

# Surfaces

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Roughly speaking, a regular surface in  $\mathbb{R}^3$  is obtained by taking pieces of a plane, deforming them, and arranging them in such a way that the resulting figure has no sharp points, edges, or self-intersections and so that it makes sense to speak of a tangent plane at points of the figure.

The idea is to define a set that is, in a certain sense, two-dimensional and that also is smooth enough so that the usual notions of calculus can be extended to it.

## Definition 2.1 (Regular Surface)

A subset  $S \subset \mathbb{R}^3$  is a regular surface if for each  $p \in S$ , there exists a neighborhood  $V \subset \mathbb{R}^3$ , an open set  $U \subset \mathbb{R}^2$  and an onto map  $\mathcal{X} : U \rightarrow V \cap S$  such that

(1)  $\mathcal{X}$  is differentiable, i.e. if we write

$$\mathcal{X}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in U$$

Then the functions  $x(u, v), y(u, v), z(u, v)$  have continuous partial derivatives of all orders in  $U$ .

(2)  $\mathcal{X}$  is a homeomorphism, since  $\mathcal{X}$  is continuous by condition (1), this means that  $\mathcal{X}^{-1} : V \cap S \rightarrow U$  is continuous.

(3) (regularity condition) For each  $q \in U$ , the differential  $d\mathcal{X}_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is one-to-one.

The mapping  $\mathcal{X}$  is called a parametrization or system of (local) coordinates in a neighborhood of  $p$ . The neighborhood  $V \cap S$  of  $p$  in  $S$  is called a coordinate neighborhood.

To give condition (3) a more familiar form, let us compute the matrix of the linear map  $d\mathcal{X}_q$  in the canonical bases  $e_1 = (1, 0)$ ,  $e_2 = (0, 1)$  of  $\mathbb{R}^2$  with coordinates  $(u, v)$  and  $f_1 = (1, 0, 0)$ ,  $f_2 = (0, 1, 0)$ ,  $f_3 = (0, 0, 1)$  of  $\mathbb{R}^3$ , with coordinates  $(x, y, z)$ .

Let  $q = (u_0, v_0)$ , the vector  $e_1$  is tangent to the curve  $\alpha : \mathbb{R} \rightarrow U \subset \mathbb{R}^2, u \mapsto (u, v_0)$  whose image under  $\mathcal{X}$  is the curve

$$\beta : \mathbb{R} \rightarrow \mathbb{R}^3, \quad u \mapsto (x(u, v_0), y(u, v_0), z(u, v_0))$$

This image curve (called the coordinate curve  $v = v_0$ ) lies on  $S$  and has the tangent vector at  $\mathcal{X}(q)$ , which is defined by

$$\left( \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0) \right) = \frac{\partial \mathcal{X}}{\partial u}(u_0, v_0)$$

By the definition of differential,

$$d\mathcal{X}_q(e_1) = \frac{\partial \mathcal{X}}{\partial u}(u_0, v_0)$$

$$d\mathcal{X}_q(e_2) = \frac{\partial \mathcal{X}}{\partial v}(u_0, v_0)$$

Thus, the matrix of the linear map  $d\mathcal{X}_q$  in the referred basis is

$$d\mathcal{X}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}_q$$

**Example 2.1 (unit sphere)**

The unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

is a regular surface.

**Proof.**

We first verify that the map  $\mathcal{X}_1 : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by

$$\mathcal{X}_1(x, y) = (x, y, \sqrt{1 - (x^2 + y^2)}), (x, y) \in U$$

is a parametrization of  $S^2$ . Observe that  $\mathcal{X}_1(U)$  is the open part of  $S^2$  above the  $xOy$  plane.

Since  $x^2 + y^2 < 1$ , the function  $\sqrt{1 - (x^2 + y^2)}$  has continuous partial derivatives of all orders. Thus,  $\mathcal{X}_1$  is differentiable and condition (1) holds.

To check condition (2), we observe that  $\mathcal{X}_1$  is one-to-one and that  $\mathcal{X}_1^{-1}$  is the restriction of the continuous projection  $\pi(x, y, z) = (x, y)$  to the set  $\mathcal{X}_1(U)$ . Thus,  $\mathcal{X}_1^{-1}$  is continuous in  $\mathcal{X}_1(U)$ .

To check condition (3), note that

$$d\mathcal{X}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1 - (x^2 + y^2)}} & \frac{-y}{\sqrt{1 - (x^2 + y^2)}} \end{pmatrix}$$

It's easy to see that  $d\mathcal{X}_1$  is one-to-one.

Similarly we can verify the above conditions for other parametrizations like

$$\mathcal{X}_2(x, z) = (x, \sqrt{1 - (x^2 + z^2)}, z)$$

$$\mathcal{X}_3(y, z) = (\sqrt{1 - (y^2 + z^2)}, y, z) \quad \square$$

**Proposition 2.1**

If  $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is a differentiable function in an open set  $U$ , then the graph of  $f$ , that is, the subset of  $\mathbb{R}^3$  given by  $(x, y, f(x, y))$  for  $(x, y) \in U$ , is a regular surface.

**Proof.**

It suffices to show that the map  $\mathcal{X} : U \rightarrow \mathbb{R}^3$  given by

$$\mathcal{X}(u, v) = (u, v, f(u, v))$$

is a parametrization of the graph whose coordinate neighborhood covers every point of the graph.

It's clearly that  $\mathcal{X}$  is differentiable.

Also note that

$$\mathcal{X}^{-1} = \pi|_G$$

where  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is bounded and thus continuous,  $G = \{(x, y, f(x, y)) : (x, y) \in U\}$ . So  $\mathcal{X}^{-1}$  is continuous and thus  $\mathcal{X}$  is a homeomorphism.

Finally, since

$$d\mathcal{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{df}{du} & \frac{df}{dv} \end{pmatrix}$$

has rank 2 at any point  $(u, v) \in U$ , it follows that  $\mathcal{X}$  is a parametrization.  $\square$

**Definition 2.2 (critical and regular point)**

Given a differentiable map  $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  defined in an open set  $U$  of  $\mathbb{R}^n$  we say that  $p \in U$  is a critical point of  $F$  if the differential  $dF_p : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is not a surjective (onto) mapping. The image  $F(p) \in \mathbb{R}^m$  of a critical point is called a critical value of  $F$ . A point of  $\mathbb{R}^m$  which is not a critical value is called a regular value of  $F$ .

**Proposition 2.2**

Let  $f : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$  be a differentiable function,  $U$  is an open subset of  $\mathbb{R}^3$ ,  $a$  is a regular value of  $f$ , then  $f^{-1}(a)$  is a regular surface.

**Proof.**

Pick any  $p \in f^{-1}(a)$ , since  $a$  is a regular value, we can assume without loss of generality that  $\frac{\partial f}{\partial z} \neq 0$ . Then define  $F : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by

$$F(x, y, z) = (x, y, f(x, y, z))$$

Note that

$$dF_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}_p$$

is invertible. So by inverse function theorem, there exists a neighborhood  $V_p \subset \mathbb{R}^3$  of  $p$  and  $W_p = F(V_p) \subset \mathbb{R}^3$  such that  $F : V_p \rightarrow W_p$  is invertible and  $F^{-1} : W_p \rightarrow V_p$  is differentiable. Note that  $F^{-1}(x, y, a) = (x, y, z)$ , hence  $F^{-1} : W_p \rightarrow V_p$  can be written as

$$F^{-1}(x, y, a) = (x, y, h(x, y))$$

where  $z = h(x, y)$  is a differentiable function here.

Therefore,  $V_p = \{(x, y, h(x, y)) : (x, y) \in W_p\}$  is the graph of  $h(x, y)$  and by *Proposition 2.1* we know that  $V_p$  is a regular surface. Since  $p$  is arbitrary in  $f^{-1}(a)$ ,  $f^{-1}(a)$  is a regular surface.  $\square$

**Proposition 2.3**

Let  $S \subset \mathbb{R}^3$  be a regular surface and  $p \in S$ , then there exists a neighborhood  $V$  of  $p \in S$  such that  $V$  is the graph of a differentiable function which has one of the following three forms:  $z = f(x, y), y = g(x, z), x = h(y, z)$ .

**Proof.**

Since  $S$  is a regular surface, we know that there exists an open set  $U \subset \mathbb{R}^2$ ,  $V = \mathcal{X}(U) \subset \mathbb{R}^3$  and a diffeomorphism  $\mathcal{X} : U \rightarrow V$  such that

$$d\mathcal{X}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}_q$$

has rank 2, where  $\mathcal{X}(q) = p, q \in U$ . Without loss of generality, we can assume that

$$\left(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\right), \left(\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}\right)$$

are linearly independent at point  $q$ .

Consider projection  $\pi : S \subset \mathbb{R}^3 \rightarrow \mathbb{R}^2$ , then  $\pi \circ \mathcal{X} : U \rightarrow W = \pi(\mathcal{X}(U)) \subset \mathbb{R}^2$  is differentiable and

$$d(\pi \circ \mathcal{X}) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

whose determinant is non-zero at point  $q$  by the assumption above.

Therefore by inverse function theorem, we can shrink  $U$  and claim that  $(\pi \circ \mathcal{X})^{-1} : W \rightarrow U$  exists and is differentiable.

Now let  $F = \mathcal{X} \circ (\pi \circ \mathcal{X})^{-1}$ , then  $F : W \subset \mathbb{R}^2 \rightarrow V \subset \mathbb{R}^3, (x, y) \mapsto (x, y, z)$  is differentiable, so there exists a differentiable function  $f : W \rightarrow \mathbb{R}$  such that  $z = f(x, y)$  for each  $(x, y) \in W$ .

Moreover, since  $\mathcal{X}$  is a homeomorphism,  $V = \mathcal{X}(U)$  is a (open) neighborhood of  $p$ , therefore,

$$V = \{(x, y, f(x, y)) : (x, y) \in W\} \quad \square$$

**Remark.**

The condition that  $\mathcal{X}$  is a homeomorphism is required, otherwise we may not find an open neighborhood of  $p$ .

**Proposition 2.4**

Let  $p$  be a point of a regular surface  $S$  and let  $\mathcal{X} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a map with  $p \in \mathcal{X}(U) \subset S$  such that conditions 1(differentiable) and 3(regularity condition) hold. Assume that  $\mathcal{X}$  is one-to-one, then  $\mathcal{X}^{-1}$  is continuous.