Volume Estimation of High-Dimensional Convex Bodies

MCMC Approach

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Problem: Calculating the volume of a convex body is NP-hard.

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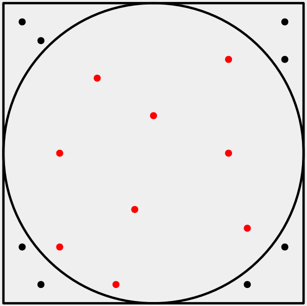
Solution: Use randomized approximation algorithm (MCMC).

Idea: Sample points randomly to estimate volume



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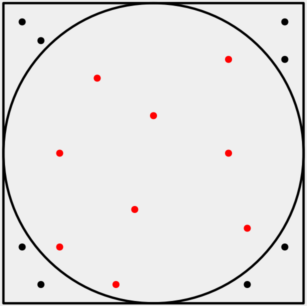
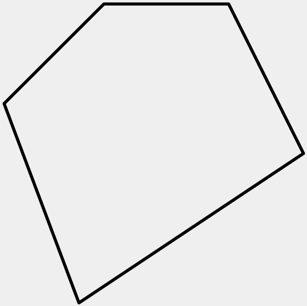
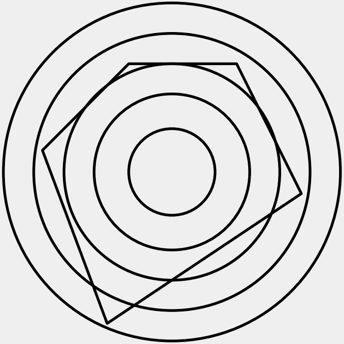


New problem: $\frac{Vol(\text{Unit ball})}{Vol(\text{Unit cube})} = O(2^{-d})$, where d is the dimension

Consider the following convex body

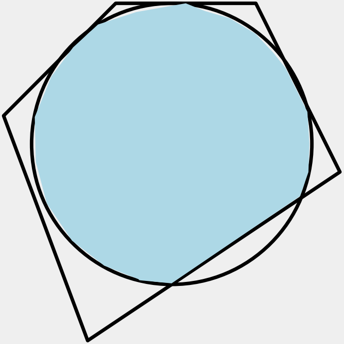


Solution: Subdivide the body into zones to keep ratio constant



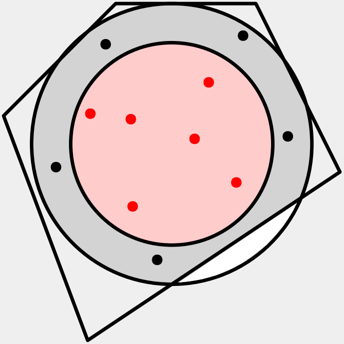
Note: The smallest ball is fully contained and the biggest ball contains the body.

Solution: Subdivide the body into zones to keep ratio constant



Each zone is the intersection of a ball with the body

Then sample in one zone and count how many points fall into the smaller zone



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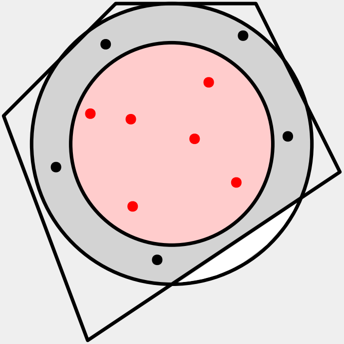


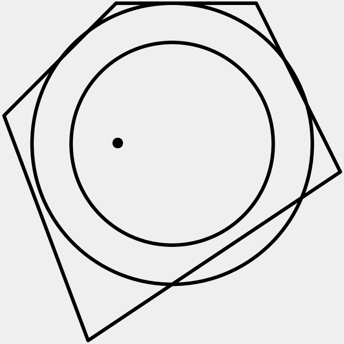
This gives us the ratio $\frac{\#\text{Total samples}}{\#\text{Samples in }\text{Zone}_{\text{i}}} \approx \frac{Vol(\text{Zone}_{\text{i+1}})}{Vol(\text{Zone}_{\text{i}})}$

Doing this for all zones we can calculate the volume of the body:

$$Vol(\text{Body}) = \frac{Vol(\text{Zone}_n)}{Vol(\text{Zone}_{n-1})} \cdot \ldots \cdot \frac{Vol(\text{Zone}_2)}{Vol(\text{Zone}_1)} \cdot Vol(\text{Zone}_1)$$

where $Vol(\text{Zone}_1)$ is just the volume of the innermost ball.

How to sample uniformly at random in a zone:



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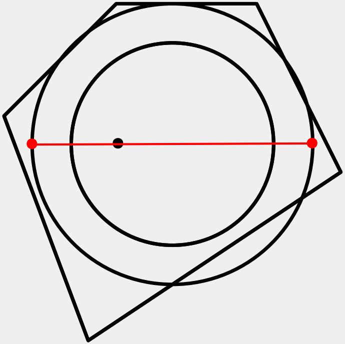

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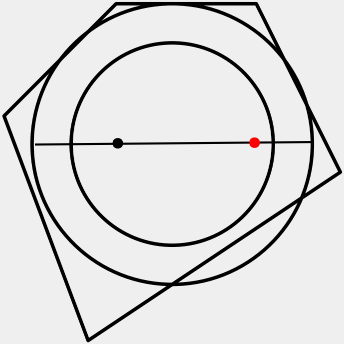

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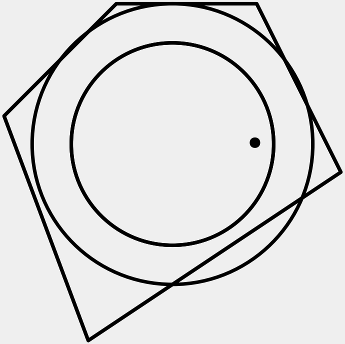

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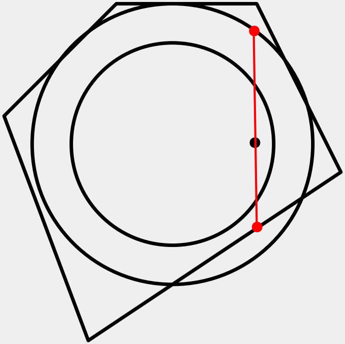

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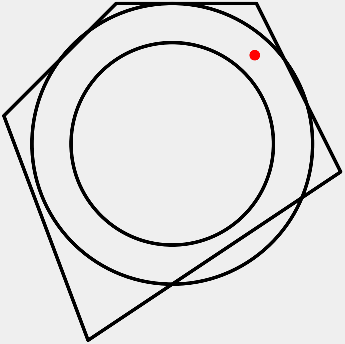
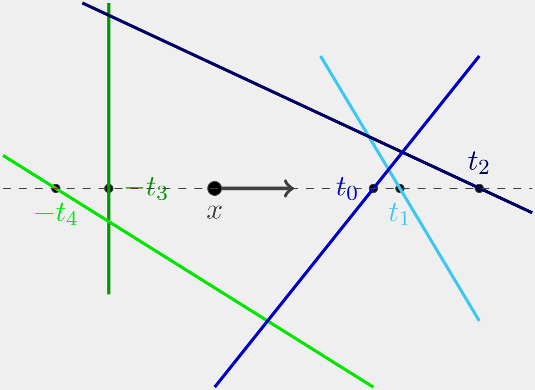


Compute intersections



$$\begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} \\ a_{3,0} & a_{3,1} \\ a_{4,0} & a_{4,1} \end{pmatrix} \cdot \begin{pmatrix} x_1 + t \\ x_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

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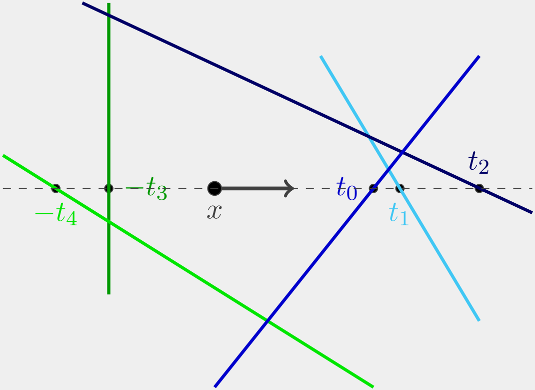

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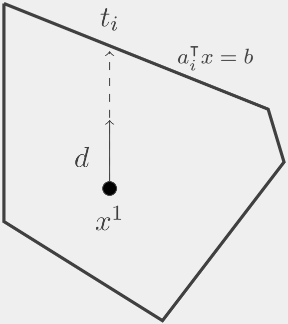
solve for t_i for each constraint a_i

$$\min\{t_0, t_1, t_2\}$$

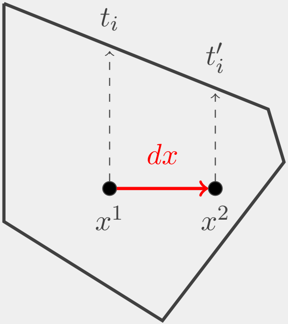
$$\max\{-t_3, -t_4\}$$

Compute intersections



$$x^1 = \begin{pmatrix} x_1^1 \\ \vdots \\ x_k^1 \\ \vdots \\ x_n^1 \end{pmatrix}$$

$$t_i = \frac{b_i - a_i^\intercal x^1}{a_i^\intercal d} = \frac{b_i - a_i^\intercal x^1}{a_{i,j}}$$



$$x^1 = \begin{pmatrix} x^1_1 \\ \vdots \\ x^1_k \\ \vdots \\ x^1_n \end{pmatrix}, \qquad x^2 = \begin{pmatrix} x^1_1 \\ \vdots \\ x^1_k + \|dx\| \\ \vdots \\ x^1_n \end{pmatrix}$$

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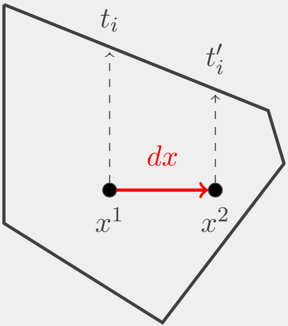
$$t'_i = \frac{b_i - a_i^\intercal x^1 - a_{i,k} \cdot \|dx\|}{a_{i,j}}$$



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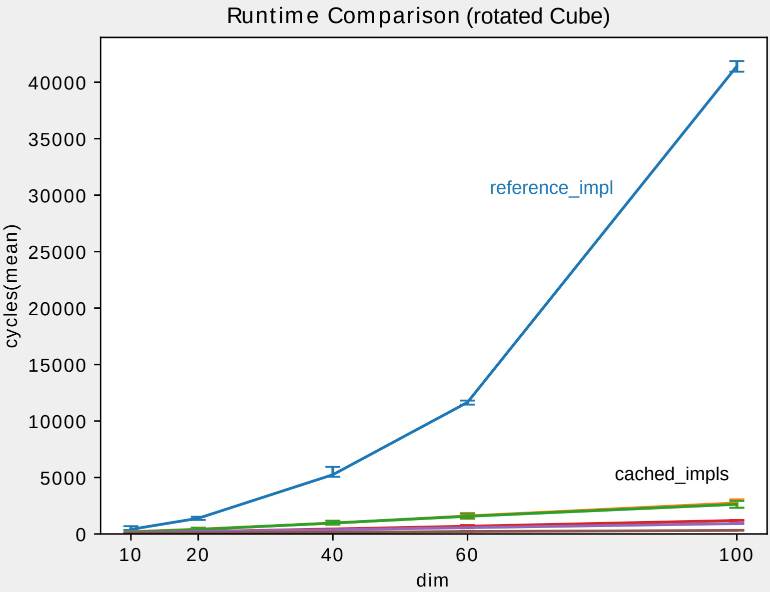
Benchmark: Setup

- Platform: Intel Haswell i7-4870HQ
- Compiler: gcc version 9.3.0
- Compilation flags: -march=native -mfma -ffast-math -O3
- L1d cache: 128kB
- L1i cache: 128kB

Benchmark: Intersect



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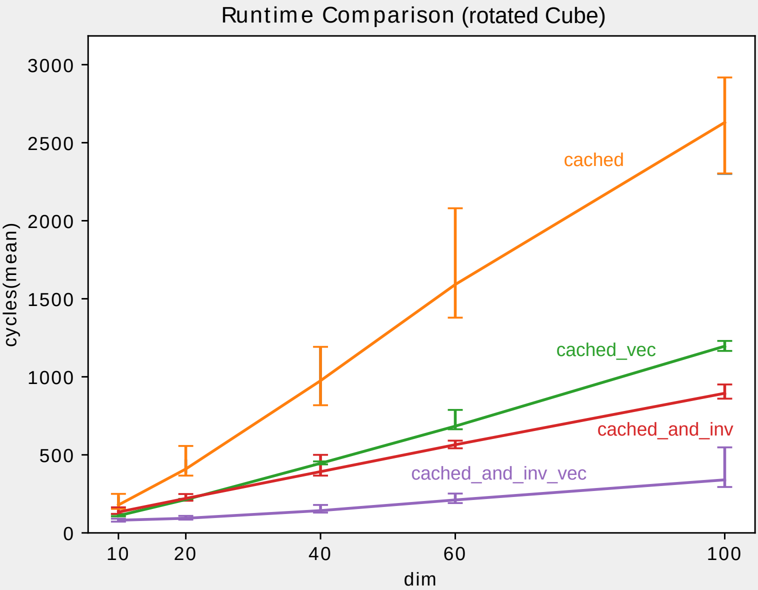

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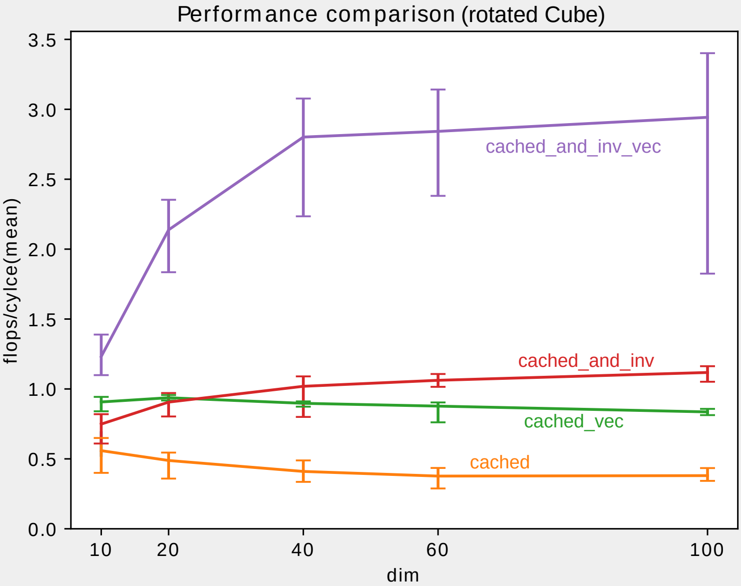

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Sparse Scenario

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- Now assume each constraint contains only a few variables (still NP-hard)

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$$
\begin{gathered}
a_1 \cdot x_1 + a_4 \cdot x_4 \leq b \\
a_0 \cdot x_0 + a_5 \cdot x_5 \leq b \\
\vdots \\
\\
\vdots \\
a_0 \cdot x_1 + a_4 \cdot x_4 \leq b
\end{gathered}
$$

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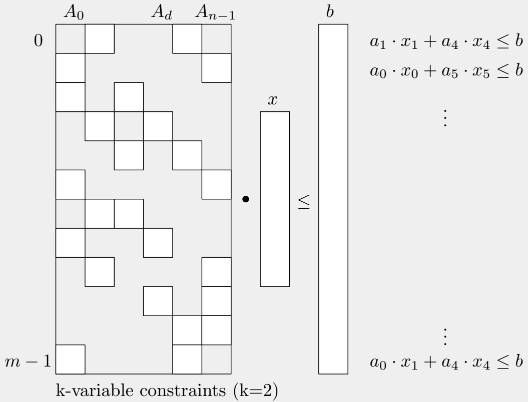
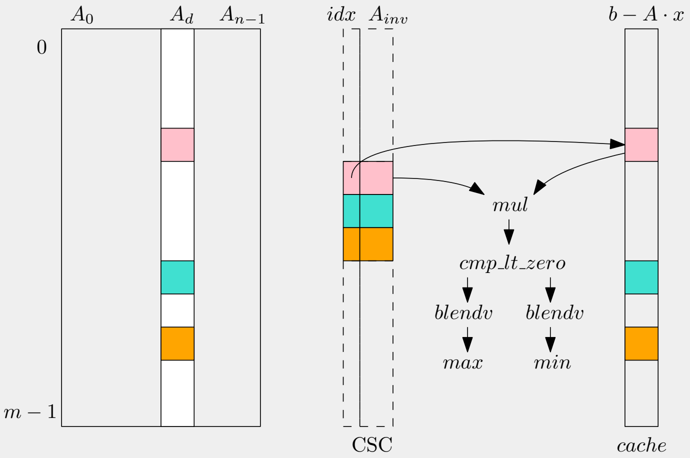
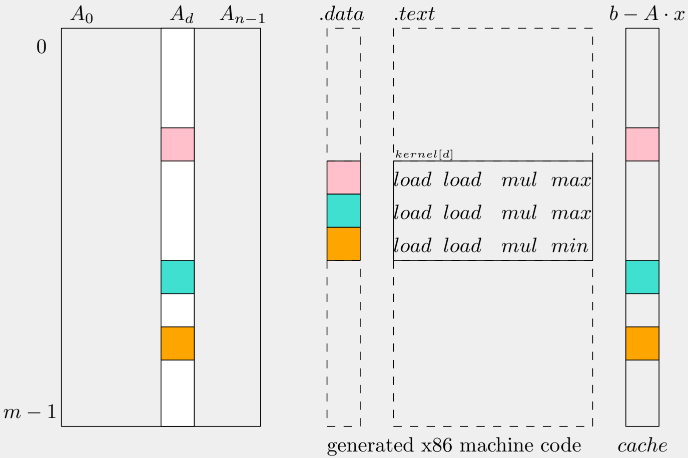


- Sparse matrix format, column major.

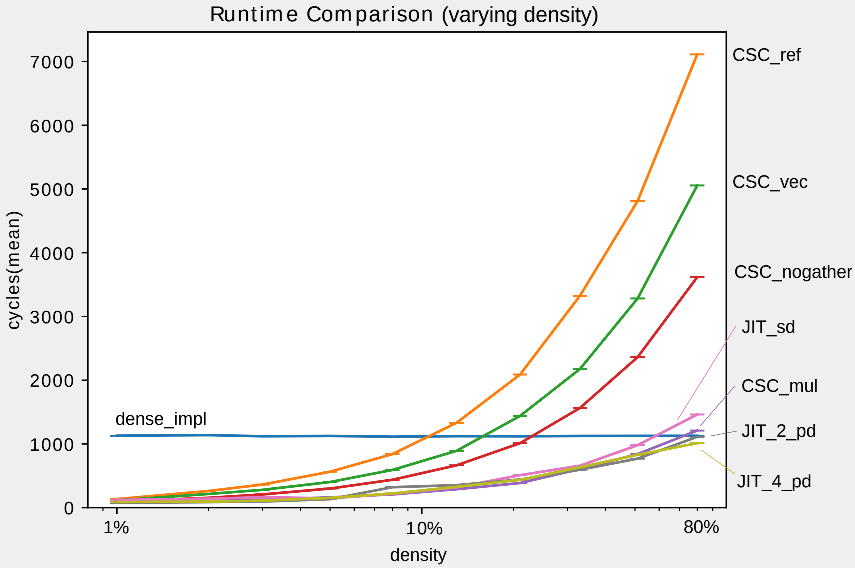


Sparsity JIT

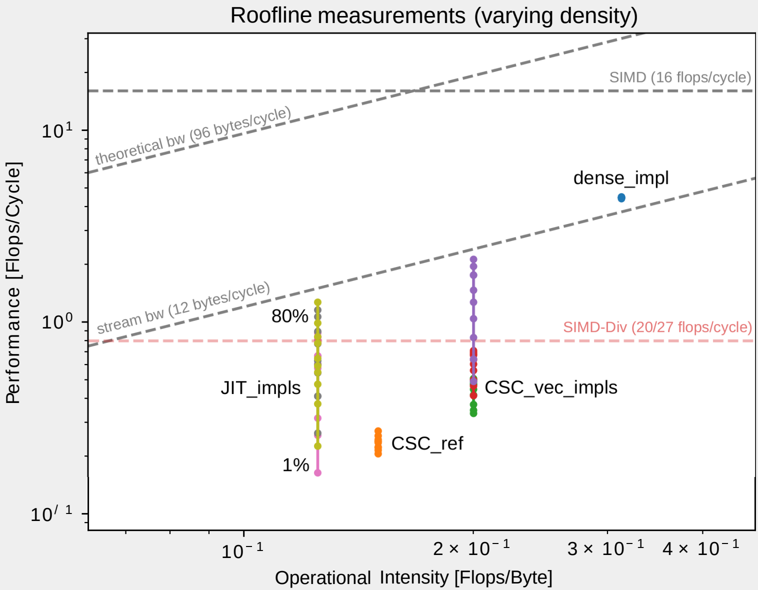
- Run same matrix many times: kernel?
- Generate code at run time (just-in-time). No blendv, no cmp.



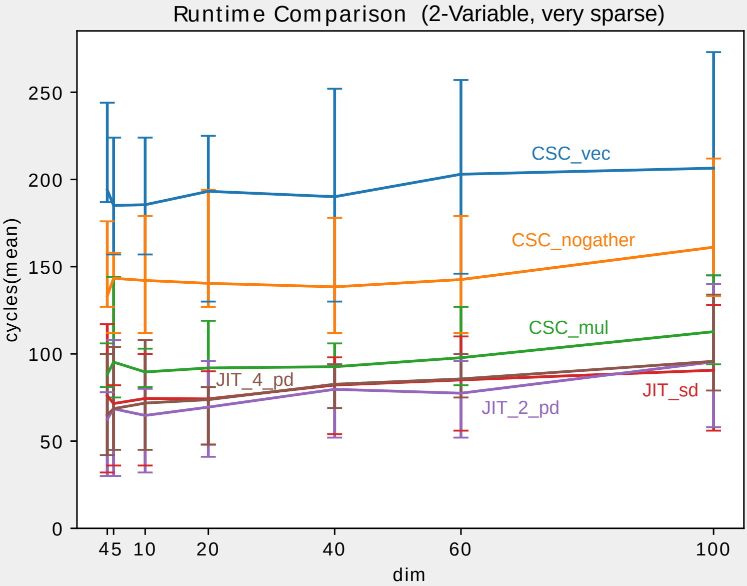
Benchmark: Runtime Density



Benchmark: Roofline Density



Benchmark: Runtime Latency



End

Thank you very much!

Benchmark: Dense Update