STSong

Homework 1

15338200

Mar 1, 2019

1-2 Ex.2
Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is a point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Solution.
Let $s(t) = |\alpha(t)|$, t_0 is the minimum point of $s(t)$ since $\alpha(t_0)$ is the closest point to the origin on the trace of α. We know that $\alpha(t) = (x(t), y(t), z(t))$ is differentiable and doesn't pass through the origin, so

$$s(t) = |\alpha(t)| = \sqrt{x^2(t) + y^2(t) + z^2(t)} > 0$$

is also differentiable. Then we have

$$\begin{aligned} s'(t) &= \frac{d}{dt}\sqrt{x^2(t) + y^2(t) + z^2(t)} \\ &= \frac{x(t)x'(t) + y(t)y'(t) + z(t)z'(t)}{\sqrt{x^2(t) + y^2(t) + z^2(t)}} \\ &= \frac{\alpha(t) \cdot \alpha'(t)}{s(t)} \end{aligned}$$

Noticed that t_0 is the minimum point of $s(t)$, It follows

$$s'(t_0) = \frac{\alpha(t_0) \cdot \alpha'(t_0)}{s(t_0)} = 0$$

which implies $\alpha(t_0) \cdot \alpha'(t_0) = 0$, i.e. $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. $\square$

1-2 Ex.4
Let $\alpha(t) : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assumed that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is also orthogonal to v for all $t \in I$.

Solution.
Suppose $I = (a, b), 0 \in (a, b)$, then $\alpha(t)$ can be written as

$$\alpha(t) = \int_a^t \alpha'(s)ds, \quad a < t < b$$

Thus we have

$$(\alpha(t) - \alpha(0)) \cdot v = \int_0^t \alpha'(s)ds \cdot v = \int_0^t \alpha'(s) \cdot v ds$$

It follows

$$\alpha(t) \cdot v = \alpha(0) \cdot v + \int_0^t \alpha'(s) \cdot v ds = 0 + \int_0^t 0 ds = 0 \quad \square$$

1-3 Ex.4
Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = (\sin t, \cos t + \log \tan \frac{t}{2})$$

where t is the angle that the y axis makes with the vector $\alpha'(t)$. The trace of α is called the tractrix. Show that
a. α is a differentiable parametrized curve, regular except at $t = \frac{\pi}{2}$.
b. The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Solution.
a. Since $x(t) = \sin t$ and $y(t) = \cos t + \log \tan \frac{t}{2}$ are both differentiable in $(0, \pi)$, $\alpha(t)$ is a differentiable map from $(0, \pi)$ to $\mathbb{R}^2$, so α is a differentiable parametrized curve. Note that

$$\alpha'(t) = (\cos t, -\sin t + \frac{1}{\sin t})$$

Let $|\alpha'(t_0)| = 0$, it follows $\cos t_0 = 0$, $\sin t_0 = \frac{1}{\sin t_0}$ and we have $t_0 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.
So $t_0 = \frac{\pi}{2}$ is the only solution in $(0, \pi)$. Therefore, α is regular in $(0, \pi)$ except at $t = \frac{\pi}{2}$. $\square$

b. Let $(x(t), y(t))$ denote the point of tangency. Since we know that t is the angle that the y axis makes with the vector $\alpha'(t)$, the segment length can be calculated by

$$l(t) = \frac{x(t)}{\sin t} = \frac{\sin t}{\sin t} = 1 \quad \square$$

1-3 Ex.10
(Straight Lines as Shortest) Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve. Let $[a, b] \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.
a. Show that, for any constant vector v, $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt$$

b. Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt$$

That is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Solution.
a. Since α is differentiable,

$$q - p = \alpha(b) - \alpha(a) = \int_a^b \alpha'(t) dt$$

Thus,

$$(q - p) \cdot v = \int_a^b \alpha'(t) dt \cdot v = \int_a^b \alpha'(t) \cdot v dt$$

For each $t \in (a, b)$, $\alpha'(t) \cdot v \leq |\alpha'(t)||v| = |\alpha'(t)|$, so

$$\int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt \quad \square$$

b. According to the conclusion above, take $v = \frac{q-p}{|q-p|}$ and it follows immediately that

$$|\alpha(b) - \alpha(a)| = |q - p| = (q - p) \cdot v \leq \int_a^b |\alpha'(t)| dt \quad \square$$

1-4 Ex.2
A plane P contained in $\mathbb{R}^3$ is given by the equation $ax+by+cz+d=0$. Show that the vector $v=(a,b,c)$ is perpendicular to the plane and that $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$ measures the distance from the plane to the origin (0, 0, 0).
Proof.
For each point (x,y,z) in plane P, the equation $ax+by+cz+d=0$ holds. Hence for each vector u contained in P, it can be denoted by $u=(x_2-x_1, y_2-y_1, z_2-z_1)$ where (x_1,y_1,z_1) and (x_2,y_2,z_2) are points in P. Therefore,

$$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0$$

That is,

$$v\cdot u=(a,b,c)\cdot(x_2-x_1,y_2-y_1,z_2-z_1)=0$$

Suppose v_0 is the shortest vector from the origin to P, it's easy to see that v_0 and v are linear dependent, so v_0 can be written as λv, where $\lambda\in\mathbb{R}$, therefore, for each point $(x,y,z)\in P$,

$$((x,y,z)-v_0)\cdot v_0=(x-\lambda a, y-\lambda b, z-\lambda c)\cdot\lambda(a,b,c)=0$$

i.e.

$$\lambda a(x-\lambda a)+\lambda b(y-\lambda b)+\lambda c(x-\lambda c)=-(a^2+b^2+c^2)\lambda^2+(ax+by+cz)\lambda=0$$

$$(a^2+b^2+c^2)\lambda^2+d\lambda=0$$

this implies $\lambda=-\frac{d}{a^2+b^2+c^2}$(when $\lambda=0$, $d=0$), so $|v_0|=|lambda||v|=\frac{|d|}{\sqrt{a^2+b^2+c^2}}$, which is exactly the distance from the plane to the origin (0, 0, 0).

1-4 Ex.11
a. Show that the volume V of a parallelepiped generated by three linearly independent vectors $u,v,w\in\mathbb{R}^3$ is given by $V=|(u\wedge v)\cdot w|$, and introduce an oriented volume in $\mathbb{R}^3$.
b. Prove that

$$V^2=\begin{vmatrix} u\cdot u & u\cdot v & u\cdot w\\ v\cdot u & v\cdot v & v\cdot w\\ w\cdot u & w\cdot v & w\cdot w\end{vmatrix}$$

.

Proof.
a.

$$V=S\cdot h=|u||v|\sin\langle u,v\rangle h=|u\wedge v|\rangle\frac{|(u\wedge v)\cdot w|}{|u\wedge v|}=|u\wedge v|\cdot w\quad\square$$

b. Note that both sides of the equation

$$V^2=\begin{vmatrix} u\cdot u & u\cdot v & u\cdot w\\ v\cdot u & v\cdot v & v\cdot w\\ w\cdot u & w\cdot v & w\cdot w\end{vmatrix}$$

are linear in u,v,w. So it suffices to show that the equation holds for all basis vectors e_1,e_2,e_3, moreover, it's easy to see that, if u,v,w are linearly dependent, then both sides will equal to 0. Thus it only remains to verify the cases that (u,v,w) is a substitution of (e_1,e_2,e_3). In these cases,

$$V^2=|(u\wedge v)\cdot w|^2=1$$

$$\begin{vmatrix} u\cdot u & u\cdot v & u\cdot w\\ v\cdot u & v\cdot v & v\cdot w\\ w\cdot u & w\cdot v & w\cdot w\end{vmatrix}=\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}=1\quad\square$$