q) = G_0(\mathbf{F}_q)/\mathbf{T}_0(\mathbf{F}_q)$. Thus every Borel pair $(T_0 \subset B_0)$ is $G_0(\mathbf{F}_q)$-conjugate to $(\mathbf{T}_0 \subset \mathbf{B}_0)$. □

Things are more complicated for the variety of maximal tori. Indeed this can be identified with the homogeneous space $G_0/N_{G_0}(\mathbf{T}_0)$, using $\mathbf{T}_0$ as a base point, and so we expect to see the (non-abelian) Galois cohomology set

$$H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q, N_{G_0}(\mathbf{T}_0)(\overline{\mathbb{F}}_q))$$

as an obstruction. Since $\mathbf{T}_0$ is connected the Galois cohomology $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q, \mathbf{T}_0)(\overline{\mathbb{F}}_q))$ vanishes by Lang's lemma. Therefore the long exact sequence in Galois cohomology for

$$1 \to \mathbf{T}_0 \to N_{G_0}(\mathbf{T}_0) \to \underline{W} \to 1$$

tells us that the natural map

$$H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), N_{G_0}(\mathbf{T}_0)(\overline{\mathbb{F}}_q)) \to H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), W(\overline{\mathbb{F}}_q))$$

is an isomorphism.

Concretely, if E is the variety of maximal tori T and $\widetilde{E}$ is the variety of Borel pairs $T \subset B$ there is a map

$$\widetilde{E} \xrightarrow{\pi} E$$

sending the point $\tilde{e}_0 = (\mathbf{T}_0 \subset \mathbf{B}_0)$ to $e_0 = \mathbf{T}_0$. This can be identified as the map $G_0/\mathbf{T}_0 \to G_0/N_{G_0}(\mathbf{T}_0)$ using chosen the base points. Let us write $Y = \pi^{-1}(E(\mathbb{F}_q))$, considered as an $\mathbb{F}_q$-scheme. We want to understand $A = G_0(\mathbf{F}_q)\backslash E(\mathbf{F}_q)$ and $\widetilde{A} = G_0(\mathbf{F}_q)\backslash Y(\overline{\mathbb{F}}_q)$.

Proposition 11.1. *The choice of $\widetilde{e_0}$ induces two bijections:*

$$\widetilde{A} \xrightarrow{\sim} \underline{W}(\mathbb{F}_q)$$

and

$$A \xrightarrow{\sim} \frac{\underline{W}(\overline{\mathbf{F}_q})}{Ad_F \underline{W}(\overline{\mathbf{F}_q})}$$

Here, Ad_F denotes the action $w.(w') = ww'F(w)^{-1}$ of $\underline{W}$ on itself and the the fractional notation denotes taking the quotient.

Note: $\mathrm{Ad}_F \underline{W}(\overline{\mathbf{F}_q}) = H^1_{\mathrm{et}}(\mathbf{F}_q, \underline{W})$. So a fancy proof of the second bijection is to look at the set of isomorphism classes over $\mathbf{F}_q$-points of the stack $[G_0 \backslash E] = [\frac{1}{N_{G_0}(\mathbf{T}_0)}]$, so this set is just

$$H^1_{\mathrm{et}}(\mathbf{F}_q, N_{G_0}(\mathbf{T}_0)) = H^1_{\mathrm{et}}(\mathbf{F}_q, N_{G_0}(\mathbf{T}_0)/\mathbf{T}_0) = H^1_{\mathrm{et}}(\mathbf{F}_q, \underline{W}).$$

[This étale cohomology is the same as the Galois cohomology from above].

Proof of the Proposition. We are trying to compute the quotient $G_0(\mathbb{F}_q)\backslash Y(\overline{\mathbb{F}}_q)$, where $Y \subset \tilde{E}$ is the inverse image of $E(\mathbb{F}_q)$ under π.

The space $Y := \pi^{-1}(E(\mathbf{F}_q))$ can be identified, using the base point $\tilde{e_0}$, with

$$\{g \in G \mid \pi(g.\tilde{e_0}) \in E(\mathbf{F}_q)\}/\mathbf{T}_0 = \{g \in G \mid g^{-1}F(g) \in N_{G_0}(\mathbf{T}_0)\}/\mathbf{T}_0$$

because $F(g^{-1}\mathbf{T}_0 g) = g^{-1}\mathbf{T}_0 g$ is equivalent to $g^{-1}F(g) \in N_{G_0}(\mathbf{T}_0)$. The upshot of this discussion is that the following diagram is a pullback diagram

$$\begin{array}{ccc} Y & \subset & G_0/\mathbf{T}_0 \\ \downarrow & & \downarrow{\scriptstyle g \mapsto g^{-1}F(g)} \\ \frac{N_{G_0}(\mathbf{T}_0)}{\mathbf{T}_0} & \subset & G_0/\mathbf{T}_0. \end{array}$$

In other words, Y is the inverse image of $\frac{N_{G_0}(\mathbf{T}_0)}{\mathbf{T}_0} \subset \frac{G_0}{\mathbf{T}_0}$ under the map $\frac{G_0}{\mathbf{T}_0} \to \frac{G_0}{\mathbf{T}_0}$ induced by the Lang isogeny $g \mapsto g^{-1}F(g)$.

The reason we did all this is that $G_0(\mathbb{F}_q) \subset G_0$ is the kernel of the Lang isogeny $g \mapsto g^{-1}F(g)$. Thus the map induced by the Lang isogeny induces an isomorphism

$$G_0(\mathbb{F}_q)\backslash Y \simeq \frac{N_{G_0}(\mathbf{T}_0)}{\mathbf{T}_0} =: \underline{W}.$$

In particular, there is an isomorphism

$$G_0(\mathbb{F}_q)\backslash Y(\overline{\mathbb{F}}_q) \simeq \underline{W}(\overline{\mathbb{F}}_q),$$

which proves the first part of the Proposition. The proof of the second part is omitted.

□

References

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