Surfaces

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Roughly speaking, a regular surface in $\mathbb{R}^3$ is obtained by taking pieces of a plane, deforming them, and arranging them in such a way that the resulting figure has no sharp points, edges, or self-intersections and so that it makes sense to speak of a tangent plane at points of the figure.
The idea is to define a set that is, in a certain sense, two-dimensional and that also is smooth enough so that the usual notions of calculus can be extended to it.

Definition 2.1 (Regular Surface)
A subset $S \subset \mathbb{R}^3$ is a regular surface if for each $p \in S$, there exists a neighborhood $V \subset \mathbb{R}^3$, an open set $U \subset \mathbb{R}^2$ and an onto map $\mathcal{X} : U \to V \cap S$ such that
(1) $\mathcal{X}$ is differentiable, i.e. if we write

$$\mathcal{X}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in U$$

Then the functions $x(u, v), y(u, v), z(u, v)$ have continuous partial derivatives of all orders in U.
(2) $\mathcal{X}$ is a homeomorphism, since $\mathcal{X}$ is continuous by condition (1), this means that $\mathcal{X}^{-1} : V \cap S \to U$ is continuous.
(3) (regularity condition) For each $q \in U$, the differential $d\mathcal{X}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

The mapping $\mathcal{X}$ is called a parametrization or system of (local) coordinates in a neighborhood of p. The neighborhood $V \cap S$ of p in S is called a coordinate neighborhood.

To give condition (3) a more familiar form, let us compute the matrix of the linear map $d\mathcal{X}_q$ in the canonical bases $e_1 = (1, 0)$, $e_2 = (0, 1)$ of $\mathbb{R}^2$ with coordinates (u, v) and $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 0, 1)$ of $\mathbb{R}^3$, with coordinates (x, y, z).
Let $q = (u_0, v_0)$, the vector e_1 is tangent to the curve $\alpha : \mathbb{R} \to U \subset \mathbb{R}^2, u \mapsto (u, v_0)$ whose image under $\mathcal{X}$ is the curve

$$\beta : \mathbb{R} \to \mathbb{R}^3, \quad u \mapsto (x(u, v_0), y(u, v_0), z(u, v_0))$$

This image curve (called the coordinate curve $v = v_0$) lies on S and has the tangent vector at $\mathcal{X}(q)$, which is defined by

$$(\frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)) = \frac{\partial \mathcal{X}}{\partial u}(u_0, v_0)$$

By the definition of differential,

$$d\mathcal{X}_q(e_1) = \frac{\partial \mathcal{X}}{\partial u}(u_0, v_0)$$

$$d\mathcal{X}_q(e_2) = \frac{\partial \mathcal{X}}{\partial v}(u_0, v_0)$$

Thus, the matrix of the linear map $d\mathcal{X}_q$ in the referred basis is

$$d\mathcal{X}_q = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}_q$$

Example 2.1 (unit sphere)
The unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

is a regular surface.
Proof.
We first verify that the map $\mathcal{X}_1 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathcal{X}_1(x, y) = (x, y, \sqrt{1 - (x^2 + y^2)}), (x, y) \in U$$

is a parametrization of S^2. Observe that $\mathcal{X}_1(U)$ is the open part of S^2 above the xOy plane.
Since $x^2 + y^2 < 1$, the function $\sqrt{1 - (x^2 + y^2)}$ has continuous partial derivatives of all orders. Thus, $\mathcal{X}_1$ is differentiable and condition (1) holds.
To check condition (2), we observe that $\mathcal{X}_1$ is one-to-one and that $\mathcal{X}_1^{-1}$ is the restriction of the continuous projection $\pi(x, y, z) = (x, y)$ to the set $\mathcal{X}_1(U)$. Thus, $\mathcal{X}_1^{-1}$ is continuous in $\mathcal{X}_1(U)$.
To check condition (3), note that

$$d\mathcal{X}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{-x}{\sqrt{1 - (x^2 + y^2)}} & \dfrac{-y}{\sqrt{1 - (x^2 + y^2)}} \end{pmatrix}$$

It's easy to see that $d\mathcal{X}_1$ is one-to-one.
Similarly we can verify the above conditions for other parametrizations like

$$\mathcal{X}_2(x, z) = (x, \sqrt{1 - (x^2 + z^2)}, z)$$

$$\mathcal{X}_3(y, z) = (\sqrt{1 - (y^2 + z^2)}, y, z) \quad \square$$

Proposition 2.1
If $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function in an open set U, then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$, is a regular surface.
Proof.
It suffices to show that the map $\mathcal{X} : U \to \mathbb{R}^3$ given by

$$\mathcal{X}(u, v) = (u, v, f(u, v))$$

is a parametrization of the graph whose coordinate neighborhood covers every point of the graph.
It's clearly that $\mathcal{X}$ is differentiable.
Also note that

$$\mathcal{X}^{-1} = \pi|_G$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is bounded and thus continuous, $G = \{(x, y, f(x, y)) : (x, y) \in U\}$. So $\mathcal{X}^{-1}$ is continuous and thus $\mathcal{X}$ is a homeomorphism.

Finally, since

$$d\mathcal{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{df}{du} & \dfrac{df}{dv} \end{pmatrix}$$

has rank 2 at any point $(u, v) \in U$, it follows that $\mathcal{X}$ is a parametrization. $\square$

Definition 2.2 (critical and regular point)
Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$ we say that $p \in U$ is a critical point of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective (onto) mapping. The image $F(p) \in \mathbb{R}^m$ of a critical point is called a critical value of F. A point of $\mathbb{R}^m$ which is not a critical value is called a regular value of F.

Proposition 2.2
Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function, U is an open subset of $\mathbb{R}^3$, a is a regular value of f, then $f^{-1}(a)$ is a regular surface.
Proof.
Pick any $p \in f^{-1}(a)$, since a is a regular value, we can assume without loss of generality that $\frac{\partial f}{\partial z} \neq 0$.
Then define $F : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x, y, z) = (x, y, f(x, y, z))$$

Note that

$$dF_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{pmatrix}_p$$

is invertible. So by inverse function theorem, there exists a neighborhood $V_p \subset \mathbb{R}^3$ of p and $W_p = F(V_p) \subset \mathbb{R}^3$ such that $F : V_p \to W_p$ is invertible and $F^{-1} : W_p \to V_p$ is differentiable. Note that $F^{-1}(x, y, a) = (x, y, z)$, hence $F^{-1} : W_p \to V_p$ can be written as

$$F^{-1}(x, y, a) = (x, y, h(x, y))$$

where $z = h(x, y)$ is a differentiable function here.
Therefore, $V_p = \{(x, y, h(x, y)) : (x, y) \in W_p\}$ is the graph of $h(x, y)$ and by $Proposition 2.1$ we know that V_p is a regular surface. Since p is arbitrary in $f^{-1}(a)$, $f^{-1}(a)$ is a regular surface. $\square$

Proposition 2.3
Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$, then there exists a neighborhood V of $p \in S$ such that V is the graph of a differentiable function which has one of the following three forms: $z = f(x, y), y = g(x, z), x = h(y, z)$.
Proof.
Since S is a regular surface, we know that there exists an open set $U \subset \mathbb{R}^2$, $V = \mathcal{X}(U) \subset \mathbb{R}^3$ and a diffeomorphism $\mathcal{X} : U \to V$ such that

$$d\mathcal{X}_q = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}_q$$

has rank 2, where $\mathcal{X}(q) = p, q \in U$ Without loss of generality, we can assume that

$$(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}), (\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v})$$

are linearly independent at point q.
Consider projection $\pi : S \subset \mathbb{R}^3 \to \mathbb{R}^2$, then $\pi \circ \mathcal{X} : U \to W = \pi(\mathcal{X}(U)) \subset \mathbb{R}^2$ is differentiable and

$$d(\pi \circ \mathcal{X}) = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}$$

whose determinant is non-zero at point q by the assumption above.
Therefore by inverse function theorem, we can shrink U and claim that $(\pi \circ \mathcal{X})^{-1} : W \to U$ exists and is differentiable.
Now let $F = \mathcal{X} \circ (\pi \circ \mathcal{X})^{-1}$, then $F : W \subset \mathbb{R}^2 \to V \subset \mathbb{R}^3, (x, y) \mapsto (x, y, z)$ is differentiable, so there exists a differentiable function $f : W \to \mathbb{R}$ such that $z = f(x, y)$ for each $(x, y) \in W$.
Moreover, since $\mathcal{X}$ is a homeomorphism, $V = \mathcal{X}(U)$ is a (open) neighborhood of p, therefore,

$$V = \{(x, y, f(x, y)) : (x, y) \in W\} \quad \square$$

Remark.
The condition that $\mathcal{X}$ is a homeomorphism is required, otherwise we may not find an open neighborhood of p.

Proposition 2.4
Let p be a point of a regular surface S and let $\mathcal{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a map with $p \in \mathcal{X}(U) \subset S$ such that conditions 1(differentiable) and 3(regularity condition) hold. Assume that $\mathcal{X}$ is one-to-one, then $\mathcal{X}^{-1}$ is continuous.
Proof.
Assume that $\mathcal{X}(u, v) = (x(u, v), y(u, v), z(u, v))$, then by condition 1 and condition3, x, y, z are differentiable and

$$d\mathcal{X} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}$$

has rank 2 in U.
Similarly we can assume without loss of generality that $\dfrac{\partial(x, y)}{\partial(u, v)} \neq 0$, then consider

$$\pi : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (x, y)$$

Note that $\pi \circ \mathcal{X} : U \to \pi(\mathcal{X}(U)) \subset \mathbb{R}^2$ is differentiable and

$$d(\pi \circ \mathcal{X}) = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}$$

which is invertible in q. Therefore, by inverse function theorem, the inverse

$$(\pi \circ \mathcal{X})^{-1} : \pi(\mathcal{X}(U)) \to U$$

exists (with shrunk U) and is differentiable, hence continuous. Therefore, $\mathcal{X}^{-1} = (\pi \circ \mathcal{X})^{-1} \circ \pi$, as a composition of two continuous funciton, is also continuous. $\square$

Proposition 2.5 (change of parameters)
Let p be a point of a regular surface S, and let $\mathcal{X} : U \subset \mathbb{R}^2 \to S, \mathcal{Y} : V \subset \mathbb{R}^2 \to S$ be two parametrizations of S such that $p \in \mathcal{X}(U) \cap \mathcal{Y}(V) = W$. Then

$$h = \mathcal{X}^{-1} \circ \mathcal{Y} : \mathcal{Y}^{-1}(W) \to \mathcal{X}^{-1}(W)$$

is a diffeomorphism.
Proof.
It's clear that h as a composition of two homeomorphisms is a homeomorphism.
Let $r \in \mathcal{Y}^{-1}(W)$ and set $q = h(r)$. Since $\mathcal{X}(u,v) = (x(u,v), y(u,v), z(u,v))$ is a parametrization, we can assume, by renaming the axes if necessary, that

$$\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$$

Then, we extend $\mathcal{X}$ to a map $F : U \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$F(u,v,t) = (x(u,v), y(u,v), z(u,v) + t), (u,v) \in U, t \in \mathbb{R}$$

It's clear that F is differentiable and that the restriction $F|_{U\times\{0\}} = \mathcal{X}$, note that

$$det(dF_q) = \frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$$

So we can apply the inverse function theorem here:
There exists

Definition 2.3
Let $f : V \subset S \to \mathbb{R}$ be a function defined in an open subset V of a regular surface S. Then f is said to be differentiable at $p \in V$ if, for some parametrization $\mathcal{X} : U \subset \mathbb{R}^2 \to S$ with $p \in \mathcal{X}(U) \subset V$, the composition $f \circ \mathcal{X} : U \subset \mathbb{R}^2$ is differentiable at $\mathcal{X}^{-1}(p)$. f is differentiable in V if it is differentiable at all points of V.