Homework 2

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2-2 Ex.1
Show that the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is a regular surface, and find parametrizations whose coordinate neighborhoods cover it.

Solution.
For each point $p = (x_0, y_0, z_0) \in S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$, without loss of generality we can assume that $y_0 > 0$, then let $q = (x_0, 0, z_0)$ and U a sufficiently small neighborhood of q in xOz plane. Define $\mathcal{X} : U \to S$ by $\mathcal{X}(x, z) = (x, \sqrt{1 - x^2}, z)$, it's clearly that $\mathcal{X}$ is differentiable, one-to-one, and its inverse is also continuous. Moreover,

$$d\mathcal{X} = \begin{pmatrix} 1 & 0 \\ \dfrac{x}{\sqrt{1-x^2}} & 0 \\ 0 & 1 \end{pmatrix}$$

has rank 2 at any point (x_0, z_0). Thus $\mathcal{X}$ is a parametrization in U. $\square$

2-2 Ex.9
Let V be an open set in the xy plane. Show that the set

$$\{(x, y, z) \in \mathbb{R}^3 : z = 0, (x, y) \in V\}$$

is a regular surface.

Solution.
Let S denote the set above and define $\mathcal{X} : V \to S$ by

$$\mathcal{X}(x, y) = (x, y, 0)$$

Then it's easy to verify that $\mathcal{X}$ satisfies condition 1, 2 and 3. So $\mathcal{X}$ is a parametrization and S is a regular surface. $\square$

2-2 Ex.12
Show that $\mathcal{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathcal{X}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u), a, b, c \neq 0,$$

where $0 < u < \pi$, $0 < v < 2\pi$ is a parametrization for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Describe geometrically the curves $u = const.$ on the ellipsoid.

Solution.
It's easy to see that $\mathcal{X}$ is a diffeomorphism, also note that

$$d\mathcal{X} = \begin{pmatrix} a\cos u\cos v & -a\sin u\sin v \\ b\cos u\sin v & b\sin u\cos v \\ -c\sin u & 0 \end{pmatrix}$$

always has rank 2 for all $u \in (0,\pi), v \in (0, 2\pi)$. Therefore $\mathcal{X}$ is a parametrization.
The curves $u = const.$ is a set of ellipses with axes $2a\sin u$ and $2b\sin u$. $\square$

2-2 Ex.16
One way to define a system of coordinates for the sphere S^2, given by $x^2+y^2+(z-1)^2 = 1$, is to consider the so-called stereographic projection $\pi : S^2 - \{N\} \to \mathbb{R}^2$ which carries a point $p = (x, y, z)$ of the sphere S^2 minus the north pole $N = (0, 0, 2)$ onto the intersection of the xy plane with the straight line which connects N to p. Let $(u, v) = \pi(x, y, z)$, where $(x, y, z) \in S^2 - \{N\}$ and $(u, v) \in xy$ plane.
a. Show that $\pi^{-1} : \mathbb{R}^2 \to S^2$ is given by

$$\pi^{-1} = \begin{cases} x = \frac{4u}{u^2+v^2+4} \\ y = \frac{4v}{u^2+v^2+4} \\ z = \frac{2u^2+2v^2}{u^2+v^2+4} \end{cases}$$

b. Show that it is possible, using stereographic projection, to cover the sphere with two coordinate neighborhoods.

Solution.
a. By definition of π, $(0, 0, 2), (x, y, z)$ and $(u, v, 0)$ are in the same line. Hence

$$(x, y, z-2) = \lambda(u, v, -2), \lambda \in \mathbb{R}$$

Also we know that $x^2 + y^2 + (z-1)^2 = 1$, it follows that

$$\lambda^2u^2 + \lambda^2v^2 + (2\lambda-1)^2 = 1$$

$$\lambda^2u^2 + \lambda^2v^2 + 4\lambda^2 - 4\lambda = 0$$

Note that $\lambda \neq 0$ because $(x, y, z) \neq N$, so we have

$$\lambda = \frac{4}{u^2+v^2+4}$$

Therefore,

$$x = \lambda u = \frac{4u}{u^2+v^2+4}, \quad y = \lambda v = \frac{4v}{u^2+v^2+4}, \quad z = 2-2\lambda = \frac{2u^2+2v^2}{u^2+v^2+4} \quad \square$$

b. The conclusion above shows that for each point p in $S^2 - \{N\}$, we can find a corresponding parametrization. Particularly, for $p = N$, consider map $\mathcal{X} : B(0, 1) \to S^2$ defined by

$$\mathcal{X}(x, y) = (x, y, 1 + \sqrt{1-x^2-y^2}), (x, y) \in B(0, 1)$$

where $B(0, 1)$ is the unit ball in $\mathbb{R}^2$. It's easy to verify that $\mathcal{X}$ is a parametrization. Hence we can cover S^2 using two coordinate neighborhoods. $\square$

2-3 Ex.3
Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to a plane.

Solution.
Let S denote the paraboloid. Consider map $\mathcal{X} : \mathbb{R}^2 \to S$:

$$\mathcal{X}(x, y) = (x, y, x^2 + y^2)$$

whose inverse is $\pi|_S : S \to \mathbb{R}^2$ and it's easy to verify that both of them are differentiable. $\square$

2-3 Ex.6
Prove that the definition of a differentiable map between surfaces does not depend on the parametrizations chosen.

Solution.
We know that if $\mathcal{X}$ and $\mathcal{Y}$ are both parametrizations of a regular surface, then

$$\mathcal{X}^{-1} \circ \mathcal{Y}$$

is a diffeomorphism.
If ϕ is a differentiable map between two regular surface S_1 and S_2, then there exists two parametrizations $\mathcal{X}_1, \mathcal{X}_2$ of two regular surfaces respectively and

$$\mathcal{X}_2^{-1} \circ \phi \circ \mathcal{X}_1$$

is a differentiable. Suppose that $\mathcal{Y}_1, \mathcal{Y}_2$ are another two parametrizations of S_1, S_2 respectively, then both $\mathcal{Y}_2^{-1} \circ \mathcal{X}_2$ and $\mathcal{X}_1^{-1} \circ \mathcal{Y}_1$ are diffeomorphism so that

$$\mathcal{Y}_2^{-1} \circ \phi \circ \mathcal{Y}_1 = \mathcal{Y}_2^{-1} \circ \mathcal{X}_2 \circ \mathcal{X}_2^{-1} \circ \phi \circ \mathcal{X}_1 \circ \mathcal{X}_1^{-1}$$

is also a diffeomorphism, which shows that the differentiable map between surfaces is well-defined. $\square$

2-3 Ex.10
Let C be a plane regular curve which lies in one side of a straight line r of the plane and meets r at the points p, q. What conditions should C satisfy to ensure that the rotation of C about r generates an extended (regular) surface of revolution.

Solution.
C should be smooth and the tangent vector at p, q should be perpendicular to the straight line r.

2-3 Ex.14
Let $A \subset S$ be a subset of a regular surface S. Prove that A is itself a regular surface if and only if A is open in S, that is, $A = U \cap S$, where U is an open set in $\mathbb{R}^3$.

Solution.
" $\Rightarrow$ ":
Given that A is a regular surface, then for each $p \in A$, we can find a neighborhood $V \subset S$ of p thus A is open.
" $\Leftarrow$ ":
Assume that A is open. Since we know that S is a regular surface, for each point $p \in A \subset S$, we can find a neighborhood V of p in S and we can always assume that V is sufficiently small such that it is contained in A. Also there exists $U \subset \mathbb{R}^2$ and an onto map $\mathcal{X} : U \to V$ satisfying 3 conditions therefore A is also a regular surface. $\square$

2-3 Ex.16
Let $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 : z = -1\}$ be identified with the complex plane $\mathbb{C}$ by setting $(x, y, -1) = x + iy = \zeta \in \mathbb{C}$. Let $P : \mathbb{C} \to \mathbb{C}$ be the complex polynomial

$$P(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + ... + a_n, a_0 \neq 0, a_i \in \mathbb{C}, i = 0, ..., n$$

Denote by π_N the stereographic projection on $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ from the north pole $N = (0, 0, 1)$ onto $\mathbb{R}^2$. Prove that the map $F : S^2 \to S^2$ given by

$$F(p) = \pi_N^{-1} \circ P \circ \pi_N(p), \text{ if } p \in S^2 - \{N\}$$

$$F(N) = N$$

is differentiable.

Solution.
By the conclusion of **2-2 Ex.16** we know that π_N is a parametrization of $S^2 - \{N\}$ and π_N^{-1} is also differentiable, so it suffices to verify that F is differentiable at N.