Introduction to ℓ-adic cohomology Talk 4 (Pol)

Def Let X be a (smooth projective) variety over $\mathbb{F}_p$, then define

$$Z_X = \exp\Big(\sum_{n\geq 1} \#X(\mathbb{F}_{p^n}) \frac{T^n}{n}\Big) \in \mathbb{Q}[[T]]$$

example • $X = pt$, $\#X(\mathbb{F}_{p^n}) = 1$

$$Z_X = \operatorname{Exp}\Big(\sum \frac{T^n}{n}\Big) = \operatorname{Exp}(-\log(1-T)) = \frac{1}{1-T}$$

• $X = \mathbb{P}^1$

$$Z_X = \frac{1}{(1-T)(1-pT)}$$

• $X = E$ is an elliptic curve

$$Z_X = \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-pT)}$$

where $\alpha, \beta \in \bar{\mathbb{Z}}$ s.t.

$|\alpha| = \sqrt{p} = |\beta|$ and $\alpha \cdot \beta = p$

• can do many more computations like this

Conjecture (Weil) Z_X is always a rational function

(he proved it for curves)

"Proof" Let M be a compact oriented manifold and $\psi: M \to M$ a continuous map with isolated fixed points

"Lefschetz fixed point theorem" Then $\#\operatorname{Fix}\psi = \sum_i (-1)^i \operatorname{Tr}(\psi_*, H^i_{sing}(M, \mathbb{R}))$

$$\| \\ \Delta \cdot \Gamma_\psi$$

inside $M \times M$

Apply this with $M = X_{\bar{\mathbb{F}}_p}$ variety over $\mathbb{F}_p$

$\psi = F_q: X \to X$

Basically $X/\mathbb{C}$ a smooth projective variety,

$$H^i_{sing}(X^{an}, \mathbb{Q})$$

Can we define this algebraically?

A: No.

$$H^1_{sing}(X^{an}, \mathbb{Z}) = \mathcal{H}_1(X^{an})^{ab}$$

E.G. $H^1(\mathbb{C}^\times, \mathbb{Z}) = \mathbb{Z}$

$\exp: \mathbb{C} \to \mathbb{C}^\times$ (not algebraic)

is a $\mathbb{Z}$-covering

<u>Riemann Existence</u>

If $X \to X^{an}$ is a finite covering ($\mathbb{C}^\times \to \mathbb{C}^\times$, $x \mapsto x^n$) then Y "is also an algebraic variety".

So $H^1_{A/G}(X, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1_{sing}(X, \mathbb{Z}/\ell\mathbb{Z})$

but can't do "$H^1_{Alg}(X, \mathbb{Z})$"

<u>Problem II</u> (Serre)

There is no cohomology theory for varieties over $\overline{\mathbb{F}_p}$ with coeffs in $\mathbb{Q}$ s.t.

1) $H^1_?(E, \mathbb{Q})$ is $\mathbb{Q}^{\oplus 2}$

From now on $\ell \neq p$.

We can construct a cohomology theory with $\mathbb{Z}/\ell^n\mathbb{Z}$ coeffs for all n.

$$H^1_{\acute{e}t}(X, \mathbb{Q}_\ell) := \left(\varprojlim_n H^1_{\acute{e}t}(X, \mathbb{Z}/\ell^n\mathbb{Z})\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

<u>Facts:</u> It is a functor

$$H^i_{\acute{e}t}: SmP/_{\mathbb{F}_p}^{op} \to \left\{ \begin{matrix} \mathbb{Q}_\ell \\ \text{vector} \\ \text{spaces} \end{matrix} \right\}$$

satisfying

1) $H^i_{\acute{e}t}(X, \mathbb{Q}_\ell)$ is finite dimensional

2) $H^i_{et}(X, \mathbb{Q}_\ell) = 0$ for $i < 0$
$i > 2\dim X$

3) If X lifts to char 0,
then $H^i_{sing}(\tilde{X}, \mathbb{Q}_\ell) \cong H^i_{et}(X, \mathbb{Q}_\ell)$

$$\begin{gathered} H^i_{sing}(\tilde{X}, \mathbb{Q}_\ell) \\ \| \\ H^i_{sing}(\tilde{X}, \mathbb{Q}) \otimes \mathbb{Q}_\ell \end{gathered}$$

4) Poincaré duality
$$H^i_{et}(X)^\vee \cong H^{2n-i}_{et}(X)$$

5) $\# X(\mathbb{F}_q) = \sum_i (-1)^i \operatorname{Tr}(F_q, H^i_{et}(X))$

$\overset{\text{Weil II}}{\Longrightarrow}$ $\operatorname{Tr}(F_q, H^i_{et}(X))$ does not depend on ℓ
(neither does $\dim_{\mathbb{Q}_\ell} H^i_{et}(X)$)

There is an extension
$$H^i_c : \left\{ \begin{matrix} \text{varieties over } \overline{\mathbb{F}_p} \\ \text{with proper map} \end{matrix} \right\}^{op} \to \{ \mathbb{Q}_\ell \text{ vector spaces} \}$$

s.t. $H^i_c(X) = H^i(X)$ if X is proper

1) finite dimensionality
2) vanishing for $i \in [0, 2\dim X]$
3) <u>Artin vanishing</u>
Suppose X is smooth and affine, then
$H^i_c = 0$ for $0 \leq i < \dim X$
4) If $Z \subset X$ closed, with open complement U,
$$\cdots \to H^i_c(U) \xrightarrow{\text{"ext. by 0"}} H^i_c(X) \xrightarrow{\text{"restriction"}} H^i_c(Z) \to \cdots$$
[Excision?]

5) $\# X(\mathbb{F}_q) = \sum_i (-1)^i \operatorname{Tr}\left(F_q, H^i_c(X)\right)$

$$\# X(\mathbb{F}_q) = \# Z(\mathbb{F}_q) + \# U(\mathbb{F}_q)$$

$$E_c(X) = \sum_i (-1)^i H^i_c(X)$$

in $K_0(\mathbb{Q}_\ell[\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)])$

for $(X/\mathbb{F}_p)$

Then 4) implies

$$E_c(X) = E_c(Z) + E_c(U)$$

in $K_0(\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p))$

How to prove anything?
Prove everything for curves, the induct on $\dim X$ and use Noetherian induction.

For curves $\Delta \cdot \Gamma_F \subset C \times C$
(apply Hodge index thm and Riemann Roch)

If $\varphi : X \to X$ ← smooth proj. is a morphism with isolated fixed points, then

$$\Delta \cdot \Gamma_\varphi = \sum (-1)^i \operatorname{Tr}(\varphi, H^i(X))$$

($\varphi = F$, then $\Delta \cdot \Gamma_F = \# X(\mathbb{F}_q)$)