Homework 3

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2-3 Ex.3
Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to a plane.

Solution.
Consider map

$$\mathcal{X}(u, v) = (u, v, u^2 + v^2)$$

It's easy to see that $\mathcal{X}$ is differentiable, bijective, and $\frac{\partial(u,v)}{\partial(u,v)} = 1$, so it suffices to show that $\mathcal{X}^{-1}$ is continuous. Since $\mathcal{X}^{-1}$ can be seen as a restriction of $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ to $S = \{(x, y, z) : z = x^2 + y^2\}$, $\mathcal{X}^{-1}$ is also continuous. $\square$

2-3 Ex.6
Prove that the definition of a differentiable map between surfaces does not depend on the parametrizations chosen.

Solution.
Suppose $\phi : S_1 \to S_2$ is a differentiable map where S_1, S_2 are regular surfaces. By definition we know that given $p \in S_1$, there exists open sets $q \in U \subset \mathbb{R}^2$, $\bar{q} \in \bar{U} \subset \mathbb{R}^2$ and parametrizations $\mathcal{X} : U \to V \cap S_1$, $\bar{\mathcal{X}} : \bar{U} \to V \cap S_2$ such that $p = \mathcal{X}(q), \phi(p) = \bar{\mathcal{X}}(\bar{q})$ and $f = \bar{\mathcal{X}}^{-1} \circ \phi \circ \mathcal{X}$ is differentiable at q.
Note that for $p \in S_1$ and $\phi(p) \in S_2$, we can find another two parametrizations $\mathcal{Y}$ and $\bar{\mathcal{Y}}$ of S_1 at p and S_2 at $\phi(p)$ respectively and moreover, $\mathcal{X} \circ \mathcal{Y}^{-1}$ and $\bar{\mathcal{Y}} \circ \bar{\mathcal{X}}^{-1}$ are both diffeomorphism. Therefore

$$g = \bar{\mathcal{Y}}^{-1} \circ \phi \circ \mathcal{Y}$$

is also differentiable at q, which implies that the definition doesn't depend on the parametrizations chosen. $\square$

2-3 Ex.10
Let C be a plane regular curve which lies in one side of a straight line r of the plane and meets r at the points p, q. What conditions should C satisfy to ensure that the rotation of C about r generates an extended regular surface of revolution?

Solution.
For simplicity, we can assume that C is parametrized by

$$\alpha : [0, 1] \to C$$

and r is the rotation axis. where α is smooth and injective(hence C is simple). $\alpha(t) = (\alpha_1(t), \alpha_2(t))$. and $\alpha(0) = p, \alpha(1) = q$.
We have known that the surface of revolution denoted by S is regular outside p, q since C is regular. Now assume that S is also regular at p and q. We shall notice that the tangent plane of S at p, q, denoted by $T_p(S)$ and $T_q(S)$ respectively, should stay invariant under rotation

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Therefore, the equation of $T_p(S)$ is given by

$$x = \alpha_1(0)$$

Then let $\tilde{C} = S \cap \{z = 0\}$, naturally we can also find a parametrization of $\tilde{C}$ by

$$\tilde{\alpha}(t) = \begin{cases} (\alpha_1(t), \alpha_2(t)), & t \geq 0 \\ (\alpha_1(-t), -\alpha_2(-t)), & t \leq 0 \end{cases}$$

2-3 Ex.14

Let $A \subset S$ be a subset of a regular surface S. Prove that A is itself a regular surface if and only if A is open in S, that is, $A = U \cap S$, where U is an open set in $\mathbb{R}^3$.

Solution.

" $\Rightarrow$ " :

Suppose A is a regular surface.

" $\Leftarrow$ " :

Suppose A is open in S, then there exists $U \subset \mathbb{R}^3$ such that $A = U \cap S$ where U is an open set.

For each point $p \in A \subset S$, there exists a parametrization $\mathcal{X} : O \to W \cap S$ satisfying three conditions since S is a regular surface. Note that U is open so we can assume that W is sufficiently small such that W is contained in $A = W \cap S$. Hence A is also a regular surface.

2-3 Ex.16

Let $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 : z = -1\}$ be identified with the complex plane $\mathbb{C}$ by setting $(x, y, -1) = x + iy = \xi \in \mathbb{C}$, let $P : \mathbb{C} \to \mathbb{C}$ be the complex polynomial

$$P(\xi) = a_0\xi^n + a_1\xi^{n-1} + ... + a_n$$

where $a_0 \neq 0, a_i \in \mathbb{C}$. Denote by π_N the stereographic projection of $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ from the north pole $N = (0, 0, 1)$ onto $\mathbb{R}^2$. Prove that the map $F : S^2 \to S^2$ given by

$$F(p) = \pi_N^{-1} \circ P \circ \pi_N(p), \forall p \in S^2 - \{N\}$$

$$F(N) = N$$

is differentiable.

Solution.

For $p \in S^2 - \{N\}$, it's easy to verify that F is differentiable at p since π_N is a diffeomorphism and P is holomorphic.

Consider map $G : \mathbb{C} \to \mathbb{C}$ given by

$$G(p) = \pi_S \circ \pi_N^{-1} \circ P \circ \pi_N \circ \pi_S^{-1}(p)$$

where π_S is defined similar to π_N

It suffices to show that G is differentiable at 0.

First observe that

$$\pi_N \circ \pi_S^{-1}(\xi) = \frac{1}{\bar{\xi}}, \quad \pi_S \circ \pi_N^{-1}(\eta) = \frac{1}{\bar{\eta}}$$

Hence

$$\begin{aligned} G(\xi) &= \frac{1}{\overline{P \circ \pi_N \circ \pi_S^{-1}(\xi)}} = \frac{1}{\overline{P(\frac{1}{\bar{\xi}})}} \\ &= \frac{1}{\bar{P}(\frac{1}{\xi})} = \frac{1}{\bar{a}_0\xi^n + \bar{a}_1\xi^{n-1} + ... + \bar{a}_n} = \frac{\xi^n}{\bar{a}_0 + \bar{a}_1\xi + ... + \bar{a}_n\xi^n} \end{aligned}$$

which is differentiable at 0.

Then, since π_S is a diffeomorphism,

$$F(p) = \pi_S^{-1} \circ G \circ \pi_S$$

is differentiable at N. $\square$

2-4 Ex.1
Show that the equation of the tangent plane at (x_0, y_0, z_0) of a regular surface given by $f(x, y, z) = 0$,where 0 is a regular value of f,is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

Solution.
Suppose w is a tangent vector of $S = f^{-1}(0)$ at $p = (x_0, y_0, z_0)$ and $\alpha : (-\epsilon, \epsilon) \to S$ is a differentiable curve such that $\alpha(0) = p$, $\alpha'(0) = w$. Let $g = f \circ \alpha$, then $g(t) = f(\alpha(t)) = 0$ for all t. Hence $g'(0) = (f_x(p), f_y(p), f_z(p)) \cdot w = 0$. Since w is arbitrary, it follows that the equation of the tangent plane is

$$f_x(p)(x - x_0) + f_y(p)(y - y_0) + f_z(p)(z - z_0) = 0$$

$\square$

2-4 Ex.2
Determine the tangent planes of $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0)$ and show that they are all parallel to the z axis.

Solution.
Using the conclusion above we know that the equation of the tangent plane at $(x_0, y_0, 0)$ is

$$x_0 x + y_0 y - 1 = 0$$

Thus the normal vector of the tangent plane is $(x_0, y_0, 0)$, which is normal to $(0, 0, 1)$, hence z axis is parallel to the tangent plane at $(x_0, y_0, 0)$ for all x_0, y_0. $\square$

2-4 Ex.13
A critical point of a differentiable function $f : S \to \mathbb{R}$ defined on a regular surface S is a point $p \in S$ such that $df_p = 0$.

a. Let $f : S \to \mathbb{R}$ be given by $f(p) = |p - p_0|$, $p \in S$, $p_0 \notin S$. Show that p is a critical point of f if and only if the line joining p and p_0 is normal to S at p.
b. Let $h : S \to \mathbb{R}$ be given by $h(p) = p \cdot v$, where $v \in \mathbb{R}^3$ is a unit vector. Show that $p \in S$ is a critical point of f if and only if v is a normal vector of S at p.

Solution.
a. Suppose p is a critical point, then for each $w \in T_p(S)$

$$df_p(w) = (\frac{x - x_0}{|p - p_0|}, \frac{y - y_0}{|p - p_0|}, \frac{z - z_0}{|p - p_0|})(w) = \frac{p - p_0}{|p - p_0|}(w) = 0$$

Thus $p - p_0$ is penpendicular to $T_p(S)$ and also S.
It's easy to verify inversely. $\square$

b.
Observe that

$$dh_p(w) =< v, w >, w \in T_p(S)$$

It follows that $dh_p = 0$ if and only if v is a normal vector of S at p.

2-4 Ex.15
Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

Soluiton.
Suppose the fixed point is denoted by p_0, then for each point $p \in S$, $p - p_0$ is normal to $T_p(S)$.
Let $f(p) = |p - p_0|^2$, then

$$df_p(w) = 2(p - p_0)(w) = 0, \forall w \in T_p(S)$$

Then we show that $f(p) = C$, for each $p_1, p_2 \in S$, we can find a curve $\alpha : I \to S$ such that $\alpha(t_1) = p_1, \alpha(t_2) = p_2$. Consider $g = f \circ \alpha$, then

$$g(t_2) - g(t_1) = \int_{t_1}^{t_2} g'(t)dt$$

Since $g'(t) = df_{\alpha(t)}(\alpha'(t)) = 0$ for all $t \in I$. Therefore $g(t_1) = g(t_2)$, i.e. $f(p_1) = f(p_2)$. Hence $f(p) = C$ for some constant C, which implies that $S \subset \{p \in \mathbb{R}^3 : |p - p_0|^2 = C\}$. □

2-4 Ex.16

Let w be a tangent vector to a regular surface S at a point $p \in S$ and let $\mathcal{X}(u, v)$ and $\bar{\mathcal{X}}(\bar{u}, \bar{v})$ be two parametrizations at p. Suppose that the expressions of w in the bases associated to $\mathcal{X}(u, v)$ and $\bar{\mathcal{X}}(\bar{u}, \bar{v})$ are

$$w = \alpha_1 \mathcal{X}_u + \alpha_2 \mathcal{X}_v$$

and

$$w = \beta_1 \bar{\mathcal{X}}_{\bar{u}} + \beta_2 \bar{\mathcal{X}}_{\bar{v}}$$

Show that the coordinates of w are related by

$$\beta_1 = \alpha_1 \frac{\partial \bar{u}}{\partial u} + \alpha_2 \frac{\partial \bar{u}}{\partial v}$$

$$\beta_2 = \alpha_1 \frac{\partial \bar{v}}{\partial u} + \alpha_2 \frac{\partial \bar{v}}{\partial v}$$

where $\bar{u} = \bar{u}(u, v)$ and $\bar{v} = \bar{v}(u, v)$ are the expressions of the change of coordinates.

Solution.

Note that

$$(\mathcal{X}_u, \mathcal{X}_v) = (\bar{\mathcal{X}}_{\bar{u}}, \bar{\mathcal{X}}_{\bar{v}}) \cdot \begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{u}}{\partial v} \\ \frac{\partial \bar{v}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix}$$

Hence

$$(\bar{\mathcal{X}}_{\bar{u}}, \bar{\mathcal{X}}_{\bar{v}}) \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = (\mathcal{X}_u, \mathcal{X}_v) \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = (\bar{\mathcal{X}}_{\bar{u}}, \bar{\mathcal{X}}_{\bar{v}}) \cdot \begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{u}}{\partial v} \\ \frac{\partial \bar{v}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{u}}{\partial v} \\ \frac{\partial \bar{v}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

2-4 Ex.18

Prove that if a regular surface S meets a plane P in a single point p, then this plane coincides with the tangent plane of S at p.

Solution.

Suppose the normal vector of P is $n = (a, b, c) \neq 0$. Then let $f(q) = (q - p) \cdot n$, where $q \in S$.

Assume that $df_p \neq 0$, then there exists some $w \in T_p(S)$ such that $df_p(w) \neq 0$, then we can find $\beta : (-\epsilon, \epsilon) to S$ such that $\beta(0) = p$, $\beta'(0) = w$, let $h = f \circ \beta$, then $h'(0) = df_p(w) \neq 0$, thus by inverse function theorem, there exists $t_1, t_2 \in (-\epsilon, \epsilon)$ such that $h(t_1)h(t_2) < 0$. Hence there exists some t_0 such that $h(t_0) = 0$. Since h is arbitrary, there are more than one point in $P \cap S$, leading a contradiction.

Hence $df(p) = 0$. Now for each $w \in T_p(S)$, we can find a curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p, \alpha'(0) = w$. Now let $g = f \circ \alpha$, then $g : (-\epsilon, \epsilon) \to S$ is a differentiable function and

$$g'(0) = \frac{d}{dt} f(\alpha(t))|_{t=0} = n \cdot \alpha'(0) = n \cdot w = 0$$

Therefore n is perpendicular to $Tp(S)$, which implies that P is $T_p(S)$ exactly. □

2-4 Ex.19
Let $S \subset \mathbb{R}^3$ be a regular surface and $P \subset \mathbb{R}^3$ be a plane. If all points of S are on the same side of P, prove that P is tangent to S at all points of $P \cap S$.

Solution.
Similarly, given $p \in S \cap P$, define

$$f(q) = (q - p) \cdot n$$

where n is the normal vector of P. Since we know that S is on one side of P, without loss of generality we can assume that $f(q) \geq 0$ for all $q \in S$.
For each $p_0 \in S \cap P$, we have $f(p_0) = (p_0 - p) \cdot n = 0$. It can derive that $df_{p_0} = 0$, otherwise by inverse function theorem we could find some q such that $f(q) < 0$.
Now pick $w \in T_{p_0}(S)$, we can find a curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p_0$, $\alpha'(0) = w$.
Let $g = f \circ \alpha$, then

$$g'(0) = df_{p_0}(w) = n \cdot w = 0$$

Since p_0 is arbitrary, n is the normal vector of tangent planes at all points of $P \cap S$. □

2-4 Ex.24
(Chain Rule.)Show that if $\phi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ are differentiable maps and $p \in S_1$, then

$$d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$$

Solution.
For each $w \in T_p(S_1)$, we can find a curve $\alpha : (-\epsilon, \epsilon) \to S_1$ such that $\alpha(0) = p$, $\alpha'(0) = w$, let $\beta = \phi \circ \alpha$, $\gamma = \psi \circ \beta$. By definition of differential,

$$\gamma'(0) = d(\psi \circ \phi)_p(w) = d\psi_{\beta(0)}(\beta'(0))$$

$$\beta'(0) = d\phi_p(w), \beta(0) = \phi(\alpha(0)) = \phi(p)$$

Hence

$$d(\psi \circ \phi)_p(w) = d\psi_{\phi(p)}(d\phi_p(w))$$

i.e.

$$d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$$

□

2-5 Ex.1(a)
Compute the first fundamental form of the following regular surface:

$$\mathcal{X}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$$

Solution.

$$\mathcal{X}_u = (a \cos u \cos v, b \cos u \sin v, -c \sin u)$$

$$\mathcal{X}_v = (-a \sin u \sin v, b \sin u \cos v, c \cos u)$$

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