Homework 1

谢铮 15338200

Mar 1, 2019

1-2 Ex.2

Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is a point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Solution.

Let $s(t) = |\alpha(t)|$, t_0 is the minimum point of $s(t)$ since $\alpha(t_0)$ is the closest point to the origin on the trace of α. We know that $\alpha(t) = (x(t), y(t), z(t))$ is differentiable and doesn't pass through the origin, so

$$s(t) = |\alpha(t)| = \sqrt{x^2(t) + y^2(t) + z^2(t)} > 0$$

is also differentiable. Then we have

$$\begin{aligned} s'(t) &= \frac{d}{dt}\sqrt{x^2(t) + y^2(t) + z^2(t)} \\ &= \frac{x(t)x'(t) + y(t)y'(t) + z(t)z'(t)}{\sqrt{x^2(t) + y^2(t) + z^2(t)}} \\ &= \frac{\alpha(t) \cdot \alpha'(t)}{s(t)} \end{aligned}$$

Noticed that t_0 is the minimum point of $s(t)$, It follows

$$s'(t_0) = \frac{\alpha(t_0) \cdot \alpha'(t_0)}{s(t_0)} = 0$$

which implies $\alpha(t_0) \cdot \alpha'(t_0) = 0$, i.e. $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. $\quad\square$

1-2 Ex.4

Let $\alpha(t) : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assumed that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is also orthogonal to v for all $t \in I$.

Solution.

Suppose $I = (a, b), 0 \in (a, b)$, then $\alpha(t)$ can be written as

$$\alpha(t) = \int_a^t \alpha'(s)ds, \quad a < t < b$$

Thus we have

$$(\alpha(t) - \alpha(0)) \cdot v = \int_0^t \alpha'(s)ds \cdot v = \int_0^t \alpha'(s) \cdot v\, ds$$

It follows

$$\alpha(t) \cdot v = \alpha(0) \cdot v + \int_0^t \alpha'(s) \cdot v\, ds = 0 + \int_0^t 0\, ds = 0 \quad \square$$

1-3 Ex.4

Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = (\sin t, \cos t + \log \tan \frac{t}{2})$$

where t is the angle that the y axis makes with the vector $\alpha'(t)$. The trace of α is called the tractrix. Show that

a. α is a differentiable parametrized curve, regular except at $t = \frac{\pi}{2}$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Solution.

a. Since $x(t) = \sin t$ and $y(t) = \cos t + \log \tan \frac{t}{2}$ are both differentiable in $(0, \pi)$, $\alpha(t)$ is a differentiable map from $(0, \pi)$ to $\mathbb{R}^2$, so α is a differentiable parametrized curve. Note that

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\sin t})$$

Let $|\alpha'(t_0)| = 0$, it follows $\cos t_0 = 0$, $\sin t_0 = \frac{1}{\sin t_0}$ and we have $t_0 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.
So $t_0 = \frac{\pi}{2}$ is the only solution in $(0, \pi)$. Therefore, α is regular in $(0, \pi)$ except at $t = \frac{\pi}{2}$. $\quad\square$

b. Let $(x(t), y(t))$ denote the point of tangency. Since we know that t is the angle that the y axis makes with the vector $\alpha'(t)$, the segment length can be calculated by

$$l(t) = \frac{x(t)}{\sin t} = \frac{\sin t}{\sin t} = 1 \quad\square$$

1-3 Ex.10

(Straight Lines as Shortest) Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve. Let $[a, b] \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

a. Show that, for any constant vector v, $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v \, dt \leq \int_a^b |\alpha'(t)| dt$$

b. Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt$$

That is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Solution.

a. Since α is differentiable,

$$q - p = \alpha(b) - \alpha(a) = \int_a^b \alpha'(t) dt$$

Thus,

$$(q - p) \cdot v = \int_a^b \alpha'(t) dt \cdot v = \int_a^b \alpha'(t) \cdot v \, dt$$

For each $t \in (a, b)$, $\alpha'(t) \cdot v \leq |\alpha'(t)||v| = |\alpha'(t)|$, so

$$\int_a^b \alpha'(t) \cdot v \, dt \leq \int_a^b |\alpha'(t)| dt \quad\square$$

b. According to the conclusion above, take $v = \frac{q-p}{|q-p|}$ and it follows immediately that

$$|\alpha(b) - \alpha(a)| = |q - p| = (q - p) \cdot v \leq \int_a^b |\alpha'(t)| dt \quad\square$$

1-4 Ex.2

A plane P contained in $\mathbb{R}^3$ is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$

is perpendicular to the plane and that $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$ measures the distance from the plane to the origin $(0, 0, 0)$.

Proof.

For each point (x, y, z) in plane P, the equation $ax + by + cz + d = 0$ holds. Hence for each vector u contained in P, it can be denoted by $u = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ where (x_1, y_1, z_1) and (x_2, y_2, z_2) are points in P. Therefore,

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$$

That is,

$$v \cdot u = (a, b, c) \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) = 0$$

Suppose v_0 is the shortest vector from the origin to P, it's easy to see that v_0 and v are linear dependent, so v_0 can be written as λv, where $\lambda \in \mathbb{R}$, therefore, for each point $(x, y, z) \in P$,

$$((x, y, z) - v_0) \cdot v_0 = (x - \lambda a, y - \lambda b, z - \lambda c) \cdot \lambda(a, b, c) = 0$$

i.e.

$$\lambda a(x - \lambda a) + \lambda b(y - \lambda b) + \lambda c(x - \lambda c) = -(a^2 + b^2 + c^2)\lambda^2 + (ax + by + cz)\lambda = 0$$

$$(a^2 + b^2 + c^2)\lambda^2 + d\lambda = 0$$

this implies $\lambda = -\frac{d}{a^2+b^2+c^2}$ (when $\lambda = 0$, $d = 0$), so $|v_0| = |\lambda||v| = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$, which is exactly the distance from the plane to the origin $(0, 0, 0)$.

1-4 Ex.11

a. Show that the volume V of a parallelepiped generated by three linearly independent vectors $u, v, w \in \mathbb{R}^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an oriented volume in $\mathbb{R}^3$.

b. Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}$$

.

Proof.

a.

$$V = S \cdot h = |u||v|\sin\langle u, v\rangle h = |u \wedge v|\rangle\frac{|(u \wedge v) \cdot w|}{|u \wedge v|} = |u \wedge v| \cdot w \quad \square$$

b. Note that both sides of the equation

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}$$

are linear in u, v, w. So it suffices to show that the equation holds for all basis vectors e_1, e_2, e_3, moreover, it's easy to see that, if u, v, w are linearly dependent, then both sides will equal to 0. Thus it only remains to verify the cases that (u, v, w) is a substitution of (e_1, e_2, e_3). In these cases,

$$V^2 = |(u \wedge v) \cdot w|^2 = 1$$

$$\begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad \square$$