Homework 5

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4-2 Ex.2
Prove the following "converse" of Prop.1: Let $\phi : S \to \bar{S}$ be an isometry and $\mathcal{X} : U \to S$ a parametrization at $p \in S$; then $\bar{\mathcal{X}} = \phi \circ \mathcal{X}$ is a parametrization at $\phi(p)$ and $\bar{E} = E$, $\bar{F} = F$, $\bar{G} = G$.

Solution.
Since ϕ is an isometry from S to $\bar{S}$, it is also a diffeomorphism thus $\bar{\mathcal{X}} = \phi \circ \mathcal{X}$ is a parametrization at $\phi(p)$. Moreover, for any $w_1, w_2 \in T_p(S)$,

$$\langle w_1, w_2 \rangle = \langle d\phi_p(w_1), d\phi_p(w_2) \rangle$$

Particularly,

$$\langle \mathcal{X}_u, \mathcal{X}_v \rangle = \langle d\phi_p(\mathcal{X}_u), d\phi_p(\mathcal{X}_v) \rangle = \langle \bar{\mathcal{X}}_u, \bar{\mathcal{X}}_u \rangle$$

Thus $E = \bar{E}$. Similarly we can obtain $F = \bar{F}$, $G = \bar{G}$. □

4-2 Ex.3
Show that a diffeomorphism $\phi : S \to \bar{S}$ is an isometry if and only if the arc length of any parametrized curve in S is equal to the arc length of the image curve by ϕ.

Solution.
" $\Rightarrow$ " : Suppose that $\phi : S \to \bar{S}$ is an isometry, then the first fundamental form keeps invariant under ϕ and thus the arc length, which can be calculated by first fundamental form, is also invariant. □

" $\Leftarrow$ " : Pick any parametrized curve $\alpha : I \to S, \alpha(t) = \mathcal{X}(u(t), v(t))$, let $\beta = \phi \circ \alpha$, suppose that $s_\alpha = s_\beta$, i.e.

$$\int_0^T |\alpha'(t)| dt = \int_0^T |\beta'(t)| dt$$

since α is arbitrary, it follows that

$$|\alpha'(0)| = |\beta'(0)|$$

which can be written as first fundamental form

$$\sqrt{Eu'^2 + 2Fu'v' + Gv'^2} = \sqrt{\bar{E}u'^2 + 2\bar{F}u'v' + \bar{G}v'^2}$$

for all u', v'. Therefore $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$. □

4-2 Ex.5
Let $\alpha_1 : I \to \mathbb{R}^3$, $\alpha_2 : I \to \mathbb{R}^3$ be regular parametrized curves, where the parameter is the arc length. Assume that the curvatures k_1 of α_1 and k_2 of α_2 satisfy $k_1(s) = k_2(s) \neq 0, s \in I$, let

$$\mathcal{X}_1(s, v) = \alpha_1(s) + v\alpha_1'(s)$$

$$\mathcal{X}_2(s, v) = \alpha_2(s) + v\alpha_2'(s)$$

be their regular tangent surfaces and let V be a neighborhood of (s_0, v_0) such that $\mathcal{X}_1(V) \subset \mathbb{R}^3$, $\mathcal{X}_2(V) \subset \mathbb{R}^3$ are regular surfaces. Prove that $\mathcal{X}_1 \circ \mathcal{X}_2^{-1} : \mathcal{X}_2(V) \to \mathcal{X}_1(V)$ is an isometry.

Solution.
Let $k(s) = k_1(s) = k_2(s)$ Note that

$$\mathcal{X}_{1s} = \alpha_1'(s) + vk_1(s)n_1(s), \mathcal{X}_{1v} = \alpha_1'(s)$$

$$\mathcal{X}_{2s} = \alpha_2'(s) + vk_2(s)n_2(s), \mathcal{X}_{2v} = \alpha_2'(s)$$

Thus,

$$\begin{aligned}
E_1 &= \langle \alpha_1'(s) + vk_1(s)n_1(s), \alpha_1'(s) + vk_1(s)n_1(s)\rangle = |\alpha_1'(s)|^2 + v^2k_1^2(s) = 1 + v^2k^2(s) \\
F_1 &= \langle \alpha_1'(s) + vk_1(s)n_1(s), \alpha_1'(s)\rangle = |\alpha_1'(s)|^2 = 1 \\
G_1 &= \langle \alpha_1'(s), \alpha_1'(s)\rangle = |\alpha_1'(s)|^2 = 1
\end{aligned}$$

$$\begin{aligned}
E_2 &= \langle \alpha_2'(s) + vk_2(s)n_2(s), \alpha_2'(s) + vk_2(s)n_2(s)\rangle = |\alpha_2'(s)|^2 + v^2k_2^2(s) = 1 + v^2k^2(s) \\
F_2 &= \langle \alpha_2'(s) + vk_2(s)n_2(s), \alpha_2'(s)\rangle = |\alpha_2'(s)|^2 = 1 \\
G_2 &= \langle \alpha_2'(s), \alpha_2'(s)\rangle = |\alpha_2'(s)|^2 = 1
\end{aligned}$$

which shows that $\phi = \mathcal{X}_1 \circ \mathcal{X}_2^{-1}$ is an isometry. □

4-2 Ex.6
Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve with $k(t) \neq 0$, $t \in I$. Let $\mathcal{X}(t, v)$ be its tangent surface. Prove that, for each $(t_0, v_0) \in I \times (R - \{0\})$, there exists a neighborhood V of (t_0, v_0) such that $\mathcal{X}(V)$ is isometric to an open set of the plane(thus, tangent surfaces are locally isometric to planes).

Solution.
Suppose α is parametrized by arc length. Then consider a plane curve β, which is also parametrized by arc length. And let $k_\beta(s) = k_\alpha(s)$, then by the conclusion of $Ex.5$, the tangent surface of α is isometric to the tangent surface of β, which is an open set of a plane since β is a plane curve. □

4-2 Ex.8
Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be a map such that

$$|G(p) - G(q)| = |p - q|$$

for all $p, q \in \mathbb{R}^3$.
(that is, G is a distance-preserving map). Prove that there exists $p_0 \in \mathbb{R}^3$ and a linear isometry F of the vector space $\mathbb{R}^3$ such that

$$G(p) = F(p) + p_0$$

for all $p \in \mathbb{R}^3$.

Solution.
Let $p_0 = G(0)$ and define $F(p) = G(p) - p_0$. Then

$$|F(p) - F(q)| = |G(p) - G(q)| = |p - q|$$

Particularly, let $q = 0$, we get $|F(p)| = |p|$. And therefore $F(0) = 0$. Note that for each $w_1, w_2 \in \mathbb{R}^3$,

$$\begin{aligned}
|F(w_1 + w_2)|^2 - |F(w_1 - w_2)|^2 &= \langle w_1 + w_2, w_1 + w_2\rangle - \langle w_1 - w_2, w_1 - w_2\rangle \\
&= 4\langle w_1, w_2\rangle
\end{aligned}$$