Surfaces

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Roughly speaking, a regualr surface in $\mathbb{R}^3$ is obtained by taking pieces of a plane, deforming them, and arranging them in such a way that the resulting figure has no sharp points, edges, or self-intersections and so that it makes sense to speak of a tangent plane at points of the figure.
The idea is to define a set that is, in a certain sense, two-dimensional and that also is smooth enough so that the usual notions of calculus can be extended to it.

Definition 2.1 (Regular Surface)
A subset $S \subset \mathbb{R}^3$ is a regular surface if for each $p \in S$, there exists a neighborhood $V \subset \mathbb{R}^3$, an open set $U \subset \mathbb{R}^2$ and an onto map $\mathcal{X} : U \to V \cap S$ such that
(1) $\mathcal{X}$ is differentiable, i.e. if we write

$$\mathcal{X}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in U$$

Then the functions $x(u, v), y(u, v), z(u, v)$ have continuous partial derivatives of all orders in U.
(2) $\mathcal{X}$ is a homeomorphism, since $\mathcal{X}$ is continuous by condition (1), this means that $\mathcal{X}^{-1} : V \cap S \to U$ is continuous.
(3) (regularity condition) For each $q \in U$, the differential $d\mathcal{X}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

The mapping $\mathcal{X}$ is called a parametrization or system of (local) coordinates in a neighborhood of p. The neighborhood $V \cap S$ of p in S is called a coordinate neighborhood.

To give condition (3) a more familiar form, let us compute the matrix of the linear map $d\mathcal{X}_q$ in the canonical bases $e_1 = (1, 0)$, $e_2 = (0, 1)$ of $\mathbb{R}^2$ with coordinates (u, v) and $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 0, 1)$ of $\mathbb{R}^3$, with coordinates (x, y, z).
Let $q = (u_0, v_0)$, the vector e_1 is tangent to the curve $\alpha : \mathbb{R} \to U \subset \mathbb{R}^2, u \mapsto (u, v_0)$ whose image under $\mathcal{X}$ is the curve

$$\beta : \mathbb{R} \to \mathbb{R}^3, \quad u \mapsto (x(u, v_0), y(u, v_0), z(u, v_0))$$

This image curve (called the coordinate curve $v = v_0$) lies on S and has the tangent vector at $\mathcal{X}(q)$, which is defined by

$$(\frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)) = \frac{\partial \mathcal{X}}{\partial u}(u_0, v_0)$$

By the definition of differential,

$$d\mathcal{X}_q(e_1) = \frac{\partial \mathcal{X}}{\partial u}(u_0, v_0)$$

$$d\mathcal{X}_q(e_2) = \frac{\partial \mathcal{X}}{\partial v}(u_0, v_0)$$

Thus, the matrix of the linear map $d\mathcal{X}_q$ in the referred basis is

$$d\mathcal{X}_q = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}_q$$

Example 2.1 (unit sphere)
The unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

is a regular surface.
Proof.
We first verify that the map $\mathcal{X}_1 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathcal{X}_1(x, y) = (x, y, \sqrt{1 - (x^2 + y^2)}), (x, y) \in U$$

is a parametrization of S^2. Observe that $\mathcal{X}_1(U)$ is the open part of S^2 above the xOy plane.
Since $x^2 + y^2 < 1$, the function $\sqrt{1 - (x^2 + y^2)}$ has continuous partial derivatives of all orders. Thus, $\mathcal{X}_1$ is differentiable and condition (1) holds.
To check condition (2), we observe that $\mathcal{X}_1$ is one-to-one and that $\mathcal{X}_1^{-1}$ is the restriction of the continuous projection $\pi(x, y, z) = (x, y)$ to the set $\mathcal{X}_1(U)$. Thus, $\mathcal{X}_1^{-1}$ is continuous in $\mathcal{X}_1(U)$.
To check condition (3), note that

$$d\mathcal{X}_1 = \begin{pmatrix} 1 & 0 \\ \\ 0 & 1 \\ \\ \dfrac{-x}{\sqrt{1 - (x^2 + y^2)}} & \dfrac{-y}{\sqrt{1 - (x^2 + y^2)}} \end{pmatrix}$$

It's easy to see that $d\mathcal{X}_1$ is one-to-one.
Similarly we can verify the above conditions for other parametrizations like

$$\mathcal{X}_2(x, z) = (x, \sqrt{1 - (x^2 + z^2)}, z)$$

$$\mathcal{X}_3(y, z) = (\sqrt{1 - (y^2 + z^2)}, y, z)$$

Proposition 2.1 (a simple property of regular surface)
Let $S \subset \mathbb{R}^2$ be a regular surface, then S is a locally graph, i.e.
$\forall p \in S$, $\exists V$ an open subset of S containing p such that

$$V = \{(x, y, f(x, y)) : (x, y) \in U \subset \mathbb{R}^2\}$$

or

$$V = \{(x, f(x, z), z) : (x, z) \in U \subset \mathbb{R}^2\}$$

or

$$V = \{(f(y, z), y, z) : (y, z) \in U \subset \mathbb{R}^2\}$$

where U is an open set, $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable map.
Proof.

Take any $p \in S$, there exists a differentiable map $\mathcal{X} : U \subset \mathbb{R}^2 \to V \cap S \subset \mathbb{R}^3$ such that $\mathcal{X}(q) = p$, where $U \subset \mathbb{R}^2, V \subset \mathbb{R}^3$ are open sets. Moreover,

$$d\mathcal{X}_q = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}_q$$

has rank 2.
Without loss of generality, we can first assume that

$$(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}) \text{ and } (\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v})$$

are linearly independent at point q. That is,

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}_q$$

is invertible.
Consider projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $\pi(x, y, z) = (x, y)$. It's easy to verify that π is a differentiable map. Then

$$\pi \circ \mathcal{X} : U \to W = \pi(\mathcal{X}(U))$$

is also a differentiable map, and

$$d(\pi \circ \mathcal{X})_q = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}_q$$

is invertible.
Now we can apply inverse function theorem to $\pi \circ \mathcal{X}$ at q:
Hence, there exists $U_1 \subset U$, let $W_1 = \pi(X(U_1))$, then $\pi \circ \mathcal{X} : U_1 \to W_1$ is a differentiable map and

$$(\pi \circ \mathcal{X})^{-1} : W_1 \to U_1$$

exists, which is also differentiable(hence continuous, and it follows that W_1 is an open set).
Consider another map $\mathcal{Y}$

$$\mathcal{Y} = \mathcal{X} \circ (\pi \circ \mathcal{X})^{-1} : W_1 \to \mathcal{X}(U_1), (x, y) \mapsto (x, y, f(x, y))$$

Therefore, given $p \in S$, we can find an open subset $V = \mathcal{X}(U_1) \subset \mathbb{R}^3$ containing p and an open subset $U = W_1 \subset \mathbb{R}^2$ such that

$$V = \{(x, y, f(x, y)) : (x, y) \in U \subset \mathbb{R}^2\}$$

Definition 2.2 (Principle Curvature)
Let $S \subset \mathbb{R}^3$ be a regular surface.