

STSong

# Homework 1

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## 1-2 Ex.2

Let  $\alpha(t)$  be a parametrized curve which does not pass through the origin. If  $\alpha(t_0)$  is a point of the trace of  $\alpha$  closest to the origin and  $\alpha'(t_0) \neq 0$ , show that the position vector  $\alpha(t_0)$  is orthogonal to  $\alpha'(t_0)$ .

### *Solution.*

Let  $s(t) = |\alpha(t)|$ ,  $t_0$  is the minimum point of  $s(t)$  since  $\alpha(t_0)$  is the closest point to the origin on the trace of  $\alpha$ . We know that  $\alpha(t) = (x(t), y(t), z(t))$  is differentiable and doesn't pass through the origin, so

$$s(t) = |\alpha(t)| = \sqrt{x^2(t) + y^2(t) + z^2(t)} > 0$$

is also differentiable. Then we have

$$\begin{aligned} s'(t) &= \frac{d}{dt} \sqrt{x^2(t) + y^2(t) + z^2(t)} \\ &= \frac{x(t)x'(t) + y(t)y'(t) + z(t)z'(t)}{\sqrt{x^2(t) + y^2(t) + z^2(t)}} \\ &= \frac{\alpha(t) \cdot \alpha'(t)}{s(t)} \end{aligned}$$

Noticed that  $t_0$  is the minimum point of  $s(t)$ , It follows

$$s'(t_0) = \frac{\alpha(t_0) \cdot \alpha'(t_0)}{s(t_0)} = 0$$

which implies  $\alpha(t_0) \cdot \alpha'(t_0) = 0$ , i.e.  $\alpha(t_0)$  is orthogonal to  $\alpha'(t_0)$ .  $\square$

## 1-2 Ex.4

Let  $\alpha(t) : I \rightarrow \mathbb{R}^3$  be a parametrized curve and let  $v \in \mathbb{R}^3$  be a fixed vector. Assumed that  $\alpha'(t)$  is orthogonal to  $v$  for all  $t \in I$  and that  $\alpha(0)$  is also orthogonal to  $v$ . Prove that  $\alpha(t)$  is also orthogonal to  $v$  for all  $t \in I$ .

### *Solution.*

Suppose  $I = (a, b)$ ,  $0 \in (a, b)$ , then  $\alpha(t)$  can be written as

$$\alpha(t) = \int_a^t \alpha'(s) ds, \quad a < t < b$$

Thus we have

$$(\alpha(t) - \alpha(0)) \cdot v = \int_0^t \alpha'(s) ds \cdot v = \int_0^t \alpha'(s) \cdot v ds$$

It follows

$$\alpha(t) \cdot v = \alpha(0) \cdot v + \int_0^t \alpha'(s) \cdot v ds = 0 + \int_0^t 0 ds = 0 \quad \square$$

**1-3 Ex.4**

Let  $\alpha : (0, \pi) \rightarrow \mathbb{R}^2$  be given by

$$\alpha(t) = (\sin t, \cos t + \log \tan \frac{t}{2})$$

where  $t$  is the angle that the y axis makes with the vector  $\alpha'(t)$ . The trace of  $\alpha$  is called the tractrix. Show that

**a.**  $\alpha$  is a differentiable parametrized curve, regular except at  $t = \frac{\pi}{2}$ .

**b.** The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

**Solution.**

**a.** Since  $x(t) = \sin t$  and  $y(t) = \cos t + \log \tan \frac{t}{2}$  are both differentiable in  $(0, \pi)$ ,  $\alpha(t)$  is a differentiable map from  $(0, \pi)$  to  $\mathbb{R}^2$ , so  $\alpha$  is a differentiable parametrized curve. Note that

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\sin t})$$

Let  $|\alpha'(t_0)| = 0$ , it follows  $\cos t_0 = 0$ ,  $\sin t_0 = \frac{1}{\sin t_0}$  and we have  $t_0 = \frac{\pi}{2} + k\pi$ ,  $k \in \mathbb{Z}$ .

So  $t_0 = \frac{\pi}{2}$  is the only solution in  $(0, \pi)$ . Therefore,  $\alpha$  is regular in  $(0, \pi)$  except at  $t = \frac{\pi}{2}$ .  $\square$

**b.** Let  $(x(t), y(t))$  denote the point of tangency. Since we know that  $t$  is the angle that the y axis makes with the vector  $\alpha'(t)$ , the segment length can be calculated by

$$l(t) = \frac{x(t)}{\sin t} = \frac{\sin t}{\sin t} = 1 \quad \square$$

**1-3 Ex.10**

(Straight Lines as Shortest) Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve. Let  $[a, b] \subset I$  and set  $\alpha(a) = p$ ,  $\alpha(b) = q$ .

**a.** Show that, for any constant vector  $v$ ,  $|v| = 1$ ,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt$$

**b.** Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt$$

That is, the curve of shortest length from  $\alpha(a)$  to  $\alpha(b)$  is the straight line joining these points.

**Solution.**

**a.** Since  $\alpha$  is differentiable,

$$q - p = \alpha(b) - \alpha(a) = \int_a^b \alpha'(t) dt$$

Thus,

$$(q - p) \cdot v = \int_a^b \alpha'(t) dt \cdot v = \int_a^b \alpha'(t) \cdot v dt$$

For each  $t \in (a, b)$ ,  $\alpha'(t) \cdot v \leq |\alpha'(t)| |v| = |\alpha'(t)|$ , so

$$\int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt \quad \square$$

**b.** According to the conclusion above, take  $v = \frac{q-p}{|q-p|}$  and it follows immediately that

$$|\alpha(b) - \alpha(a)| = |q - p| = (q - p) \cdot v \leq \int_a^b |\alpha'(t)| dt \quad \square$$

**1-4 Ex.2**

A plane  $P$  contained in  $\mathbb{R}^3$  is given by the equation  $ax + by + cz + d = 0$ . Show that the vector  $v = (a, b, c)$  is perpendicular to the plane and that  $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$  measures the distance from the plane to the origin  $(0, 0, 0)$ .

**Proof.**

For each point  $(x, y, z)$  in plane  $P$ , the equation  $ax + by + cz + d = 0$  holds. Hence for each vector  $u$  contained in  $P$ , it can be denoted by  $u = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$  where  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  are points in  $P$ . Therefore,

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$$

That is,

$$v \cdot u = (a, b, c) \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) = 0$$

Suppose  $v_0$  is the shortest vector from the origin to  $P$ , it's easy to see that  $v_0$  and  $v$  are linear dependent, so  $v_0$  can be written as  $\lambda v$ , where  $\lambda \in \mathbb{R}$ , therefore, for each point  $(x, y, z) \in P$ ,

$$((x, y, z) - v_0) \cdot v_0 = (x - \lambda a, y - \lambda b, z - \lambda c) \cdot \lambda(a, b, c) = 0$$

i.e.

$$\begin{aligned} \lambda a(x - \lambda a) + \lambda b(y - \lambda b) + \lambda c(z - \lambda c) &= -(a^2 + b^2 + c^2)\lambda^2 + (ax + by + cz)\lambda = 0 \\ (a^2 + b^2 + c^2)\lambda^2 + d\lambda &= 0 \end{aligned}$$

this implies  $\lambda = -\frac{d}{a^2+b^2+c^2}$  (when  $\lambda = 0$ ,  $d = 0$ ), so  $|v_0| = |\lambda v| = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$ , which is exactly the distance from the plane to the origin  $(0, 0, 0)$ .

**1-4 Ex.11**

**a.** Show that the volume  $V$  of a parallelepiped generated by three linearly independent vectors  $u, v, w \in \mathbb{R}^3$  is given by  $V = |(u \wedge v) \cdot w|$ , and introduce an oriented volume in  $\mathbb{R}^3$ .

**b.** Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}$$

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**Proof.**

**a.**