Reductive groups (Will) Talk 2

§1.) Intro

q prime power, G reductive group over $\overset{\text{coeffs in } \mathbb{F}_q}{\mathbb{F}_q}$

'68 : MacDonald's conjecture

$\exists$ well-def. correspondence $\overset{\text{irr. reps}}{G^F} \overset{\text{fixed pts}}{}$ $\longrightarrow$ pairs (T, θ)

$F : G \to G$ q^{th} power map

T, F-stable $\overset{\text{maxl}}{}$ torus

$\hookrightarrow F$ preserves T

θ, a character of T^F

'76: D-L proved by constructing "virtual rep", R_T^θ of G^F

Lemma $g \in G$ commutator $\Leftrightarrow$ $\displaystyle\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} \neq 0$

§2 $GL_2(q) = G$ (G^F relabelled to G)

not^n : (φ, V), $\varphi : G \to GL(V)$

$\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{\varepsilon})$

$|G| = q(q+1)(q-1)^2$

$q^2 - 1$ conj. classes

	Class rep. g	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	$\begin{pmatrix} \lambda & \varepsilon\mu \\ \mu & \lambda \end{pmatrix}$
	details	$\lambda \in \mathbb{F}_q^*$	$\lambda \in \mathbb{F}_q^*$	$\lambda \neq \mu$	$\lambda, \mu \in \mathbb{F}_q$, $\mu \neq 0$
	$\# g^G$	$q-1$	$q-1$	$\frac{1}{2}(q-1)(q-2)$	$\frac{1}{2}q(q-1)$
	$\lvert g^G \rvert$	1	q^2-1	q^2+q	q^2-q
1	χ_{V_α}	$\alpha(\lambda)^2$	$\alpha(\lambda)^2$	$\alpha(\lambda\mu)$	$\alpha(\lambda^2-\varepsilon\mu^2)$
$q-1$	χ_V	q	0	1	-1
	χ_{U_α}	$q\alpha(\lambda)^2$	0	$\alpha(\lambda\mu)$	$-\alpha(\varepsilon)$
	$\chi_{W_{\alpha,\beta}}$	$(q+1)\alpha(\lambda)\beta(\lambda)$	$\alpha(\lambda)\beta(\lambda)$	$\alpha(\lambda)\beta(\mu) + \alpha(\mu)\beta(\lambda)$	0
	χ_ϕ	$(q-1)\phi(\lambda)$	$-\phi(\lambda)$	0	$-(\phi(\varepsilon)+\phi(\varepsilon)^q)$

$\dfrac{q(q-1)}{2}$

§3 - One-dim reps

$\exists$ bijection $\{1\text{-dim reps}\} \longleftrightarrow \widehat{G/[G,G]}$

not^n $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$

$[G, G] = SL_2(q) = \ker(\det)$

$$G/\widehat{SL_2(q)} \cong \widehat{\mathbb{F}_q^*}$$

$$\alpha \in \widehat{\frac{G}{SL_2(q)}} \longleftrightarrow \alpha \circ \det$$

$(\varphi_\alpha, V_\alpha)$

§4 Standard rep

$H \curvearrowright X$, permutation rep. (ρ, W)

$W = \mathrm{span}(w_i \mid i \in X)$

$\rho(g) w_x = w_{gx} \quad g \in H$

$G \curvearrowright \mathbb{P}^1(\mathbb{F}_q) = \{0, 1, \ldots, q-1\} \cup \{\infty\}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = (ax+b)(cx+d)^{-1}$

$\chi_W(g) = |\mathrm{fix}_X(g)|$

$Tr = \mathrm{span}\{\sum w_i\}$

$V \oplus Tr = W$ complement to Tr

$\langle \chi_V, \chi_V \rangle = 1$ so V irred.

§5 Tensor product

$U_\alpha = V_\alpha \otimes V$ (one is 1-dim, other irred., so U_α irred.)

§6. Parabolically induced

<u>Defⁿ</u> Borel $B = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G\}$

Unip. Rad. $N = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G\}$

Max'l torus $T = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G\}$

$N = [B, B] \trianglelefteq B$, $T < B$, $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$

Weyl group $W = N(T)/T = \{1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: w\}$

$G = B \sqcup BwB$

$B/N \cong T$

$B/_{[B,B]} \cong \widehat{\mathbb{F}_q^\times \times \mathbb{F}_q^\times} \ni (\alpha, \beta)$

$\rho_{\alpha,\beta}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \alpha(a)\ \beta(d)$

$\pi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi \longrightarrow$ Induce $I(\alpha,\beta) = \mathrm{Ind}_B^G(\pi_{\alpha,\beta})$

$W_{\alpha,\beta} = \mathrm{Ind}\ V_{\alpha,\beta}$

Prop $\alpha,\beta \in \widehat{\mathbb{F}_q^*}$, $I(\alpha,\beta)$ irred. deg. $q+1$

unless $\alpha = \beta$, $W_{\alpha,\alpha} = U_\alpha \oplus U_\alpha$

$I(\alpha,\beta) \cong I(\gamma,\delta) \iff \pi_{\alpha,\beta} = \pi_{\gamma,\delta}$ or $\pi_{\delta,\gamma}$

Until not $\exists\ \frac{1}{2}(q-1)(q-2)$ irreps.

§7 Cuspidal reps

"virtual reps"

Lemma χ virtual character, $\chi(1) > 0$

$\langle \chi, \chi \rangle = 1 \Rightarrow \chi$ irreducible

$K = \left\{ \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix} \in G \right\} \cong (\mathbb{F}_{q^2})^*$

$[G:K] = q(q-1)$

$\phi \in \hat{K}$, $\mathrm{Ind}_K^G(\phi)$

$\Rightarrow \chi_\phi = \chi_{V \otimes W_{\alpha,1}} - \chi_{W_{\alpha,1}} - \chi_{\mathrm{Ind}(\phi)}$

Irred $\iff \phi \neq \phi^q$