Lecture 10 – Regression and Time-series Analysis

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Roadmap

- **Linear Regression**
  - Scenarios where linear regression will be helpful!
  - How to fit a line with least squares.
  - Residuals and homoscedasticity in fitting a line.
  - More complex data and more complex model.
  - How to measure the fit.
- **Time-series Analysis**
  - Varying time-series
  - Objectives of time-series analysis
  - Time-series vs. Regression

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Bordeaux Wine

- Large differences in price and quality between years, although wine is produced in a similar way
- Meant to be aged, so hard to tell if wine will be good when it is on the market
- Expert tasters predict which ones will be good.
- **QUESTION**. Can analytics be used to come up with a different system for judging wine?

Bordeaux Wine

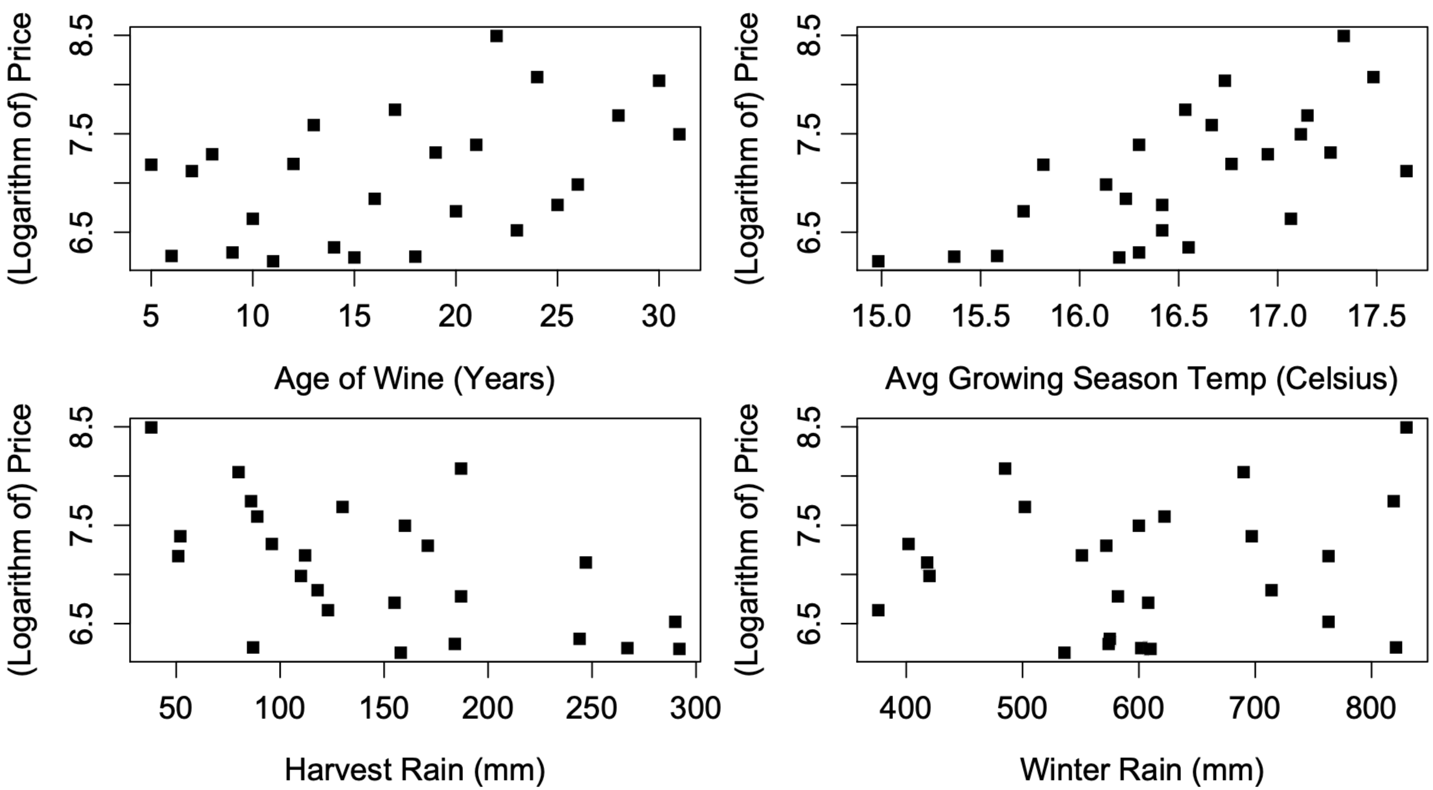
- *Orley Ashenfelter*, a Princeton economics restrictions professor, claims he can predict wine quality without tasting the wine in March 1990.
- *Robert Parker*, the world's most influential wine expert comments on *Ashenfelter*:
  - *"Ashenfelter is an absolute total sham"*
  - *"Rather like a movie critic who never goes to see the movie but tells you how good it is based on the actors and the director"*

Linear Regression vs. Bordeaux Wine

- **Two types of variables are designed:**
  - *Dependent variable* y:
    - typical price in 1990-1991 wine auctions (approximates quality)
  - *Independent variable* x:
    - Age (*older wines are more expensive*)
    - Weather
    - Average Growing Season Temperature
    - Harvest Rain
    - Winter Rain

Linear Regression vs. Bordeaux Wine

- **Relations between price (y) and diverse factors (x):**



Linear Regression

- Suppose we have collected *bivariate data* (x_i, y_i), $i = 1, \ldots, n$.
- **Goal**: to model the relationship between x and y by finding a function $y = f(x)$ that is a close fit to the data.
- **Assumptions**: x_i is *NOT* random and that y_i is a function of x_i plus some random noise.
  - x is called the *independent or predictor variable*
  - y is called the *dependent or response variable*.

Linear Regression (Example)

- **Example 1**. The cost of a first-class stamp in cents over time:

.05 (1963)	.06 (1968)	.08 (1971)	.10 (1974)	.13 (1975)	.15 (1978)	.20 (1981)	.22 (1985)
.25 (1988)	.29 (1991)	.32 (1995)	.33 (1999)	.34 (2001)	.37 (2002)	.39 (2006)	.41 (2007)
.42 (2008)	.44 (2009)	.45 (2012)	.46 (2013)	.49 (2014)			

- Using the R function *lm()* we found the *least squares fit* for a line to this data is:
  - $y = -0.06558 + 0.87574x$
  - where x is the number of years since 1960 and y is in cents.

PREDICT: What is the price for 2016 stamp?

Linear Regression (Example)

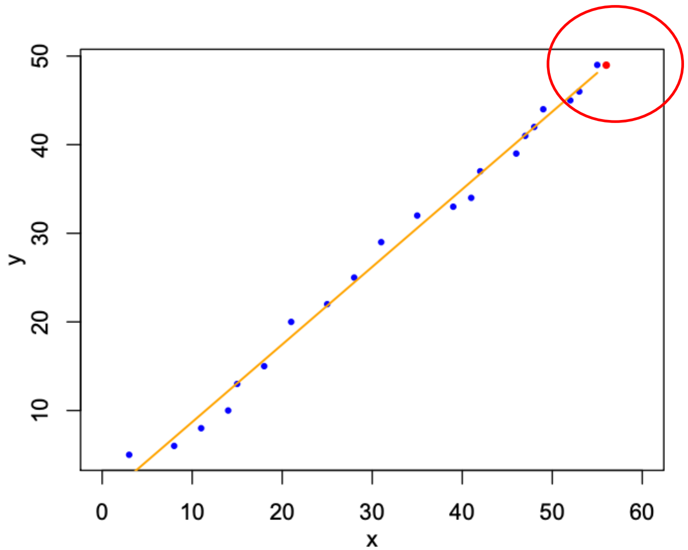
- **Example 1**. The cost of a first-class stamp in cents over time:
  - $y = -0.06558 + 0.87574x$
  - where x is *the number of years since 1960* and y is in cents.
- To predict the price for 2016 stamp, we let $x = 56$, then $y \approx 48.98$
- How to further analyze the relations of x and y?
  - Visualize the data in graphs!

QUESTION. What graph to choose? Barplots, Histograms, or scatterplots?

Linear Regression (Example)

- **Example 1**. The cost of a first-class stamp in cents over time:
  - $y = -0.06558 + 0.87574x$
  - where x is *the number of years since 1960* and y is in cents.

Red plot: the predicted price of the stamp in 2016



Stamp cost (cents) vs. time (years since 1960)

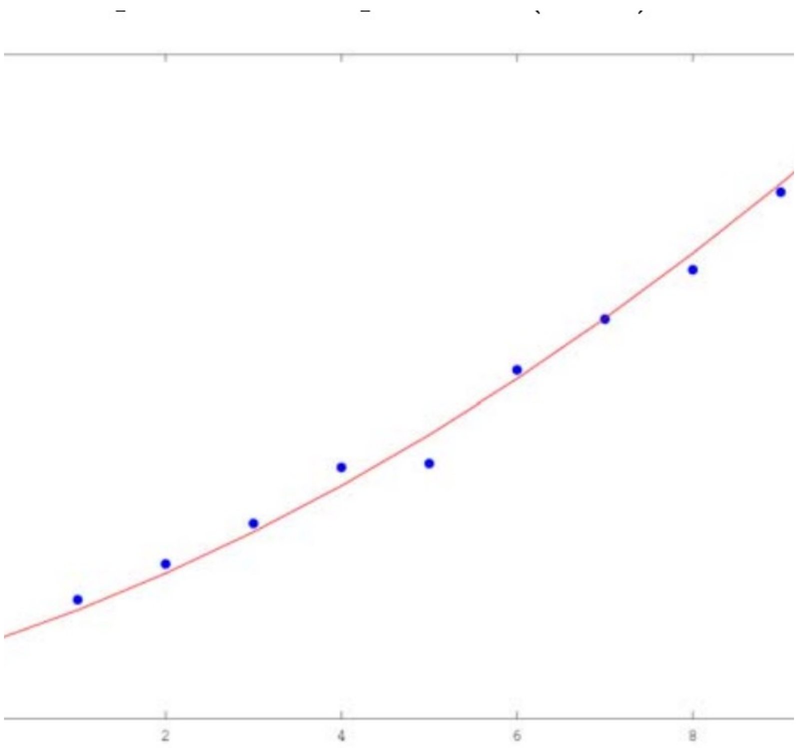
Observation.
- None of the data points actually lie on the line.
- Rather this line has the 'best fit' with respect to all the data, with a small error for each data point.

Linear Regression (More Examples)

- **Example 2**. Suppose we have n pairs of fathers and adult sons.
  - Let x_i and y_i be the heights of the i-th father and son, respectively.
  - The least squares line for this data could be used to predict the adult height of a young boy from that of his father.

Linear Regression (More Examples)

- **Example 3**. We are not limited to best fit lines (sometimes more complex model needed!).
  - For all positive d, the method of least squares may be used to find a polynomial of degree d with the *best fit* to the data.
  - Right hand side figure shows the least squares fit of a parabola (d = 2).



Fitting a parabola, $b_2x^2 + b_1x + b_0$, to data

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Fitting a Line with Least Squares

- Suppose we have several data (x_i, y_i) as above.
- **Goal**. find a line $y = \beta_1 x + \beta_0$ *best fitting* the data.

regression coefficient for the independent variable *intercept coefficient*

- **Assumption**. Each y_i is predicted by x_i up to some error ϵ_i:

Real value *Predicted value*

Error

  - $y_i = \beta_1 x_i + \beta_0 + \epsilon_i$

- **QUESTION**. How to find out the values of β_1 and β_0?

Fitting a Line with Least Squares

- **Goal**: find a line $y = \beta_1 x + \beta_0$ *best fitting* the data.
- **Assumption**: Each y_i is predicted by x_i up to some error ϵ_i:
  - $y_i = \beta_1 x_i + \beta_0 + \epsilon_i$
- **Errors**. The sum of the square errors:
  - $S(\beta_0, \beta_1) = \sum_i \epsilon_i^2 = \sum_i (y_i - \beta_1 x_i - \beta_0)^2$
  - The method of least squares finds the values $\hat{\beta}_0$ and $\hat{\beta}_1$ of β_0 and β_1 that minimize $S(\beta_0, \beta_1)$, the sum of the squared errors.
- **QUESTION**. How to find out $\hat{\beta}_0$ and $\hat{\beta}_1$?
  - **Hint**. Use the methods in *Calculus*!

Fitting a Line with Least Squares

- **Errors**. The sum of the square errors:
  - $S(\beta_0, \beta_1) = \sum_i \epsilon_i^2 = \sum_i (y_i - \beta_1 x_i - \beta_0)^2$
  - The method of least squares finds the values $\hat{\beta}_0$ and $\hat{\beta}_1$ of β_0 and β_1 that minimize $S(\beta_0, \beta_1)$, the sum of the squared errors.
- $\hat{\beta}_1 = \frac{s_{xy}}{s_{xx}}$ and $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

n is the number of data points

  - $\bar{x} = \frac{1}{n} \sum_i x_i$ and $\bar{y} = \frac{1}{n} \sum_i y_i$

Sample Mean

  - $s_{xx} = \frac{1}{n-1} \sum_i (x_i - \bar{x})^2$ and $s_{xy} = \frac{1}{n-1} \sum_i (x_i - \bar{x})(y_i - \bar{y})$

Sample Variance

Sample Covariance of x and y

Fitting a line with Least Squares

- Use least squares to fit a line to the following three data points: (0,1), (2,1), and (3,4).
  - So, $(x_1, y_1) = (0,1)$, $(x_2, y_2) = (2,1)$, $(x_3, y_3) = (3,4)$.
- **QUESTION**.
  - What are $\bar{x}, \bar{y}, s_{xx}, s_{xy}$?
  - What are $\hat{\beta}_0$ and $\hat{\beta}_1$?

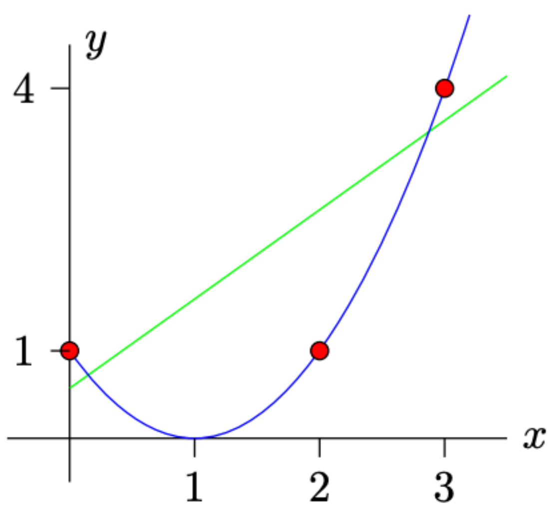
Fitting a line with Least Squares

- Use least squares to fit a line to the following three data points: (0,1), (2,1), and (3,4).
  - So, $(x_1, y_1) = (0,1)$, $(x_2, y_2) = (2,1)$, $(x_3, y_3) = (3,4)$.
- **QUESTION**.
  - $\bar{x} = \frac{5}{3}, \bar{y} = 2, s_{xx} = \frac{7}{3}, s_{xy} = 2$
  - $\hat{\beta}_0 = \frac{4}{7}$ and $\hat{\beta}_1 = \frac{6}{7}$
  - So the least squares line has equation:
    - $y = \frac{6}{7}x + \frac{4}{7}$

Fitting a line with Least Squares

- Use least squares to fit a line to the following three data points: (0,1), (2,1), and (3,4).
  - So, $(x_1, y_1) = (0,1)$, $(x_2, y_2) = (2,1)$, $(x_3, y_3) = (3,4)$.
  - The least square line: $y = \frac{6}{7}x + \frac{4}{7}$.

Which one would you prefer?

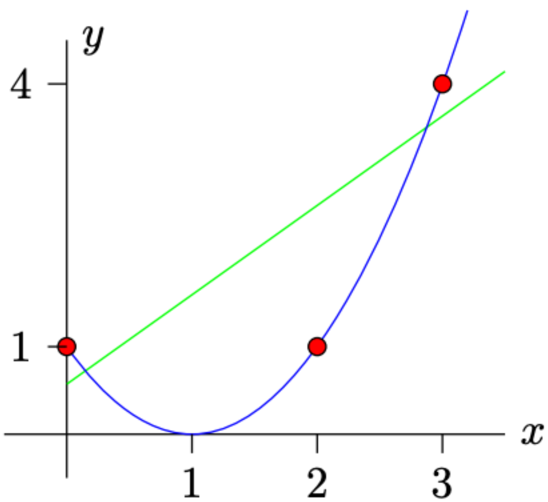


Least square fit of a line

Least square fit of a parabola

Fitting a line with Least Squares

- **Notes**.
  - The word ***"linear"*** in *linear regression* does not refer to fitting a line, though it's the most common curve to fit.
  - When we *fit a line* to *bivariate data*, it is called ***simple linear regression***.



Which one would you prefer?

Least square fit of a line

Least square fit of a parabola

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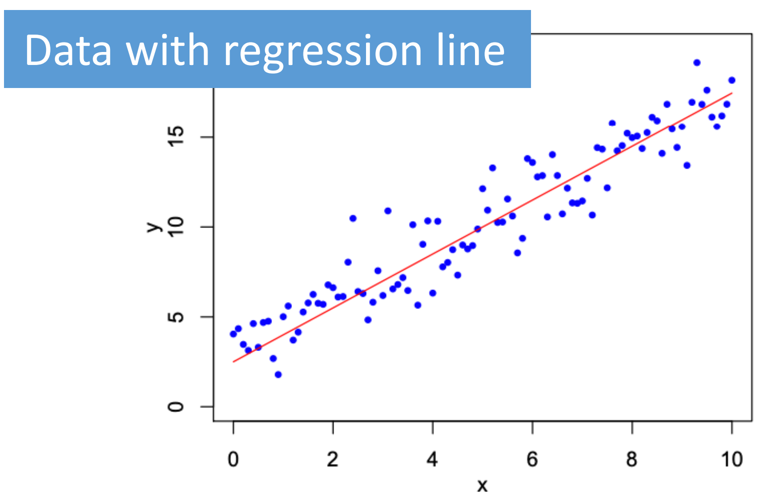
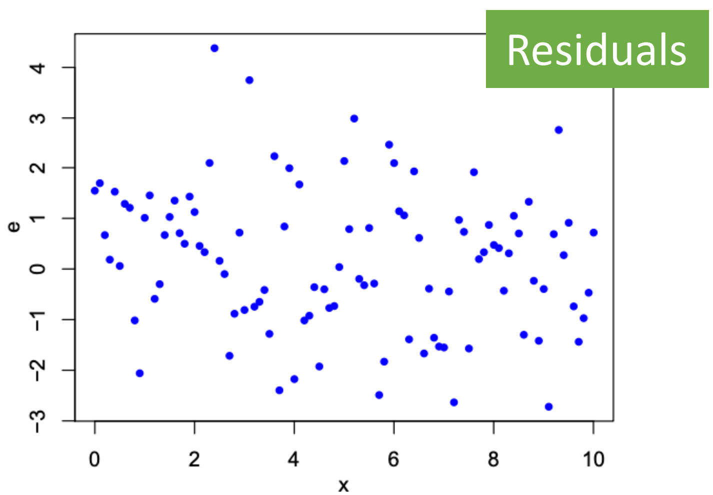
Residuals in Fitting a Line

- Suppose the model is $y_i = \hat{\beta}_1 x_i + \hat{\beta}_0 + \epsilon_i$.
  - $\hat{\beta}_1 x_i + \hat{\beta}_0$ as the predicting or explaining y_i
  - The left-over term ϵ_i is called the ***residual***.
  - Residuals as *random noise* or *measurement error*
- When plot the residuals out, the data points should hover near the regression line. The residuals should look about the same across the range of x.





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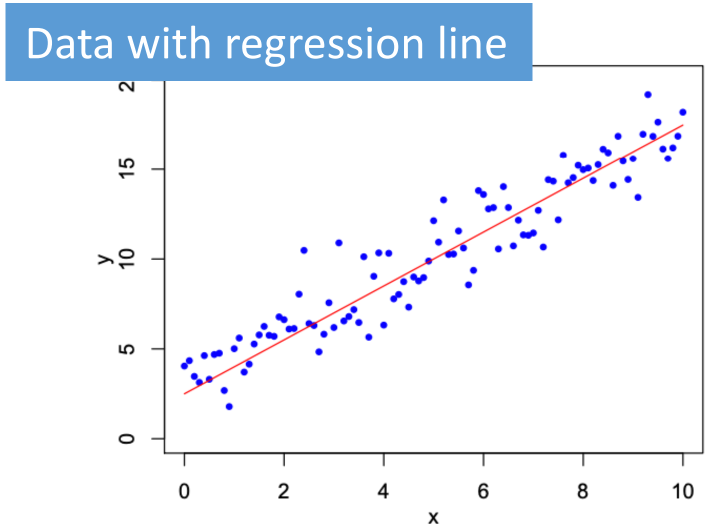
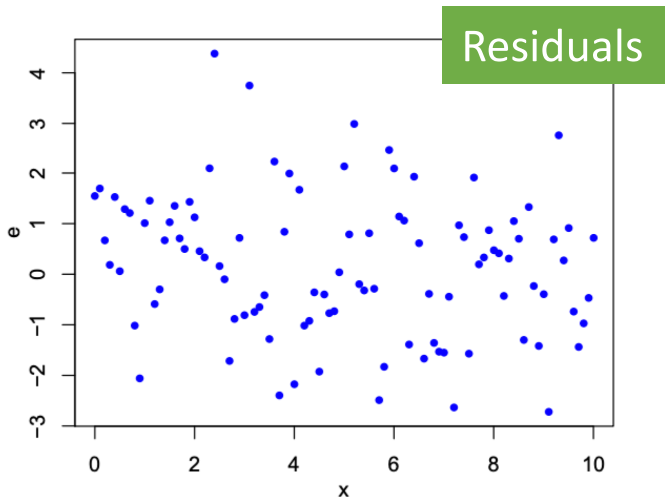


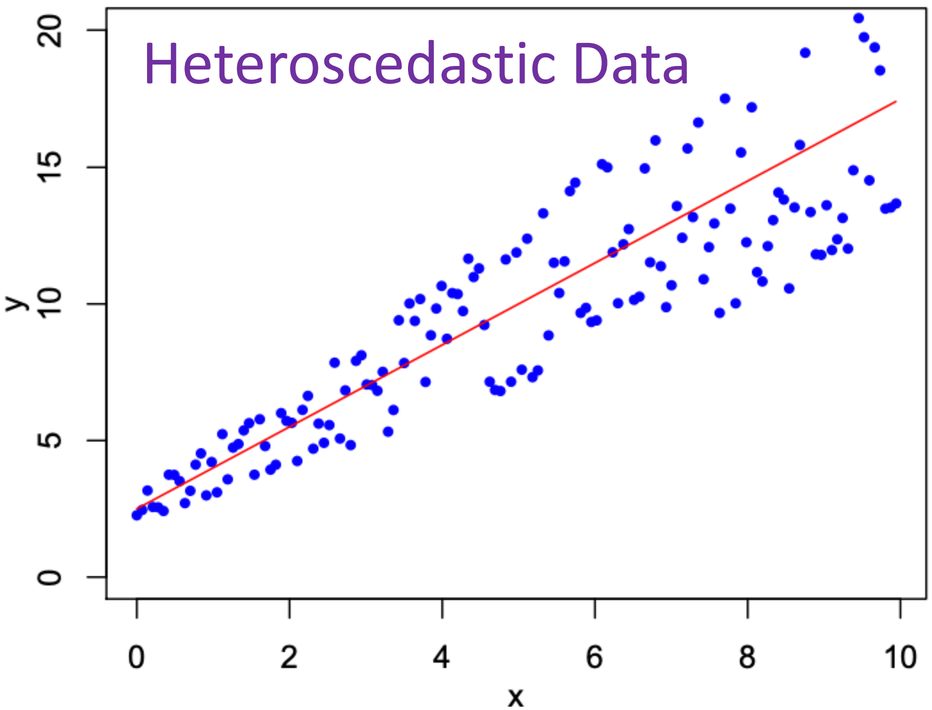


The data hovers in the band of fixed width around the regression line.

At every x the residuals have about the same vertical spread

Homoscedasticity in Fitting a Line

- **Assumption**. The residuals ϵ_i have the same variance for all i. This is called ***homoscedasticity***.
- The opposite case is called ***heteroscedasticity***.



The vertical spread of the data increases as x increases.

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Linear Regression for Multivariate

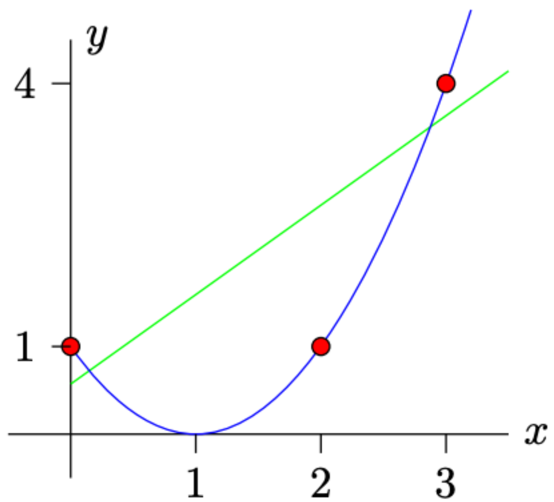
- For multivariate data: $(x_{i,1}, x_{i,2}, \dots, x_{i,m}, y_i)$
- To fit the data with a line (in high dimensional space):
  - $y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_m x_m + \beta_0$

Response Variable   Explanatory (or predictor) variables

- The total square error is:
  - $\sum_i (\beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_m x_{i,m} + \beta_0 - y_i)^2$

Fitting Polynomials

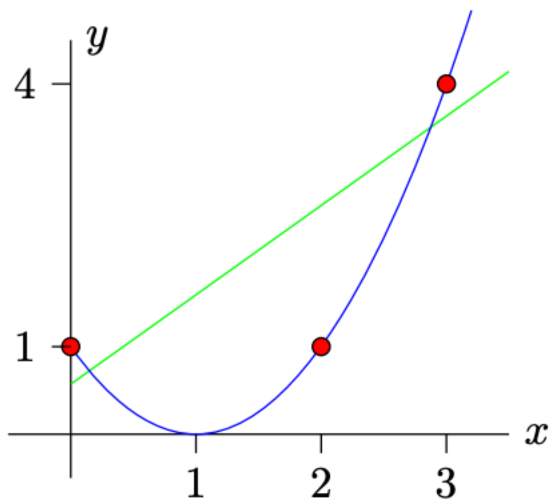
- What is the meaning of "*linear*" in *linear regression*?
  - It refers to the linear algebraic equations for the unknown parameters β_i, i.e. each β_i has exponent 1.
- Use least squares to fit a line to the following three data points: (0,1), (2,1), and (3,4).
  - So, $(x_1, y_1) = (0{,}1)$, $(x_2, y_2) = (2{,}1)$, $(x_3, y_3) = (3{,}4)$.



The parabola has the formula
$y = \beta_0 + \beta_1 x + \beta_2 x^2$

Fitting Polynomials

- The parabola has the formula $y = \beta_0 + \beta_1 x + \beta_2 x^2$
- The square error is
  - $S(\beta_0, \beta_1, \beta_2) = \sum_i \big(\overbrace{\beta_0 + \beta_1 x_i + \beta_2 x_i^2 - y_i}^{\text{The error for } (x_i, y_i)}\big)^2$



Predicted Values *Real Values*

The minimum solutions: $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ will be used to fit polynomials.

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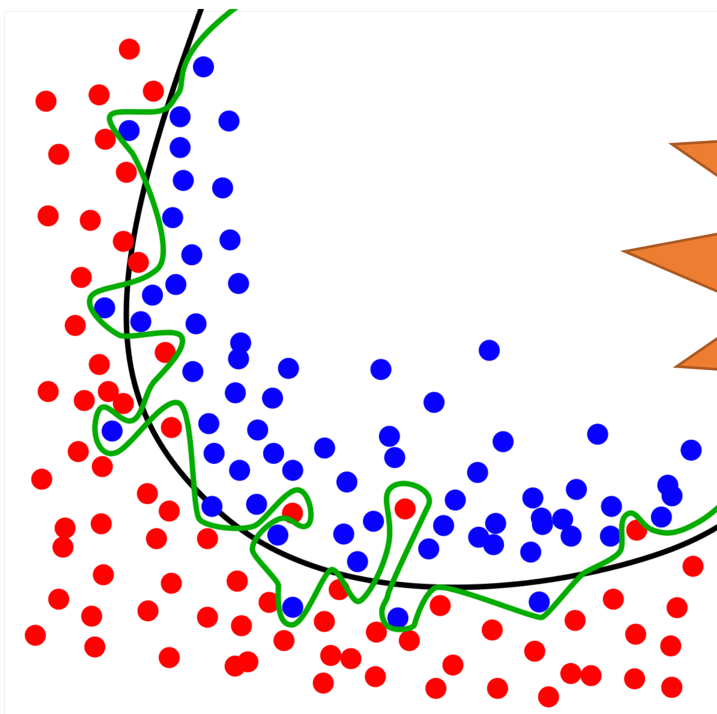
Measuring the Fit

- Data and predicted values of the response variable:
  - $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$
  - $\hat{\boldsymbol{y}} = (\hat{y}_1, \hat{y}_2, \dots, \hat{y}_n)$
- **Total Sum of Squares (TSS)**: $\sum_i (y_i - \bar{y})^2$
- **Residual Sum of Squares (RSS)**: $\sum_i (y_i - \hat{y}_i)^2$
- **The goodness of fit**: $R^2 = 1 - \frac{RSS}{TSS}$
  - More complex model, better fitness (and smaller R^2)!
  - Tradeoff between goodness of fit and complexity.

Overfitting in Regression

- More complex model, better fitness (and smaller R^2)!
- Tradeoff between goodness of fit and complexity.



Unable to fit new data!

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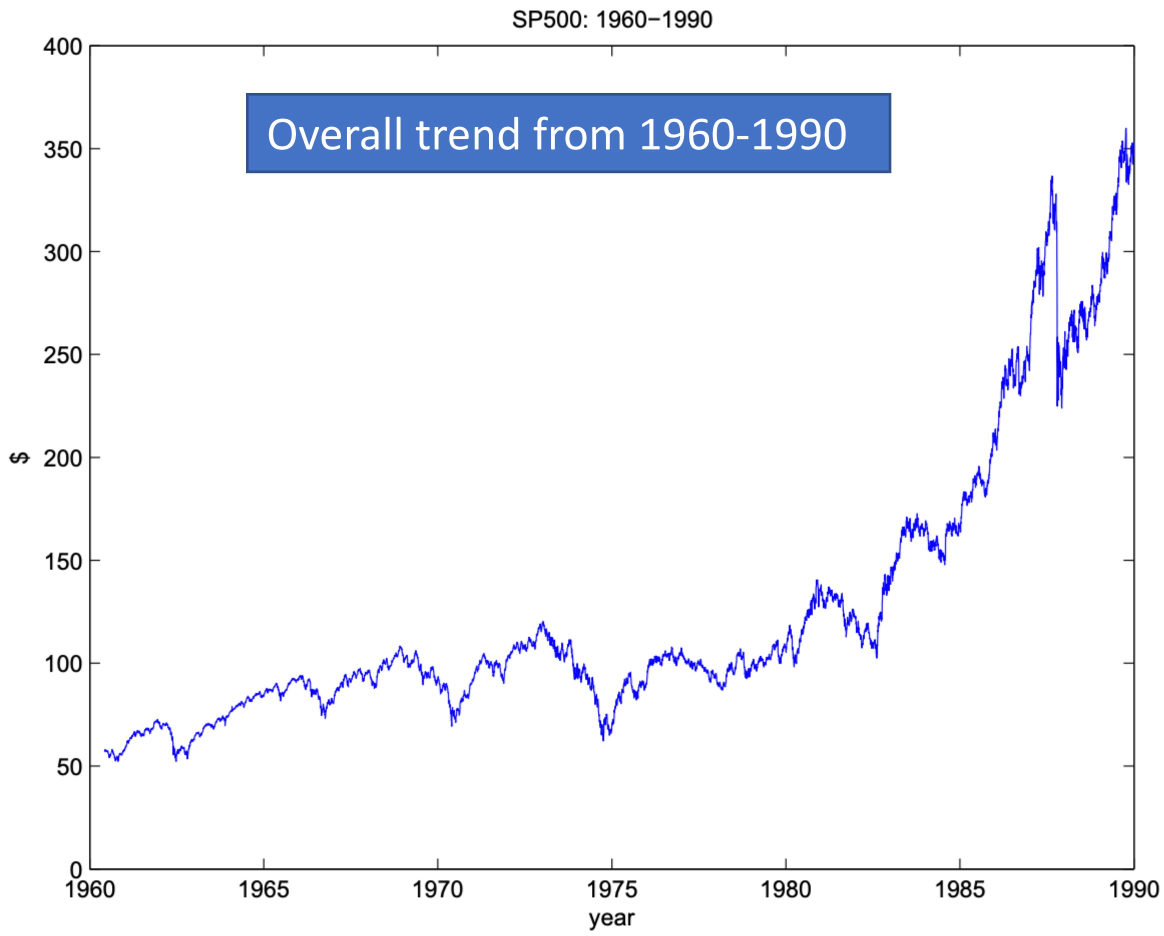
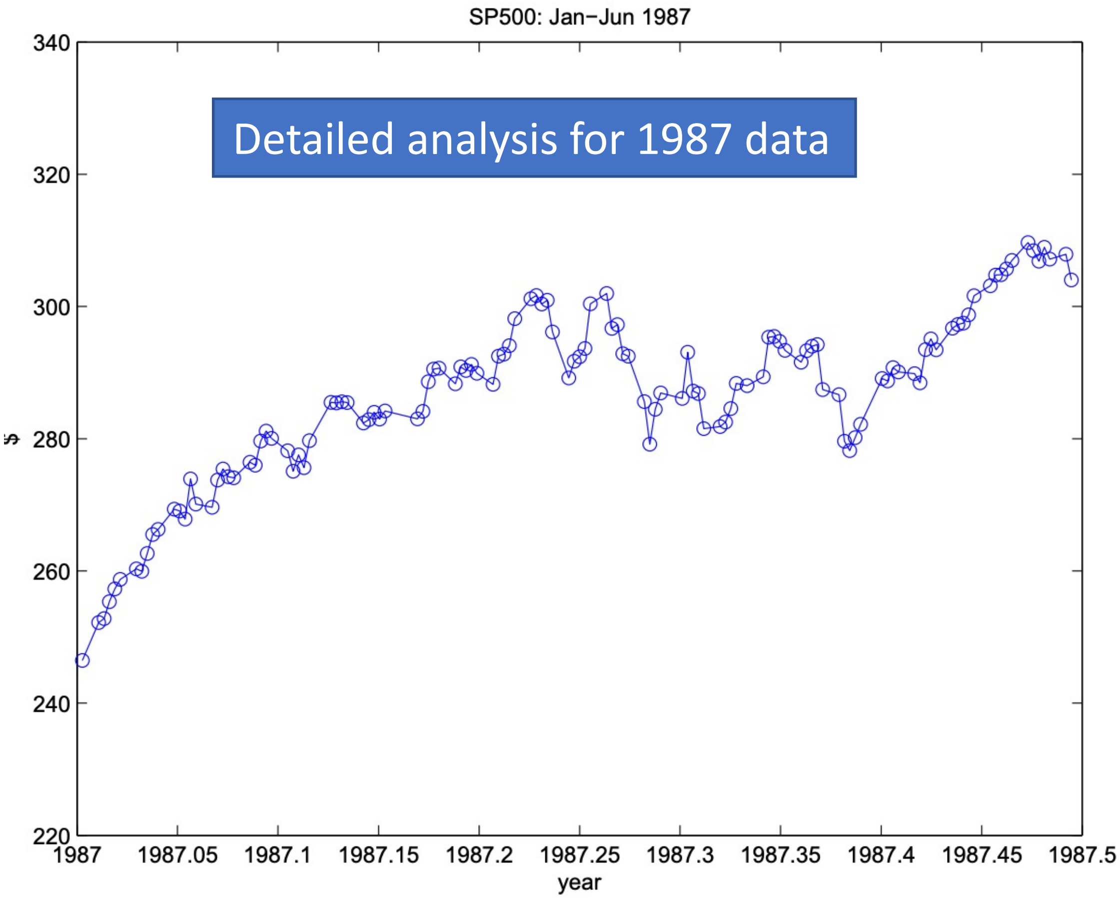
Varying Time Series



The stock performance of 500 large companies listed on stock exchanges in the United States

Varying Time Series



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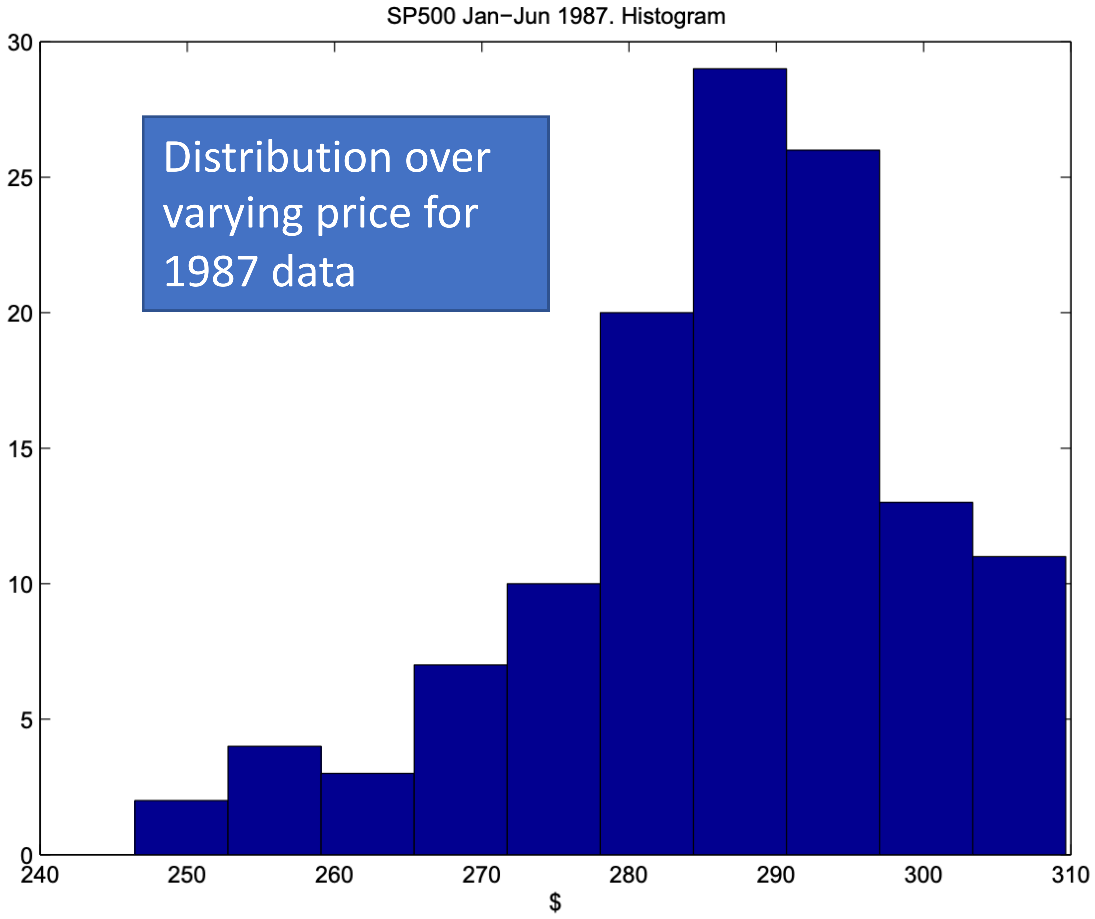

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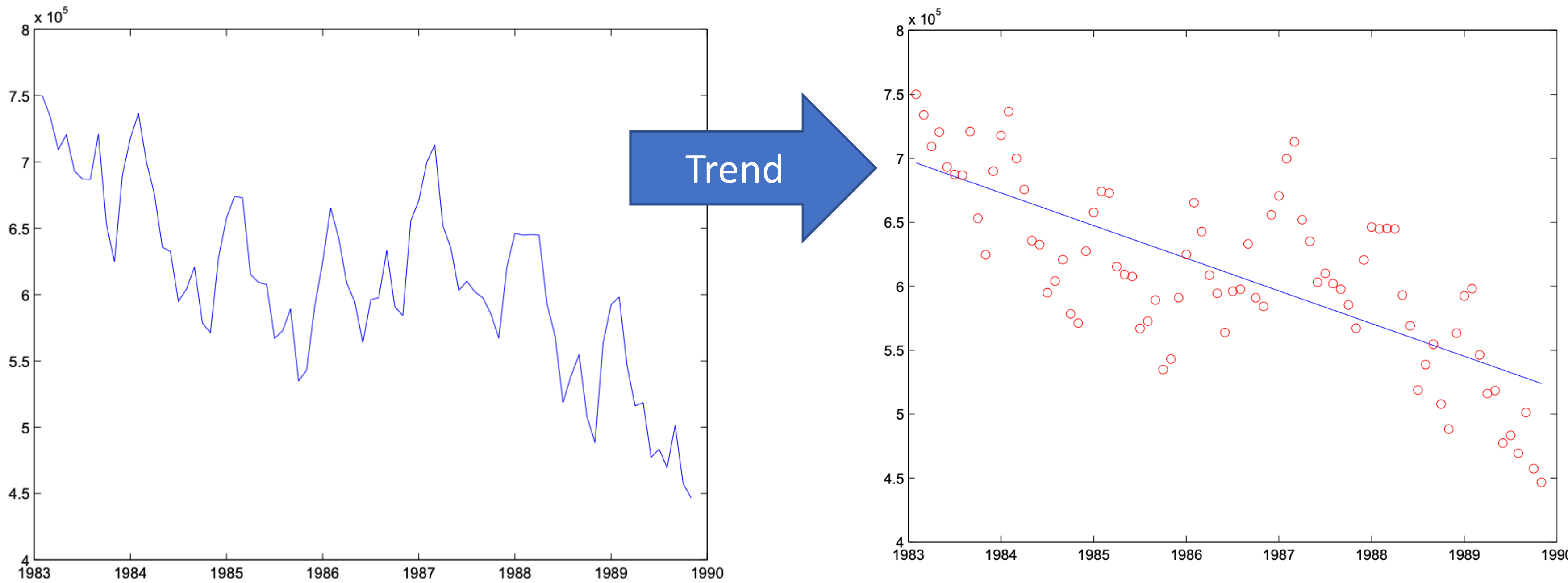
Why we analyze time-series?

- **Compact description of data**:
  - **Level**: The average value in the series.
  - **Trend**: The increasing or decreasing value in the series.
  - **Seasonality**: The repeating short-term cycle in the series.
  - **Noise**: The random variation in the series.
- **Interpretation**: e.g., *seasonal adjustment.*
- **Forecasting**: e.g., *predict unemployment.*

Example: Unemployment Data

- Monthly number of unemployed people over years in Australia. (*Hipel and McLeod, 1994*)



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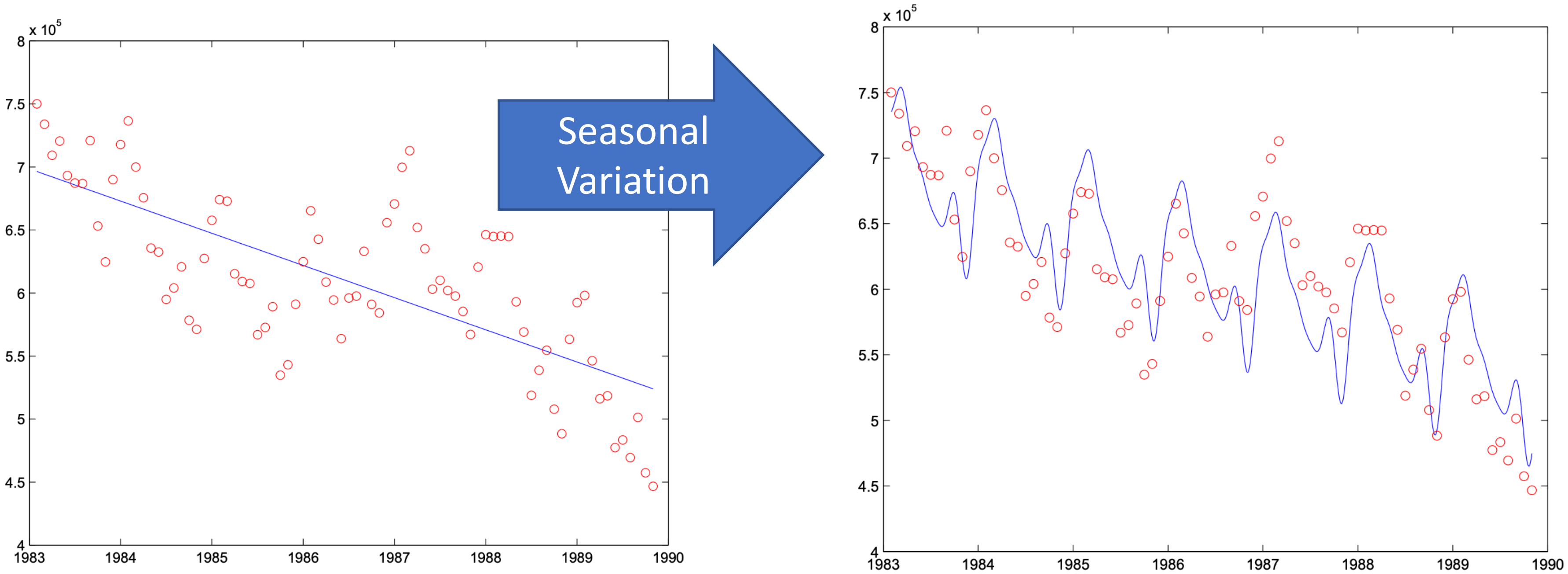

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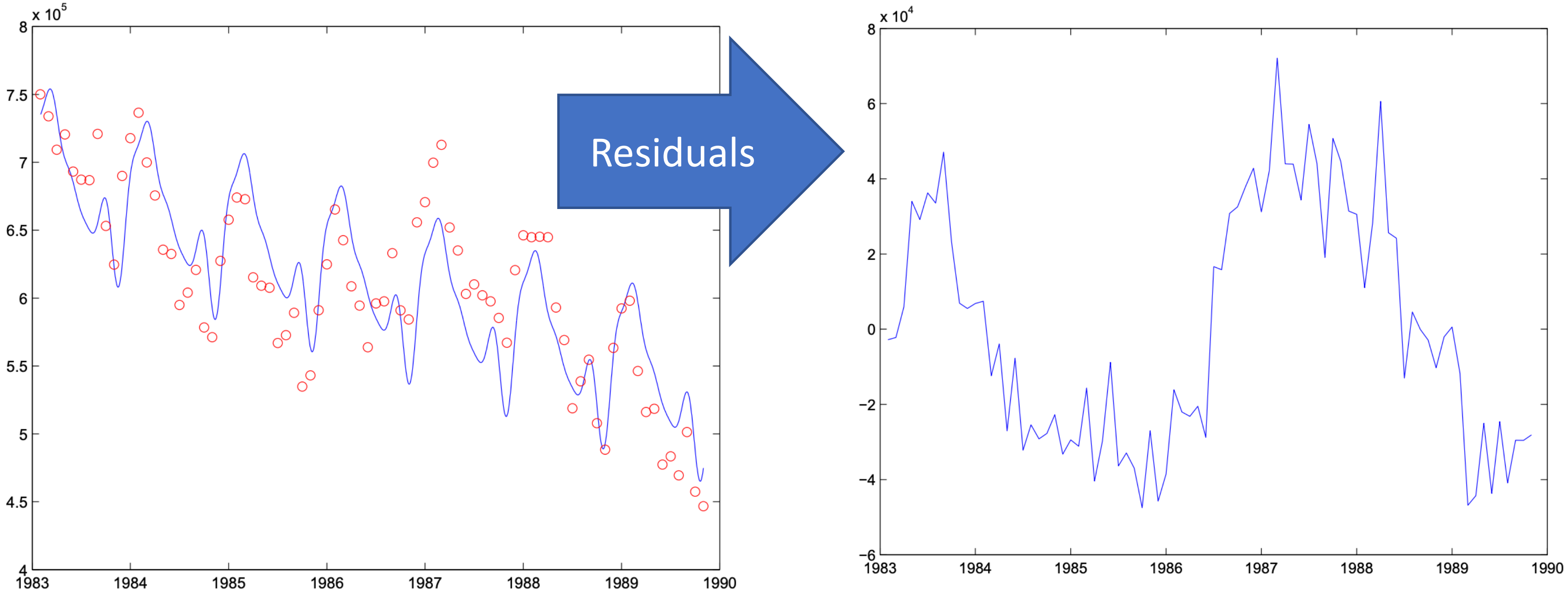

Why we analyze time-series?

- **Compact description of data**
- **Interpretation**: e.g., *seasonal adjustment.*
- **Forecasting**: e.g., *predict unemployment.*
- **Control**: e.g., analyze impact of monetary policy on unemployment.
- **Hypothesis Testing**: e.g., *global warming*.
- **Simulation**: e.g., *estimate probability of catastrophic events*.

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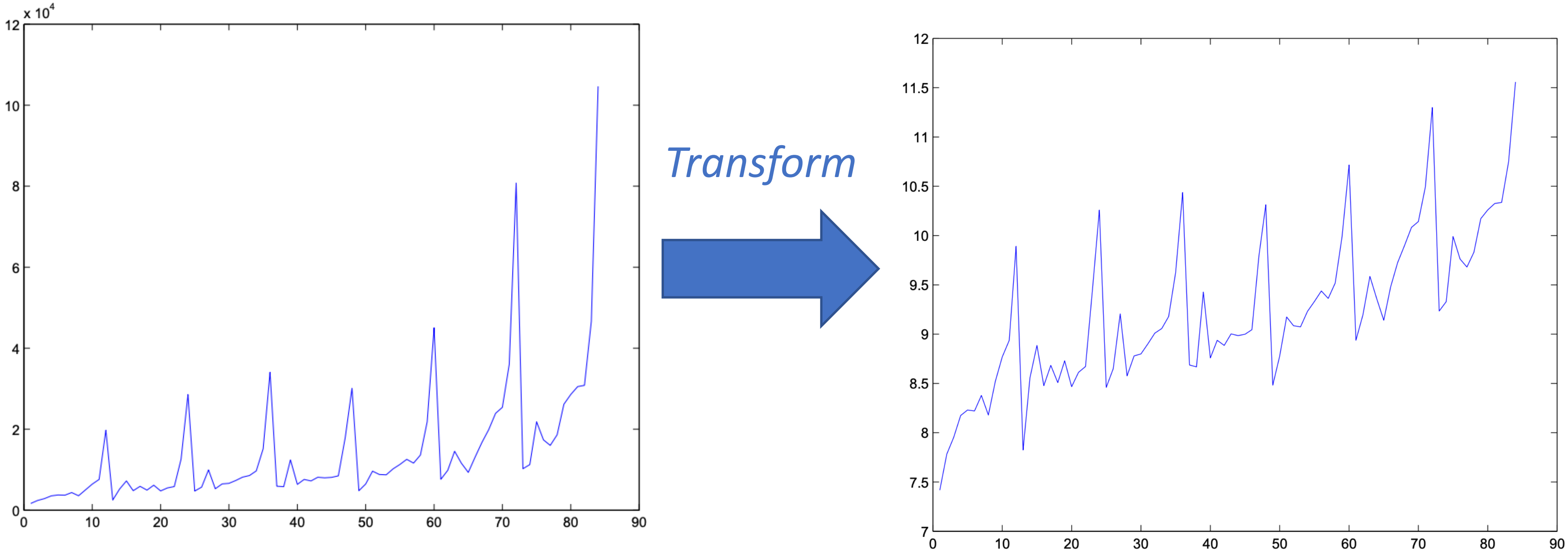
Time-series Analysis vs. Regression

- Monthly sales for a souvenir shop at a beach resort town in Queensland.
  - (*Makridakis, Wheelwright and Hyndman, 1998*)



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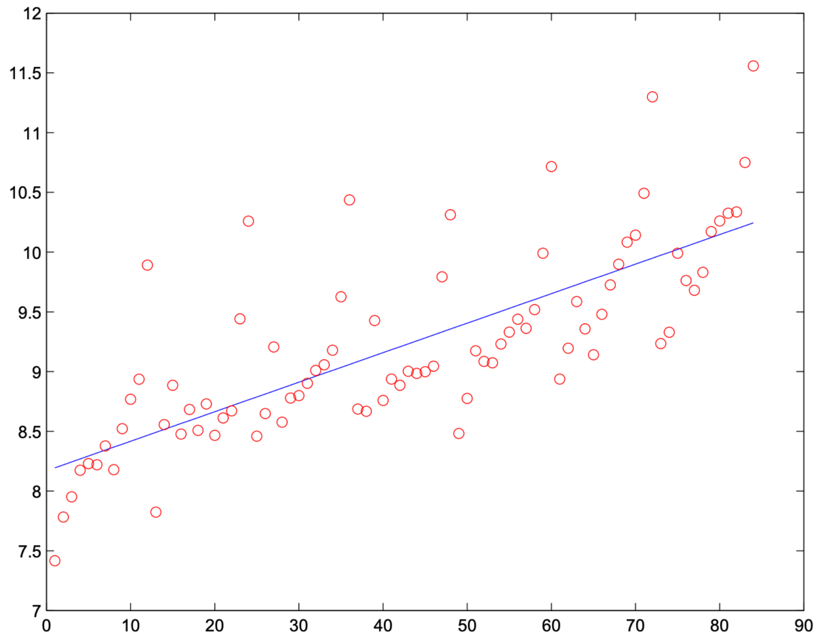
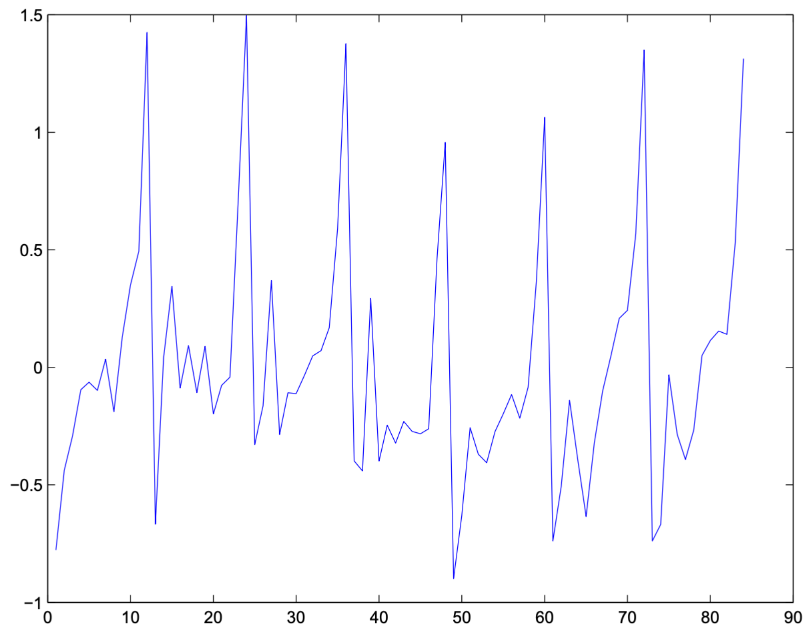
Trend: fitted line



Residuals



A slide to take away

- What is linear regression?
- How to use linear regression to fit data?
- How to evaluate the regression results?
- What are time-series?
- Why we do time-series analysis?
- What are the useful tools to analyze time series?