Surfaces

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Roughly speaking, a regualr surface in $\mathbb{R}^3$ is obtained by taking pieces of a plane, deforming them, and arranging them in such a way that the resulting figure has no sharp points, edges, or self-intersections and so that it makes sense to speak of a tangent plane at points of the figure.
The idea is to define a set that is, in a certain sense, two-dimensional and that also is smooth enough so that the usual notions of calculus can be extended to it.

Definition 2.1 (Regular Surface)
A subset $S \subset \mathbb{R}^3$ is a regular surface if for each $p \in S$, there exists a neighborhood $V \in \mathbb{R}^3$, an open set $U \in \mathbb{R}^2$ and an onto map $\mathcal{X} : U \to V \cap S$ such that
(1) $\mathcal{X}$ is differentiable, i.e. if we write

$$\mathcal{X}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in U$$

Then the functions $x(u, v), y(u, v), z(u, v)$ have continuous partial derivatives of all orders in U.
(2) $\mathcal{X}$ is a homeomorphism, since $\mathcal{X}$ is continuous by condition (1), this means that $\mathcal{X}^{-1} : V \cap S \to U$ is continuous.
(3) (regularity condition) For each $q \in U$, the differential $d\mathcal{X}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

The mapping $\mathcal{X}$ is called a parametrization or system of (local) coordinates in a neighborhood of p. The neighborhood $V \cap S$ of p in S is called a coordinate neighborhood.

To give condition (3) a more familiar form, let us compute the matrix of the linear map $d\mathcal{X}_q$ in the canonical bases $e_1 = (1, 0)$, $e_2 = (0, 1)$ of $\mathbb{R}^2$ with coordinates (u, v) and $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 0, 1)$ of $\mathbb{R}^3$, with coordinates (x, y, z).
Let $q = (u_0, v_0)$, the vector e_1 is tangent to the curve $\alpha : \mathbb{R} \to U \subset \mathbb{R}^2, u \mapsto (u, v_0)$ whose image under $\mathcal{X}$ is the curve

$$\beta : \mathbb{R} \to \mathbb{R}^3, \quad u \mapsto (x(u, v_0), y(u, v_0), z(u, v_0))$$

This image curve (called the coordinate curve $v = v_0$) lies on S and has the tangent vector at $\mathcal{X}(q)$, which is defined by

$$(\frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)) = \frac{\partial \mathcal{X}}{\partial u}(u_0, v_0)$$

By the definition of differential,

$$d\mathcal{X}_q(e_1) = \frac{\partial \mathcal{X}}{\partial u}(u_0, v_0)$$

$$d\mathcal{X}_q(e_2) = \frac{\partial \mathcal{X}}{\partial v}(u_0, v_0)$$

Thus, the matrix of the linear map $d\mathcal{X}_q$ in the referred basis is

$$d\mathcal{X}_q = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}$$

Definition 2.2 (Principle Curvature)
Let $S \subset \mathbb{R}^3$ be a regular surface.