Homework 4

Mar 22, 2019

3-2 Ex.2
Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

Solution.
Suppose that S is tangent to the plane P along the curve C. For each $p \in C$, suppose that the tangent vector of C at p is $w = ae_1 + be_2$ where e_1, e_2 are eigenvectors of dN_p and the corresponding eigenvalues are $-k_1, -k_2$. Since S is tangent to the plane P along the curve so $dN_p(w) = -k_1 ae_1 - k_2 be_2 = 0$, hence $k_1 = 0$ or $k_2 = 0$, i.e. p is parabolic or planar. □

3-2 Ex.5
Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi}\int_0^\pi k_n(\theta)d\theta$$

where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

Solution.
Suppose that the eigenvectors of dN_p are e_1, e_2, then

$$\begin{aligned}
\frac{1}{\pi}\int_0^\pi k_n(\theta)d\theta &= \frac{1}{\pi}\int_0^\pi k_n(e_1\cos\theta + e_2\sin\theta)d\theta \\
&= \frac{1}{\pi}\int_0^\pi -\langle dN_p(e_1\cos\theta + e_2\sin\theta), e_1\cos\theta + e_2\sin\theta\rangle d\theta \\
&= \frac{1}{\pi}\int_0^\pi \langle k_1\cos\theta e_1 + k_2\sin\theta e_2, e_1\cos\theta + e_2\sin\theta\rangle d\theta \\
&= \frac{1}{\pi}\int_0^\pi (k_1\cos^2\theta + k_2\sin^2\theta)d\theta \\
&= \frac{1}{\pi}\int_0^\pi (k_1\frac{1+\cos 2\theta}{2} + k_2\frac{1-\sin 2\theta}{2})d\theta \\
&= \frac{1}{\pi}\int_0^\pi \frac{k_1+k_2}{2}d\theta \\
&= \frac{k_1+k_2}{2}
\end{aligned}$$

□

3-2 Ex.9
Prove that
a. The image $N \circ \alpha$ by the Gauss map $N : S \to S^2$ of a parametrized regular curve $\alpha : I \to S$ which contains no planar or parabolic points is a parametrized regular curve on the surface S^2 (called the spherical image of α).
b. If $C = \alpha(I)$ is a line of curvature, and k it its curvature at p, then

$$k = |k_n k_N|,$$

where k_n is the normal curvature at p along the tangent line of C and k_N is the curvature of the spherical image $N(C) \subset S^2$ at $N(p)$.

Solution.
a. Let's denote $N \circ \alpha$ by β. Note that

$$\beta'(s) = dN_p(\alpha'(s)) = dN_p(e_1 \cos\theta + e_2 \sin\theta) = -k_1 e_1 \cos\theta - k_2 e_2 \sin\theta$$

where $p = \alpha(s)$, then

$$\langle \beta'(s), \beta'(s) \rangle = k_1^2 \cos^2\theta + k_2^2 \sin^2\theta > 0$$

since $k_1 \neq 0, k_2 \neq 0$. Therefore β is regular. □

b. WLOG we can assume that $dN(\alpha'(s)) = -k_1 \alpha'(s)$, then

$$k_n = -\langle dN_p(\alpha'(s)), \alpha'(s) \rangle = \langle k_1 \alpha'(s), \alpha'(s) \rangle = k_1$$

Also note that

$$\begin{aligned} k_N &= \frac{|\beta'(s) \wedge \beta''(s)|}{|\beta'(s)|^3} \\ &= \frac{|(-k_1\alpha'(s)) \wedge (-k_1\alpha''(s))|}{|k_1|^3} \\ &= \frac{|k_1^2 k|}{|k_1|^3} \end{aligned}$$

Therefore

$$|k_n k_N| = |k_1 \frac{|k_1|^2 k}{|k_1|^3}| = k$$

□

3-2 Ex.15(Theorem of Joachimstahl.)
Suppose that S_1 and S_2 intersect along a regular curve C and make an angle $\theta(p)$, $p \in C$. Assume that C is a line of curvature of S_1. Prove that $\theta(p)$ is constant if and only if C is a line of curvature of S_2.

Solution.
$" \Rightarrow "$:
Suppose that $\theta(p)$ is constant. Since we know that C is a line of curvature of S_1, so the tangent vector denoted by w of C at p satisfies

$$dN_{1p}(w) = -kw$$

for some principle curvature k. Since we know that $\theta(p)$ is constant, then $\langle N_1(p), N_2(p) \rangle$ is constant. Let $f(t) = \langle N_1(\alpha(t)), N_2(\alpha(t)) \rangle$, where $\alpha(I) = C$, $\alpha(0) = p$, $\alpha'(0) = w$. Then

$$f'(t) = \langle dN_{1p}(\alpha'(t)), N_2(\alpha(t)) \rangle + \langle N_1(\alpha(t)), dN_{2p}(\alpha'(t)) \rangle$$

Particularly,

$$\begin{aligned} f'(0) &= \langle -kw, N_2(p) \rangle + \langle N_1(p), dN_{2p}(w) \rangle \\ &= \langle N_1(p), dN_{2p}(w) \rangle = 0 \end{aligned}$$

Thus $dN_{2p}(w) \in T_p(S_1)$. And we have known that $dN_{2p}(w) \in T_p(S_2)$. Therefore $dN_{2p}(w) \in T_p(S_1) \cap T_p(S_2)$, and it follows

$$dN_{2p}(w) = \lambda w$$

So w is a principal direction at p. Since p is arbitrary, this shows that C is a line of curvature.

$" \Leftarrow "$:
Suppose that C is a line of curvature of S_2, then we have

$$dN_{1p}(w) = -k_1 w, \forall w \in T_p(S_1)$$

$$dN_{2p}(w) = -k_2 w, \forall w \in T_p(S_2)$$

for some $k_1, k_2 \in \mathbb{R}$. Suppose C is parametrized by α, then let

$$f(t) = \langle N_{1p}(\alpha(t)), N_{2p}(\alpha(t))\rangle$$

It follows

$$\begin{aligned} f'(t) &= \langle dN_{1p}(\alpha'(t)), N_{2p}(\alpha(t))\rangle + \langle N_{1p}(\alpha(t)), dN_{2p}(\alpha'(t))\rangle \\ &= \langle -k_1\alpha'(t), N_{2p}(\alpha(t))\rangle + \langle N_{1p}(\alpha(t)), -k_2\alpha'(t)\rangle \\ &= 0 + 0 = 0 \end{aligned}$$

Thus $f(t) \equiv 0$, so $\theta(p)$ is a constant. □

3-2 Ex.17

Show that if $H \equiv 0$ on S and S has no planar points, then the Gauss map $N : S \to S^2$ has the following property:

$$\langle dN_p(w_1), dN_p(w_2)\rangle = -K(p)\langle w_1, w_2\rangle$$

for all $p \in S$ and all $w_1, w_2 \in T_p(S)$. Show that the above condition implies that the angle of two intersecting curves on S and the angle of their spherical images are equal up to a sign.

Solution.

We can assume that the eigenvalues of dN_p are k and $-k$ and the corresponding eigenvectors are e_1, e_2. Suppose $w_1 = e_1 \cos\theta_1 + e_2 \sin\theta_1$, $w_2 = e_1 \cos\theta_2 + e_2 \sin\theta_2$. Then

$$\begin{aligned} \langle dN_p(w_1), dN_p(w_2)\rangle &= \langle ke_1 \cos\theta_1 - ke_2 \sin\theta_1, ke_1 \cos\theta_1 - ke_2 \sin\theta_2\rangle \\ &= k^2 \cos\theta_1 \cos\theta_2 + k^2 \sin\theta_1 \sin\theta_2 \end{aligned}$$

while

$$\begin{aligned} -K(p)\langle w_1, w_2\rangle &= -K(p)\langle e_1 \cos\theta_1 + e_2 \sin\theta_1, e_1 \cos\theta_2 + e_2 \sin\theta_2\rangle \\ &= k^2(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) \end{aligned}$$

□

3-2 Ex.18

Let $\lambda_1, ...\lambda_m$ be the normal curvatures at $p \in S$ along directions making angles $0, \frac{2\pi}{m}, ..., (m-1)\frac{2\pi}{m}$ with a principal direction, $m > 2$. Prove that

$$\lambda_1 + ... + \lambda_m = mH,$$

where H is the mean curvature at p.

Solution.

Note that

$$\begin{aligned} \lambda_k &= k_1 \cos^2\theta_k + k_2 \sin^2\theta_k \\ &= k_1 \frac{1 + \cos(2\theta_k)}{2} + k_2 \frac{1 - \cos(2\theta_k)}{2} \\ &= H + \frac{k_1 - k_2}{2}\cos(2\theta_k) \end{aligned}$$

Then

$$\begin{aligned} \lambda_1 + \lambda_2 + ... + \lambda_m &= mH + \frac{k_1 - k_2}{2}\big[\cos(2\theta_1) + \cos(2\theta_2) + ... + \cos(2\theta_m)\big] \\ &= mH + \frac{k_1 - k_2}{4}\sum_{k=1}^{m}\big[\cos(2\theta_k) + \cos(2\theta_{m-k+1})\big] \\ &= mH + \frac{k_1 - k_2}{4}\sum_{k=1}^{m}\big[\cos(\frac{4k\pi}{m}) + \cos(\frac{4(m-k+1)\pi}{m})\big] \\ &= mH + \frac{k_1 - k_2}{4}\sum_{k=1}^{m}\big[\cos(\frac{4k\pi}{m}) - \cos(\frac{4(k-1)\pi}{m})\big] \\ &= mH + \frac{k_1 - k_2}{4}\big[\cos(4\pi) - \cos(0)\big] = mH \end{aligned}$$

□

3-2 Ex.19
Let $C \subset S$ be a regular curve in S. Let $p \in C$ and $\alpha(s)$ be a parametrization of C in p by arc length so that $\alpha(0) = p$. Choose in $T_p(S)$ an orthonormal positive basis $\{t, h\}$, where $t = \alpha'(0)$. The geodesic torsion τ_g of $C \subset S$ at p is defined by

$$\tau_g = \langle \frac{dN}{ds}(0), h \rangle$$

Prove that
a. $\tau_g = (k_1 - k_2)\cos\phi \sin\phi$, where ϕ is the angle from e_1 to t and t is the unit tangent vector corresponding to the principal curvature k_1.
b. If τ is the torsion of C, n is the (principal) normal vector of C and $\cos\theta = \langle N, n \rangle$, then

$$\frac{d\theta}{ds} = \tau - \tau_g$$

c. The lines of curvature of S are characterized by having geodesic torsion identically zero.

Solution.
a. Since $t = e_1 \cos\phi + e_2 \sin\phi$, $dN_p(e_1) = -k_1 e_1$.
Note that

$$\begin{aligned}\frac{dN}{ds}(0) &= dN_p(\alpha'(0)) = dN_p(e_1\cos\phi + e_2\sin\phi) = -k_1 e_1 \cos\phi - k_2 e_2 \sin\phi \\ &= -k_1 e_1 \cos\phi - k_1 e_2 \sin\phi + (k_1 - k_2) e_2 \sin\phi \\ &= -k_1 t + (k_1 - k_2) e_2 \sin\phi\end{aligned}$$

Also note that since ϕ is the angle from e_1 to t, so the angle from e_2 to h is ϕ. Thus

$$\tau_g = \langle -k_1 t + (k_1 - k_2) e_2 \sin\phi, h \rangle = (k_1 - k_2)\sin\phi \langle e_2, h \rangle = (k_1 - k_2)\cos\phi\sin\phi$$

□

b. Note that

$$\frac{d\theta}{ds} = \frac{d\theta}{d\cos\theta}\frac{d\cos\theta}{ds} = -\frac{1}{\sin\theta}\frac{d}{ds}\langle N, n \rangle \quad = -\frac{1}{\sin\theta}(\langle \frac{dN}{ds}, n \rangle + \langle N, \frac{dn}{ds} \rangle)$$

where

$$\langle \frac{dN}{ds}, n \rangle = \langle \frac{dN}{ds}, h\sin\theta \rangle = \tau_g \sin\theta, \quad \frac{dn}{ds} = -kt - \tau b, \quad \langle N, b \rangle = \sin\theta$$

Thus

$$\frac{d\theta}{ds} = -\frac{1}{\sin\theta}(\tau_g \sin\theta - \tau\sin\theta) = \tau - \tau_g$$

□

c. By the conclusion of **a.** we know that the geodesic curvature of C is

$$\tau_g = (k_1 - k_2)\cos\phi\sin\phi$$

If C is a line of curvature of S, then $t = e_1$ or $t = e_2$, i.e. $\phi = 0$ or $\phi = \frac{\pi}{2}$, leading to $\tau_g = 0$. □

3-3 Ex.5
Consider the parametrized surface (Enneper's surface)

$$\mathcal{X}(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$$

and show that
a. The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, F = 0$$

b. The coefficients of the second fundamental form are

$$e = 2, g = -2, f = 0$$

c. The principal curvatures are

$$k_1 = \frac{2}{(1+u^2+v^2)^2}, k_2 = -\frac{2}{(1+u^2+v^2)^2}$$

d. The lines of curvature are the coordinate curves.

e. The asymptotic curves are $u + v = const.$, $u - v = const.$

Solution.
a. Note that

$$\mathcal{X}_u = (1 - u^2 + v^2, 2uv, 2u)$$
$$\mathcal{X}_v = (2uv, 1 - v^2 + u^2, -2v)$$

So we have

$$E = \langle \mathcal{X}_u, \mathcal{X}_u \rangle = (1 - u^2 + v^2)^2 + 4u^2(1 + v^2) = (1 + u^2 + v^2)^2$$
$$G = \langle \mathcal{X}_v, \mathcal{X}_v \rangle = (1 + u^2 + v^2)^2$$
$$F = \langle \mathcal{X}_u, \mathcal{X}_v \rangle = 4uv - 4uv = 0$$

□

b. Note that

$$\mathcal{X}_{uu} = (-2u, 2v, 2)$$
$$\mathcal{X}_{vv} = (2u, -2v, -2)$$
$$\mathcal{X}_{uv} = (2v, 2u, 0)$$

So we have

$$\begin{aligned} e &= \frac{(\mathcal{X}_u, \mathcal{X}_v, \mathcal{X}_{uu})}{\sqrt{EG - F^2}} = \frac{2[u^4 + 2u^2(1+v^2) + (1+v^2)^2]}{\sqrt{EG - F^2}} \\ &= \frac{2(1+u^2+v^2)^2}{(1+u^2+v^2)^2} = 2 \end{aligned}$$

Similarly we have $g = -2$.
Also,

$$f = \frac{(\mathcal{X}_u, \mathcal{X}_v, \mathcal{X}_{uv})}{\sqrt{EG - F^2}} = \frac{-4u^3v - 4uv - 4uv^3 + 4uv + 4u^3v + 4uv^3}{\sqrt{EG - F^2}} = 0$$

□

c. Since $F = f = 0$,

$$k_1 = \frac{e}{E} = \frac{2}{(1+u^2+v^2)^2}, k_2 = \frac{g}{G} = \frac{-2}{(1+u^2+v^2)^2}$$

□

d. Since $F = f = 0$, the lines of curvatures are the coordinate curves. □

e. We have known that the principal directions are coordinates, thus

$$dN_p = \begin{bmatrix} k_1 & \\ & k_2 \end{bmatrix}$$

And the equation

$$\langle dN_p(u',v'),(u',v')\rangle = k_1u'^2 + k_2v'^2 = \frac{2(u'^2 - v'^2)}{(1+u^2+v^2)^2} = 0$$

implies that $u' - v' = 0$ or $u' + v' = 0$. So we have $u - v = const.$ or $u + v = const.$. $\square$

3-3 Ex.7(Surfaces of Revolution with Constant Curvature)
$(\phi(v)\cos u, \phi(v)\sin u, \psi(v))$, $\phi \neq 0$ is given as a surface of revolution with constant Gaussian curvature K. To determine the functions ϕ and ψ, choose the parameter v in such a way that $(\phi')^2 + (\psi')^2 = 1$ (geometrically, this means that v is the arc length of the generating curve $(\phi(v), \psi(v))$).
Show that
a. ϕ satisfies $\phi'' + K\phi = 0$ and ψ is given by

$$\psi = \int \sqrt{1-(\phi')^2}dv$$

thus, $0 < u < 2\pi$, and the domain of v is such that the last integral makes sense.

b. All surfaces of revolution with constant curvature $K = 1$ which intersect perpendicularly the plane xOy are given by

$$\phi(v) = C\cos v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2\sin^2 v}dv$$

where C is a constant ($C = \phi(0)$). Determine the domain of v and draw a rough sketch of the profile of the surface in the xz plane for the cases $C = 1$, $C > 1$, $C < 1$. Observe that $C = 1$ gives a sphere.

c. All surfaces of revolution with constant curvature $K = -1$ may be given by one of the following types:
1. $\phi(v) = C\cosh v, \psi(v) = \int_0^v \sqrt{1 - C^2\sinh^2 v}dv$.
2. $\phi(v) = C\sinh v, \psi(v) = \int_0^v \sqrt{1 - C^2\cosh^2 v}dv$.
3. $\phi(v) = e^v, \psi(v) = \int_0^v \sqrt{1 - e^{2v}}dv$.

e. The only surfaces of revolution with $K \equiv 0$ are the right circular cylinder, the right circular cone, and the plane.

Solution.
a. First note that

$$\mathcal{X}_u = (-\phi(v)\sin u, \phi(v)\cos u, 0)$$
$$\mathcal{X}_v = (\phi'(v)\cos u, \phi'(v)\sin u, \psi'(v))$$
$$E = \phi^2(v), F = 0, G = \phi'^2(v) + \psi'^2(v) = 1$$

Moreover,

$$\mathcal{X}_{uu} = (-\phi(v)\cos u, -\phi(v)\sin u, 0)$$
$$\mathcal{X}_{vv} = (\phi''(v)\cos u, \phi''(v)\sin u, \psi''(v))$$
$$\mathcal{X}_{uv} = (-\phi'(v)\sin u, \phi'(v)\cos u, 0)$$
$$e = \frac{(\mathcal{X}_u, \mathcal{X}_v, \mathcal{X}_{uu})}{\sqrt{EG - F^2}} = -\phi(v)\psi'(v)$$
$$f = \frac{(\mathcal{X}_u, \mathcal{X}_v, \mathcal{X}_{uv})}{\sqrt{EG - F^2}} = 0$$
$$g = \frac{(\mathcal{X}_u, \mathcal{X}_v, \mathcal{X}_{vv})}{\sqrt{EG - F^2}} = \phi''(v)\psi'(v) - \phi'(v)\psi''(v)$$

Then

$$K = \frac{eg - f^2}{EG - F^2} = \frac{\phi\psi'(\phi'\psi'' - \phi''\psi')}{\phi^2}$$
$$= \frac{\psi'(\phi'\psi'' - \phi''\psi')}{\phi}$$

Since we know that $\phi'^2 + \psi'^2 = 1$, by differentiating this equation we obtain $\phi'\phi'' + \psi'\psi'' = 0$, thus

$$K = -\frac{\psi'^2\phi'' + \phi'^2\phi''}{\phi} = -\frac{\phi''}{\phi}$$

Hence $\phi'' + K\phi = 0$. Also,

$$\psi = \int \psi' dv = \int \sqrt{1 - \phi'^2} dv$$

$\square$

b. We have known that $\phi + \phi'' = 0$, whose solution is

$$\phi(v) = C \cos v$$

where C is a constant. It follows that

$$\psi(v) = \int_0^v \sqrt{1 - \phi'^2} dv = \int_0^v \sqrt{1 - C^2 \sin^2 v} dv$$

It's easy to see that $v \in (-\arcsin \frac{1}{|C|}, \arcsin \frac{1}{|C|})$. $\square$

c. Similarly, the equations

$$\phi'' - \phi = 0, \phi'^2 + \psi'^2 = 1$$

have the following three types of solution:
1. $\phi(v) = C \cosh v, \psi(v) = \int_0^v \sqrt{1 - C^2 \sinh^2 v} dv$.
2. $\phi(v) = C \sinh v, \psi(v) = \int_0^v \sqrt{1 - C^2 \cosh^2 v} dv$.
3. $\phi(v) = e^v, \psi(v) = \int_0^v \sqrt{1 - e^{2v}} dv$. $\square$

e. It's easy to see that $\phi'' = 0$ has the following solutions:
1. $\phi \equiv C, \psi(v) = v$, where C is a constant, S is a cylinder.
2. $\phi(v) = kv, \psi(v) = \sqrt{1 - k^2} v$, where $k \in (-1, 0) \cup (0, 1)$, S is a cone.
3. $\phi(v) = v, \psi \equiv C$, where C is a constant, S is a plane. $\square$

3-3 Ex.13
Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the map defined by $F(p) = cp$, $p \in \mathbb{R}^3$, c a positive constant. Let $S \subset \mathbb{R}^3$ be a regular surface and set $F(S) = \bar{S}$. Show that $\bar{S}$ is a regular surface, and find formulas relating the Gaussian and mean curvatures, K and H, of S with the Gaussian and mean curvatures, $\bar{K}$ and $\bar{h}$, of $\bar{S}$.

Solution.
Suppose S is parametrized by $\mathcal{X}(u, v)$, then $\bar{S}$ is parametrized by $\bar{\mathcal{X}}(u, v) = c\mathcal{X}(u, v)$. It follows that

$$\bar{\mathcal{X}}_u(u, v) = c\mathcal{X}_u(u, v), \quad \bar{\mathcal{X}}_v(u, v) = c\mathcal{X}_v(u, v)$$

$$\bar{\mathcal{X}}_{uu}(u, v) = c\mathcal{X}_{uu}(u, v), \quad \bar{\mathcal{X}}_{vv}(u, v) = c\mathcal{X}_{vv}(u, v)$$

Thus,

$$\bar{E} = c^2 E, \quad \bar{F} = c^2 F, \quad \bar{G} = c^2 G$$

And

$$\bar{e} = \frac{(\bar{\mathcal{X}}_u, \bar{\mathcal{X}}_v, \bar{\mathcal{X}}_{uu})}{\sqrt{\bar{E}\bar{G} - \bar{F}^2}} = \frac{(c\mathcal{X}_u, c\mathcal{X}_v, c\mathcal{X}_{uu})}{c^2\sqrt{EG - F^2}} = ce$$

Similarly, we have $\bar{f} = cf, \bar{g} = cg$. Finally we obtain

$$\bar{K} = \frac{\bar{e}\bar{g} - \bar{f}^2}{\bar{E}\bar{G} - \bar{F}^2} = \frac{c^2(eg - f^2)}{c^4(EG - F^2)} = \frac{K}{c^2}$$

$$\bar{H} = \frac{1}{2}\frac{\bar{e}\bar{G} - 2\bar{f}\bar{F} + \bar{g}\bar{E}}{\bar{E}\bar{G} - \bar{F}^2} = \frac{1}{2}\frac{c^3(eG - 2fF + gE)}{c^4(EG - F^2)} = \frac{H}{c}$$

$\square$

3-3 Ex.16
Show that a surface which is compact has an elliptic point.

Solution.
Since S is compact, S is bounded. Therefore, there are spheres of $\mathbb{R}^3$, centered in a fixed point $O \in \mathbb{R}^3$, such that S is contained in the interior of the region bounded by any of them. Consider the set of all such spheres. Let r be the infimum of their radius and let $\Sigma \subset \mathbb{R}^3$ be a sphere of radius r centered in O. It is clear that Σ and p has only one common point, say p, since S is compact. The tangent plane to Σ at p has only the common point p with S, in a neighborhood of p. Therefore, Σ and S are tangent at p. By observing the normal sections at p, it is easy to conclude that any normal curvature of S at p is greater than or equal to the corresponding curvature of Σ at p. Therefore, $K_S(p) \geq K_\Sigma(p) > 0$, and p is an elliptic point, as we desired. $\square$

3-3 Ex.21
Let S be a surface with orientation N. Let $V \subset S$ be an open set in S and let $f : V \subset S \to \mathbb{R}$ be any nowhere-zero differentiable function in V. Let v_1 and v_2 be two differentiable(tangent) vector fields in V such that at each point of V, v_1 and v_2 are orthonormal and $v_1 \wedge v_2 = N$.
a. Prove that the Gaussian curvature K of V is given by

$$K = \frac{\langle d(fN)(v_1) \wedge d(fN)(v_2), fN\rangle}{f^3}$$

b. Apply the above result to show that if f is the restriction of

$$\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

then the Gaussian curvature of the ellipsoid is

$$K = \frac{1}{a^2b^2c^2}\frac{1}{f^4}$$

Solution.
a. By definition, for each $w \in T_p(S)$, choose $\alpha : I \to S$ such that $\alpha(0) = p, \alpha'(0) = w$, then

$$\begin{aligned} d(f(p)N_p)(w) &= \frac{d}{dt}(f(\alpha(t))N(\alpha(t)))|_{t=0} \\ &= df_p(w)N_p + f(p)dN_p(w) \end{aligned}$$

Note that N_p is perpendicular to $T_p(S)$, it follows

$$\begin{aligned} \frac{\langle d(fN)(v_1) \wedge d(fN)(v_2), fN\rangle}{f^3} &= \frac{\langle (df_p(v_1)N_p + f(p)dN_p(v_1)) \wedge (df_p(v_2)N_p + f(p)dN_p(v_2)), fN\rangle}{f^3} \\ &= \frac{\langle f dN_p(v_1) \wedge f dN_p(v_2), fN\rangle}{f^3} \\ &= \frac{f^3\langle dN_p(v_1) \wedge dN_p(v_2), N\rangle}{f^3} \\ &= det(dN_p)\langle v_1 \wedge v_2, N\rangle = K \end{aligned}$$

$\square$

b. By differentiating the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

we obtain

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 0$$

this implies

$$n = (\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})$$

is a normal vector at (x, y, z). And $f = \|n\|$.
So we have

$$N = \frac{n}{f}$$

Then choose $v_1 = (v_{11}, v_{12}, v_{13}), v_2 = (v_{21}, v_{22}, v_{23})$ such that $v_1 \wedge v_2 = N$, it follows

$$\begin{aligned}
K &= \frac{\langle dfN(v_1) \wedge dfN(v_2), fN\rangle}{f^3} = \frac{\langle (\frac{v_{11}}{a^2}, \frac{v_{12}}{b^2}, \frac{v_{13}}{c^2}) \wedge (\frac{v_{21}}{a^2}, \frac{v_{22}}{b^2}, \frac{v_{23}}{c^2}), n\rangle}{f^3} \\
&= \frac{1}{f^3}\begin{vmatrix} \frac{v_{11}}{a^2} & \frac{v_{12}}{b^2} & \frac{v_{13}}{c^2} \\ \frac{v_{21}}{a^2} & \frac{v_{22}}{b^2} & \frac{v_{23}}{c^2} \\ \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \end{vmatrix} = \frac{1}{a^2b^2c^2f^3}\begin{vmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ x & y & z \end{vmatrix} \\
&= \frac{1}{a^2b^2c^2f^3}\langle N, (x, y, z)\rangle \\
&= \frac{1}{a^2b^2c^2f^4}\langle (\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}), (x, y, z)\rangle \\
&= \frac{1}{a^2b^2c^2f^4}
\end{aligned}$$

□

3-3 Ex.22(The Hessian.)
Let $h : S \to \mathbb{R}$ be a differentiable function on a surface S, and let $p \in S$ be a critical point of h(i.e. $dh_p = 0$). Let $w \in T_p(S)$ and let

$$\alpha : (-\epsilon, \epsilon) \to S$$

be a parametrization curve with $\alpha(0) = p$, $\alpha'(0) = w$. Set

$$H_p h(w) = \frac{d^2(h \circ \alpha)}{dt^2}|_{t=0}$$

a. Let $\mathcal{X} : U \to S$ be a parametrization of S at p, and show that (the fact that p is a critical point of h is essential here)

$$H_p h(u'\mathcal{X}_u + v'\mathcal{X}_v) = h_{uu}(p)u'^2 + 2h_{uv}(p)u'v' + h_{vv}(p)v'^2$$

Conclude that $H_p h : T_p(S) \to \mathbb{R}$ is a well-defined(i.e. it does not depend on the choice of $\mathcal{X}$) quadratic form on $T_p(S)$. $H_p h$ is called the Hessian of h at p.

b. Let $h : S \to \mathbb{R}$ be the height function of S relative to $T_p(S)$; that is, $h(q) = \langle q - p, N(p)\rangle$, $q \in S$. Verify that p is a critical point of h and thus that the Hessian $H_p h$ is well defined. Show that if $w \in T_p(S)$, $|w| = 1$, then

$$H_p h(w) = \text{normal curvature at } p \text{ in the direction of } w$$

Conclude that the Hessian at p of the height function relative to $T_p(S)$ is the second fundamental form of S at p.

Solution.
a. Let $h(u, v) = h \circ \alpha$, observe that

$$\frac{d}{dt}(h \circ \alpha)|_{t=0} = dh_p(w) = h_u u' + h_v v'$$

$$\frac{d^2}{dt^2}(h \circ \alpha)|_{t=0} = \frac{d}{dt}(h_u u' + h_v v') = h_u u'' + h_{uu} u'^2 + h_v v'' + h_{vv} v'^2 + 2h_{uv} u' v'$$

Note that $dh_p = (h_u, h_v) = 0$, so

$$\frac{d^2}{dt^2}(h \circ \alpha)|_{t=0} = h_{uu} u'^2 + 2h_{uv} u' v' + h_{vv} v'^2$$

which doesn't depend the choice of $\mathcal{X}$. $\square$

b. Consider $h_\alpha(t) = h \circ \alpha$, where $\alpha(0) = p, \alpha'(0) = w$, note that

$$h'_\alpha(0) = \langle \alpha'(0), N(p) \rangle = 0$$

Thus $dh_p(w) = h'_\alpha(0) = 0$. Since w is arbitrary, $dh_p = 0$. So $H_p h$ is well defined.
Observe that

$$H_p h(w) = \frac{d^2(h \circ \alpha)}{dt^2}|_{t=0} = \frac{d}{dt}\langle \alpha'(t), N(p) \rangle|_{t=0} = \langle \alpha''(0), N_p \rangle = k_n(w)$$

$\square$