Homework 1

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Mar 1, 2019

1-2 Ex.2
Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is a point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Solution.
Let $s(t) = |\alpha(t)|$, t_0 is the minimum point of $s(t)$ since $\alpha(t_0)$ is the closest point to the origin on the trace of α. We know that $\alpha(t) = (x(t), y(t), z(t))$ is differentiable and doesn't pass through the origin, so

$$s(t) = |\alpha(t)| = \sqrt{x^2(t) + y^2(t) + z^2(t)} > 0$$

is also differentiable. Then we have

$$\begin{aligned} s'(t) &= \frac{d}{dt}\sqrt{x^2(t) + y^2(t) + z^2(t)} \\ &= \frac{x(t)x'(t) + y(t)y'(t) + z(t)z'(t)}{\sqrt{x^2(t) + y^2(t) + z^2(t)}} \\ &= \frac{\alpha(t) \cdot \alpha'(t)}{s(t)} \end{aligned}$$

Noticed that t_0 is the minimum point of $s(t)$, It follows

$$s'(t_0) = \frac{\alpha(t_0) \cdot \alpha'(t_0)}{s(t_0)} = 0$$

which implies $\alpha(t_0) \cdot \alpha'(t_0) = 0$, i.e. $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. $\square$

1-2 Ex.4
Let $\alpha(t) : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assumed that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is also orthogonal to v for all $t \in I$.

Solution.
Suppose $I = (a, b), 0 \in (a, b)$, then given $t \in (a, b)$, $\alpha(t)$ can be written as

$$\alpha(t) = \int_a^t \alpha'(s)ds$$

Thus we have

$$(\alpha(t) - \alpha(0)) \cdot v = \int_0^t \alpha'(s)ds \cdot v = \int_0^t \alpha'(s) \cdot vds$$

It follows

$$\alpha(t) \cdot v = \alpha(0) \cdot v + \int_0^t \alpha'(s) \cdot vds = 0 + \int_0^t 0ds = 0 \quad \square$$

1-3 Ex.4
Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = (\sin t, \cos t + \log\tan\frac{t}{2})$$

where t is the angle that the y axis makes with the vector $\alpha'(t)$. The trace of α is called the tractrix. Show that
a. α is a differentiable parametrized curve, regular except at $t = \frac{\pi}{2}$.
b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Solution.
a. Since $x(t) = \sin t$ and $y(t) = \cos t + \log\tan\frac{t}{2}$ are both differentiable in $(0, \pi)$, $\alpha(t)$ is a differentiable map from $(0, \pi)$ to $\mathbb{R}^2$, so α is a differentiable parametrized curve. Note that

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\sin t})$$

Let $|\alpha'(t_0)| = 0$, it follows $\cos t_0 = 0$, $\sin t_0 = \frac{1}{\sin t_0}$ and we have $t_0 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.
So $t_0 = \frac{\pi}{2}$ is the unique solution in $(0, \pi)$. Therefore, α is regular in $(0, \pi)$ except at $t = \frac{\pi}{2}$. $\square$

b. Let $(x(t), y(t))$ denote the point of tangency. Since we know that t is the angle that the y axis makes with the vector $\alpha'(t)$, the segment length $l(t)$ can be calculated by

$$l(t) = \frac{x(t)}{\sin t} = \frac{\sin t}{\sin t} = 1 \quad \square$$

1-3 Ex.10
(Straight Lines as Shortest) Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve. Let $[a, b] \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.
a. Show that, for any constant vector v, $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt$$

b. Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt$$

That is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Solution.
a. Since α is differentiable,

$$q - p = \alpha(b) - \alpha(a) = \int_a^b \alpha'(t) dt$$

Thus,

$$(q - p) \cdot v = \int_a^b \alpha'(t) dt \cdot v = \int_a^b \alpha'(t) \cdot v dt$$

For each $t \in (a, b)$, $\alpha'(t) \cdot v \leq |\alpha'(t)||v| = |\alpha'(t)|$, so

$$\int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt \quad \square$$

b. According to the conclusion above, take $v = \frac{q-p}{|q-p|}$ and it follows immediately that

$$|\alpha(b) - \alpha(a)| = |q - p| = (q - p) \cdot v \leq \int_a^b |\alpha'(t)| dt \quad \square$$

1-4 Ex.2
A plane P contained in $\mathbb{R}^3$ is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$ measures the distance from the plane to the origin (0, 0, 0).

Solution.
For each point (x, y, z) in plane P, the equation $ax + by + cz + d = 0$ holds. Hence for each vector u contained in P, it can be denoted by $u = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ where (x_1, y_1, z_1) and (x_2, y_2, z_2) are points in P. Therefore,

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$$

That is,

$$u \cdot v = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \cdot (a, b, c) = 0$$

Suppose v_0 is the shortest vector from the origin to P, it's easy to see that v_0 and v are linear dependent, so v_0 can be written as λv, where $\lambda \in \mathbb{R}$, therefore, for each point $(x, y, z) \in P$,

$$((x, y, z) - v_0) \cdot v_0 = (x - \lambda a, y - \lambda b, z - \lambda c) \cdot \lambda(a, b, c) = 0$$

i.e.

$$\lambda a(x - \lambda a) + \lambda b(y - \lambda b) + \lambda c(x - \lambda c) = -(a^2 + b^2 + c^2)\lambda^2 + (ax + by + cz)\lambda = 0$$

$$(a^2 + b^2 + c^2)\lambda^2 + d\lambda = 0$$

this implies $\lambda = -\frac{d}{a^2+b^2+c^2}$(when $\lambda = 0$, $d = 0$), so $|v_0| = |\lambda||v| = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$, which is exactly the distance from the plane to the origin (0, 0, 0).

1-4 Ex.11
a. Show that the volume V of a parallelepiped generated by three linearly independent vectors $u, v, w \in \mathbb{R}^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an oriented volume in $\mathbb{R}^3$.
b. Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}$$

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Solution.
a. Let S and h denote the basal area and height of the parallelepiped, then

$$V = S \cdot h = |u||v| \sin\langle u, v\rangle h = |u \wedge v| \frac{|(u \wedge v) \cdot w|}{|u \wedge v|} = |(u \wedge v) \cdot w| \quad \square$$

b. Let

$$G = \begin{bmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{bmatrix}$$

If any two vectors of u, v, w are linearly dependent, then it's easy to see both sides will equal to 0. Thus it only remains to verify the cases that $\{u, v, w\}$ are linearly independent.
In these cases, $\{u, v, w\}$ is a basis of $\mathbb{R}^3$. By Gram-Schmidt process, we can find an orthonormal basis $\{\epsilon_i\}$ based on u, v, w, and there exists an upper triangular matrix P such that

$$(u, v, w) = (\epsilon_1, \epsilon_2, \epsilon_3)P = (\epsilon_1, \epsilon_2, \epsilon_3) \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ & p_{22} & p_{23} \\ & & p_{33} \end{bmatrix}$$

Hence,

$$\begin{aligned} G = \begin{bmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{bmatrix} &= \begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot \begin{bmatrix} u & v & w \end{bmatrix} \\ &= P^T \cdot \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} \cdot \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 \end{bmatrix} \cdot P \\ &= P^T \cdot I \cdot P = P^T P \end{aligned}$$

Therefore, $|G| = |P|^2 = p_{11}^2 \cdot p_{22}^2 \cdot p_{33}^2$.
On the other hand,

$$\begin{aligned} V(u,v,w) &= |(u \wedge v) \cdot w| \\ &= |(p_{11}\epsilon_1 \wedge (p_{12}\epsilon_1 + p_{22}\epsilon_2)) \cdot (p_{13}\epsilon_1 + p_{23}\epsilon_2 + p_{33}\epsilon_3)| \\ &= |p_{11}p_{22}(\epsilon_1 \wedge \epsilon_2) \cdot (p_{13}\epsilon_1 + p_{23}\epsilon_2 + p_{33}\epsilon_3)| \\ &= |p_{11}p_{22}\epsilon_3 \cdot (p_{13}\epsilon_1 + p_{23}\epsilon_2 + p_{33}\epsilon_3)| \\ &= |p_{11}p_{22}p_{33}| \end{aligned}$$

So we have $V^2 = p_{11}^2 \cdot p_{22}^2 \cdot p_{33}^2 = |G| = \begin{vmatrix} u\cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}$. $\square$

1-5 Ex.1
Given the parametrized curve (helix)

$$\alpha(s) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}), s \in \mathbb{R}$$

where $c^2 = a^2 + b^2$,
a. Show that the parameter s is the arc length.
b. Determine the curvature and the tortion of α.
c. Determine the osculating plane of α.
d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\frac{\pi}{2}$.
e. Show that the tangent line of α make a constant angle with the z axis.

Solution.
a. We only need to verify that $|\alpha(s)| \equiv 1$.

$$\alpha'(s) = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c})$$

$$|\alpha'(s)| = \sqrt{(-\frac{a}{c}\sin\frac{s}{c})^2 + (\frac{a}{c}\cos\frac{s}{c})^2 + (\frac{b}{c})^2} = \sqrt{\frac{a^2+b^2}{c^2}} = 1 \quad \square$$

b.

$$\alpha''(s) = (-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0)$$

$$k(s) = |\alpha''(s)| = \frac{|a|}{c^2}$$

$$n(s) = (-sgn(a) \cdot \cos\frac{s}{c}, -sgn(a) \cdot \sin\frac{s}{c}, 0)$$

$$b(s) = \alpha'(s) \wedge n(s) = (sgn(a) \cdot \frac{b}{c}\sin\frac{s}{c}, -sgn(a) \cdot \frac{b}{c}\cos\frac{s}{c}, \frac{a}{c})$$

$$b'(s) = (sgn(a) \cdot \frac{b}{c^2}\cos\frac{s}{c}, sgn(a) \cdot \frac{b}{c^2}\sin\frac{s}{c}, 0)$$

Hence we have $\tau(s) = \frac{b}{c^2}$. $\square$

c. The osculating plane of α is the plane spanned by $t(s)$ and $n(s)$. So the normal vector of the osculating plane is $b(s)$. Given $s \in \mathbb{R}$, the osculating plane at s is defined by the equation

$$sgn(a) \cdot \frac{b}{c}\sin\frac{s}{c}(x - a\cos\frac{s}{c}) - sgn(a) \cdot \frac{b}{c}\cos\frac{s}{c}(y - a\sin\frac{s}{c}) + \frac{a}{c}(z - b\frac{s}{c}) = 0$$

$\square$

d. Note that $n(s) \cdot (0,0,1) = 0$. $\square$

e. Note that $t(s) \cdot (0,0,1) = \frac{b}{c}$ for all $s \in \mathbb{R}$. $\square$

1-5 Ex.2
Show that the tortion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}$$

Solution.
Since $b'(s) = \tau(s)n(s)$,

$$\begin{aligned}
\tau(s) &= b'(s) \cdot n(s) = (t(s) \wedge n(s))' \cdot n(s) \\
&= (t'(s) \wedge n(s) + t(s) \wedge n'(s)) \cdot n(s) \\
&= (t(s) \wedge n'(s)) \cdot n(s) \\
&= (t(s) \wedge (\frac{\alpha''(s)}{k(s)})') \cdot \frac{\alpha''(s)}{k(s)} \\
&= \frac{(\alpha'(s) \wedge \alpha'''(s)) \cdot \alpha''(s)}{|k(s)|^2} \\
&= -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2} \quad \square
\end{aligned}$$

1-5 Ex.4
Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.

Solution.
Suppose the fixed point is denoted by p_0. Then given any $s \in I$, $p_0 - \alpha(s) = \lambda(s) \cdot n(s)$, where $0 \leq \lambda(s) \leq 1$. Take derivatives of both sides of the equation, we have

$$-t(s) = \lambda(s)n'(s) = \lambda(s)(-k(s)t(s) - \tau(s)b(s)) + \lambda'(s)n(s)$$

Since $t(s)$ is always perpendicular to $b(s)$ and $n(s)$, it follows that $\tau(s) = 0$ and $\lambda'(s) = 0$, so $t(s) = \lambda(s)k(s)t(s)$, $\lambda(s) = \frac{1}{k(s)} = r$ where r is a constant. Therefore,

$$|p_0 - \alpha(s)| = |\lambda(s)n(s)| = r$$

That is, the trace of $\alpha(s)$ is contained in a circle centered at the point p_0 with radius r.

1-5 Ex.9
Given a differentiable function $k(s)$, $s \in I$, show that the parametrized plane curve having $k(s) = k$ as curvature is given by

$$\alpha(s) = (\int \cos\theta(s)ds + a, \int \sin\theta(s)ds + b),$$

where

$$\theta(s) = \int k(s)ds + \phi$$

Solution.
Note that

$$\alpha''(s) = (-\sin(\theta(s))\theta'(s), \cos(\theta(s))\theta'(s))$$
$$|\alpha''(s)| = |\theta'(s)| = |k(s)| \quad \square$$

1-5 Ex.12
Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve (not necessarily by arc length) and let $\beta : J \to \mathbb{R}^3$ be a

reparametrization of $\alpha(I)$ by the arc length $s = s(t)$, measured from $t_0 \in I$. Let $t = t(s)$ be the inverse function of s and set $\frac{d\alpha}{dt} = \alpha'$, $\frac{d^2\alpha}{dt^2} = \alpha''$, etc. Prove that

a.

$$\frac{dt}{ds} = \frac{1}{|\alpha'|}, \quad \frac{d^2t}{ds^2} = -\frac{\alpha' \cdot \alpha''}{|\alpha'|^4}$$

b. The curvature of α at $t \in I$ is

$$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}.$$

c. The tortion of α at $t \in I$ is

$$\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}.$$

d. If $\alpha : I \to \mathbb{R}^2$ is a plane curve $\alpha(t) = (x(t), y(t))$, the signed curvature of α at t is

$$k(t) = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}}.$$

Solution.

a.

$$\frac{dt}{ds} = (\frac{ds}{dt})^{-1} = \frac{1}{|\alpha'|}$$

$$\frac{d^2t}{ds^2} = \frac{d}{ds}\frac{1}{|\alpha'(t)|} = -\frac{1}{|\alpha'(t)|^2}\frac{d}{ds}|\alpha'(t)| = -\frac{1}{|\alpha'(t)|^2} \cdot \frac{\alpha'(t) \cdot \alpha''(t)}{|\alpha'(t)|}\frac{dt}{ds} = -\frac{\alpha' \cdot \alpha''}{|\alpha'|^4} \quad \square$$

b. Note that

$$\alpha'(t) = \frac{d\beta(s)}{dt} = \frac{d\beta(s)}{ds} \cdot \frac{ds}{dt} = \beta'(s)\frac{ds}{dt}$$

$$\alpha''(t) = \frac{d\alpha'(t)}{dt} = \beta'(s)\frac{d^2s}{dt^2} + \beta''(s)(\frac{ds}{dt})^2$$

Thus

$$\begin{aligned} |\alpha'(t) \wedge \alpha''(t)| &= |(\beta'(s)\frac{ds}{dt} \wedge (\beta''(s)\frac{ds}{dt})^2)| \\ &= (\frac{ds}{dt})^3 \cdot |\beta'(s) \wedge \beta''(s)| \\ &= (|\alpha'(t)|)^3 \cdot k_\beta(s(t)) \\ &= (|\alpha'(t)|)^3 \cdot k_\alpha(t) \end{aligned}$$

So we have

$$k(t) = k_\alpha(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3} \quad \square$$

c.

$$\begin{aligned} \tau_\alpha(t) = \tau_\beta(s(t)) &= -\frac{\beta'(s) \wedge \beta''(s) \cdot \beta'''(s)}{|k_\beta(s)|^2} \\ &= -\frac{|\beta'(s)|^6 \cdot (\beta'(s) \wedge \beta''(s)) \cdot \beta'''(s)}{|\beta'(s) \wedge \beta''(s)|^2} \\ &= -\frac{(\frac{\alpha'}{|\alpha'|} \wedge \frac{\alpha''}{|\alpha'|^2}) \cdot \frac{\alpha'''}{|\alpha'|^3}}{|\frac{\alpha'}{|\alpha'|} \wedge \frac{\alpha''}{|\alpha''|}|^2} \\ &= -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2} \quad \square \end{aligned}$$

d. By the conclusion of **b**, we have

$$|k(t)| = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3} = \frac{|(x', y') \wedge (x'', y'')|}{(x'^2 + y'^2)^{\frac{3}{2}}} = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{\frac{3}{2}}}$$

According to the definition of signed curvature, $k(t) > 0$ when $det(\alpha', \alpha'') > 0$, $k(t) < 0$ when $det(\alpha', \alpha'') < 0$. Hence

$$k(t) = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}} \quad \square$$

1-5 Ex.13

Assume that $\tau(s) \neq 0$ and $k'(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$R^2 + (R')^2 T^2 = const.$$

where $R = \frac{1}{k}$, $T = \frac{1}{\tau}$, and R' is the derivative of R relative to s.

Solution.

Without loss of generality, we can assume that the sphere is centered at the origin.

"$\Rightarrow$":

Suppose $\alpha(I)$ lies on a sphere, then there exists some constant C such that

$$|\alpha(s)|^2 = C^2$$

Take derivatives on both sides of the equation, we have

$$\alpha(s) \cdot \alpha'(s) = \alpha(s) \cdot t(s) = 0$$

$$\alpha(s) \cdot \alpha''(s) + |\alpha'(s)|^2 = k(s)\alpha(s) \cdot n(s) + 1 = 0$$

$$\alpha(s) \cdot \alpha'''(s) + 3\alpha'(s) \cdot \alpha''(s) = \alpha(s) \cdot \alpha'''(s) = 0$$

For each $s \in I$, we can write $\alpha(s)$ in the form of

$$\alpha(s) = c_1 t(s) + c_2 n(s) + c_3 b(s)$$

The first equation above implies that $c_1 = 0$, the second equation implies that $c_2 = -\frac{1}{k(s)}$.

Also note that,

$$\begin{aligned}\alpha'''(s) &= (k(s)n(s))' = k'(s)n(s) + k(s)n'(s) = k'(s)n(s) - k^2(s)t(s) - k(s)\tau(s)b(s) \\ &= -k^2(s) \cdot t(s) + k'(s) \cdot n(s) - k(s)\tau(s) \cdot b(s)\end{aligned}$$

Thus the third equation implies that

$$c_2 \cdot k'(s) - k(s)\tau(s)c_3 = -\frac{k'(s)}{k(s)} - k(s)\tau(s)c_3 = 0$$

It follows

$$c_3 = -\frac{k'(s)}{k^2(s)\tau(s)}$$

Thus we have

$$\alpha(s) = -\frac{1}{k(s)}n(s) - \frac{k'(s)}{k^2(s)\tau(s)}b(s) = -Rn + R'Tb$$

And

$$|\alpha(s)|^2 = R^2 + (R'T)^2 = C^2 \quad \square$$

"$\Leftarrow$":

Let $\beta(s) = \alpha(s) + Rn - R'Tb$.

First take derivatives on $R^2 + (R'T)^2 = C^2$, we get

$$RR' + (R'T)(R'T)' = 0$$

Then, note that

$$\begin{aligned}
\beta'(s) &= t(s) + R'n + Rn' - (R'T)'b - (R'T)b' \\
&= t + R'n + R(-kt - \tau b) - (R'T)'b - (R'T)\tau n \\
&= t + R'n - R(\frac{t}{R} + \frac{b}{T}) - (R'T)'b - (R'T)\frac{n}{T} \\
&= t + R'n - t - \frac{R}{T}b - (R'T)'b - R'n \\
&= -\frac{R}{T}b - (R'T)'b = -b(\frac{R}{T} + (R'T)')
\end{aligned}$$

Hence

$$\beta'(s) \cdot R'T = -b(RR' + (R'T)(R'T)') = 0$$

Since $k' \neq 0$, $\tau \neq 0$, it implies $\beta'(s) = 0$ and thus $\beta(s)$ is a constant $p_0 \in \mathbb{R}^3$. So we have

$$|\alpha - p_0| = |\alpha - \beta| = C \quad \square$$