COMP 1433: Introduction to Data Analytics &
COMP 1003: Statistical Tools and Applications

Lecture 5 – Calculus Basics

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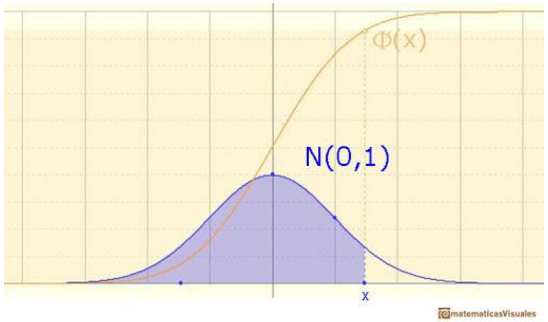
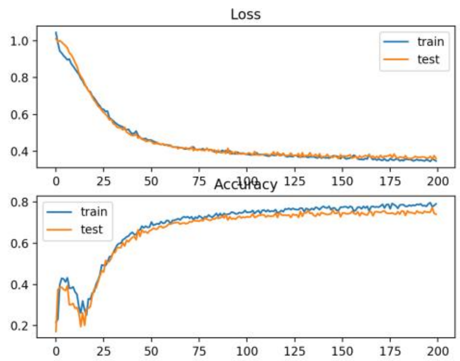


continuous random variables

trend of time-series data



machine learning

Why we learn *Calculus*?

What are *Functions*?

- y is a *function* of x, written with $y = f(x)$:
  - Every value of x corresponds to *one and only one value* of y.
  - x is the *independent variable* while y is the *dependent variable*.
  - **EXAMPLE**. Distance traveled per hour y is a function of velocity x.

Optimization of a Function

- Given a function $f(x_1, x_2, \dots, x_n)$, where $x_1, x_2, \dots, x_n$ are variables or parameters.
- **Optimization**: Find a set of variables $x_1, x_2, \dots, x_n$ that maximize or minimize $f(x_1, x_2, \dots, x_n)$.
- An optimization problem in everyday life:
  - You selected 3 classes this semester. The three classes have different effects on the GPA and you want to maximize the GPA (*the value of a function*) via priorly knowing the # of hours (*function variables*) you should spend on each of the class.

A Recipe for Machine Learning

1. Given training data:

$$\{\boldsymbol{x}_i, \boldsymbol{y}_i\}_{i=1}^N$$

Face

Face

Not a face

2. Choose each of these:
   - Decision function

$$\hat{\boldsymbol{y}} = f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)$$

   - Loss function

$$\ell(\hat{\boldsymbol{y}}, \boldsymbol{y}_i) \in \mathbb{R}$$

Examples: Linear regression, Logistic regression, Neural Network

Examples: Mean-squared error, Cross Entropy

Roadmap

- Derivatives
  - Basic Concepts of Derivatives
  - How to calculate derivatives?
  - Partial Derivatives and Gradients
  - **Application**: the training of machine learning
- Antiderivatives: Integrals
  - Basic Concepts of integrals.
  - How to calculate integrals?

Roadmap

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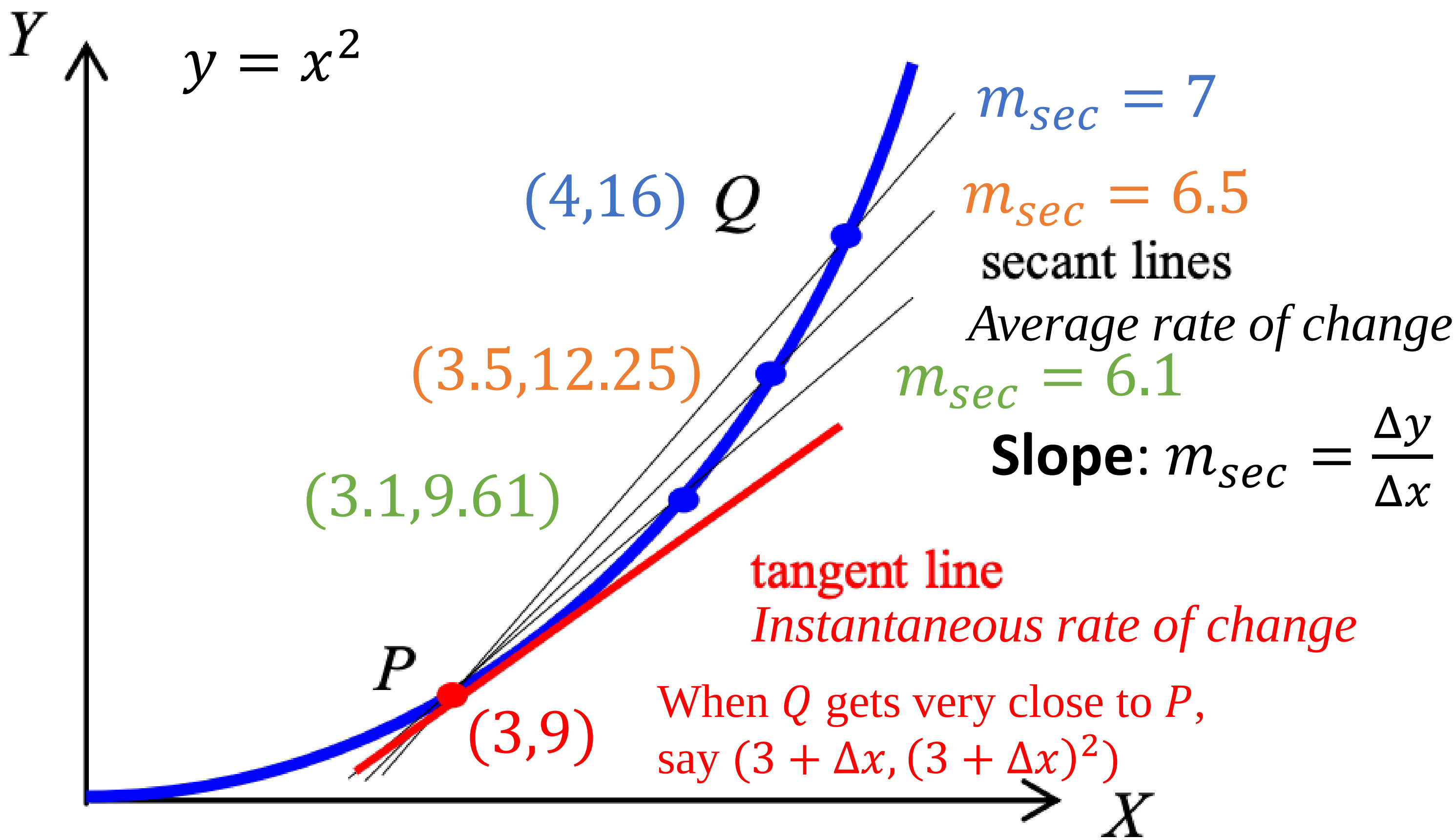
Secant Line and Tangent Line



Slope: $m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\mathrm{f(x+}\Delta x) - \mathrm{f}(x)}{\Delta x}$

What are *Derivatives*?

- The *derivative* of $f(x)$ is the *slope of tangent line* (*instantaneous rate of change*) at $(x, f(x))$
  - $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\mathrm{f(x+}\Delta x) - f(x)}{\Delta x}$
- The process of calculating the derivatives of a function is called *differentiation*.
- **Example**: $y = x^2$
  - $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(\mathrm{x+}\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$
  - Also written as: $dy = 2xdx$ (the *differential* dy in terms of the *differential* dx)
    - Used to estimate the output difference Δy in terms of a small input difference Δx

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Some Useful Derivatives

- **Power Rule**: $\frac{d(x^p)}{dx} = px^{p-1}$
  - $\frac{d(x^3)}{dx} = 3x^2$; $\frac{d(\frac{1}{x})}{dx} = -\frac{1}{x^2}$; $\frac{dx^{\frac{1}{2}}}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$
  - $\frac{dx}{dx} = 1$ ($y = x$ *has slope 1 every where*)
- **Exponential Rule**: $\frac{d(b^x)}{dx} = b^x lnb$
  - $\frac{d(e^x)}{dx} = e^x$
- **Logarithm Rule**: $\frac{d(\log_b x)}{dx} = \frac{1}{xlnb}$
  - $\frac{d(lnx)}{dx} = \frac{1}{x}$
- **Derivatives for constants**: $\frac{dC}{dx} = 0$

Properties of Derivatives

- For any constant c and any *differentiable* function $f(x)$, $\frac{d[cf(x)]}{dx} = c\frac{d[f(x)]}{dx}$
  - $\frac{d[5x^3]}{dx} = 5 \cdot 3x^2 = 15x^2$
  - $\frac{d[-3e^{2x}]}{dx} = -3 \cdot \frac{d\left[\left(e^2\right)^x\right]}{dx} = -3e^{2x}\ln(e^2) = -6e^{2x}$
- For any two *differentiable* functions: $f(x)$ and g(x)
  - **Sum and Difference Rules.**
  - **Product Rule.**
  - **Quotient Rule.**
  - **Chain Rule.**

Properties of Derivatives (cont.)

- For any two *differentiable* functions: $f(x)$ and $g(x)$
  - **Sum and Difference Rules.**
    - $\frac{d[f(x) \pm g(x)]}{dx} = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$
    - **Example**: $y = x^{\frac{3}{2}} - 7x^4 + 10e^{-3x} - 5$
      - $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 28x^3 - 30e^{-3x}$
  - **Product Rule.**
  - **Quotient Rule.**
  - **Chain Rule.**

Properties of Derivatives (cont.)

- For any two *differentiable* functions: $f(x)$ and $g(x)$
  - **Sum and Difference Rules.**
  - **Product Rule.**
    - $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
    - **Example**: $y = x^{11}e^{6x}$
      - $\frac{dy}{dx} = 11x^{10}e^{6x} + 6e^{6x}x^{11} = (11 + 6x)x^{10}e^{6x}$
  - **Quotient Rule.**
  - **Chain Rule.**

Properties of Derivatives (cont.)

- For any two *differentiable* functions: $f(x)$ and g(x)
  - **Sum and Difference Rules.**
  - **Product Rule.**
  - **Quotient Rule.**
    - $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ where $g(x) \neq 0$
    - **Example**: $y = \frac{e^{4x}}{x^7+8}$
      - $\frac{dy}{dx} = \frac{4e^{4x}(x^7+8)-e^{4x}(7x^6)}{(x^7+8)^2} = \frac{e^{4x}(4x^7+32-7x^6)}{(x^7+8)^2}$
  - **Chain Rule.**

Properties of Derivatives (cont.)

- For any two *differentiable* functions: $f(x)$ and g(x)
  - **Sum and Difference Rules.**
  - **Product Rule.**
  - **Quotient Rule.**
  - **Chain Rule.**
    - $\left[f\left(g(x)\right)\right]' = f'\left(g(x)\right) \cdot g'(x)$
    - **Example**: $y = \left(x^{\frac{2}{3}} + 2e^{-9x}\right)^6$
    - $\frac{dy}{dx} = 6\left(x^{\frac{2}{3}} + 2e^{-9x}\right)^5 \cdot (\frac{2}{3}x^{-\frac{1}{3}} - 18e^{-9x})$

Properties of Derivatives (cont.)

- For any two *differentiable* functions: $f(x)$ and g(x)
  - **Sum and Difference Rules.**
  - **Product Rule.**
  - **Quotient Rule.**
  - **Chain Rule.**
    - $\left[f\left(g(x)\right)\right]' = f'\left(g(x)\right) \cdot g'(x)$
    - **Example**: $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
    - $\frac{dy}{dx} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot (-x) = -\frac{1}{\sqrt{2\pi}} x \cdot e^{-\frac{x^2}{2}}$

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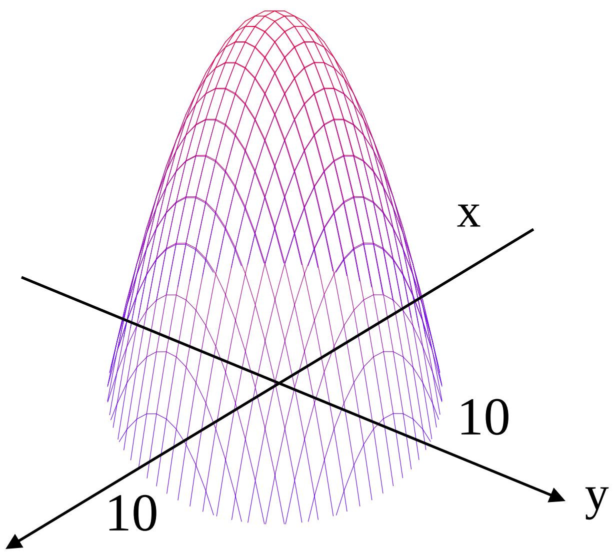
Partial Derivatives



$$f\left(x, y\right) = 100 - x^2 - y^2$$

$$\frac{\partial f}{\partial x} = -2x \qquad \frac{\partial f}{\partial x} = -2y$$

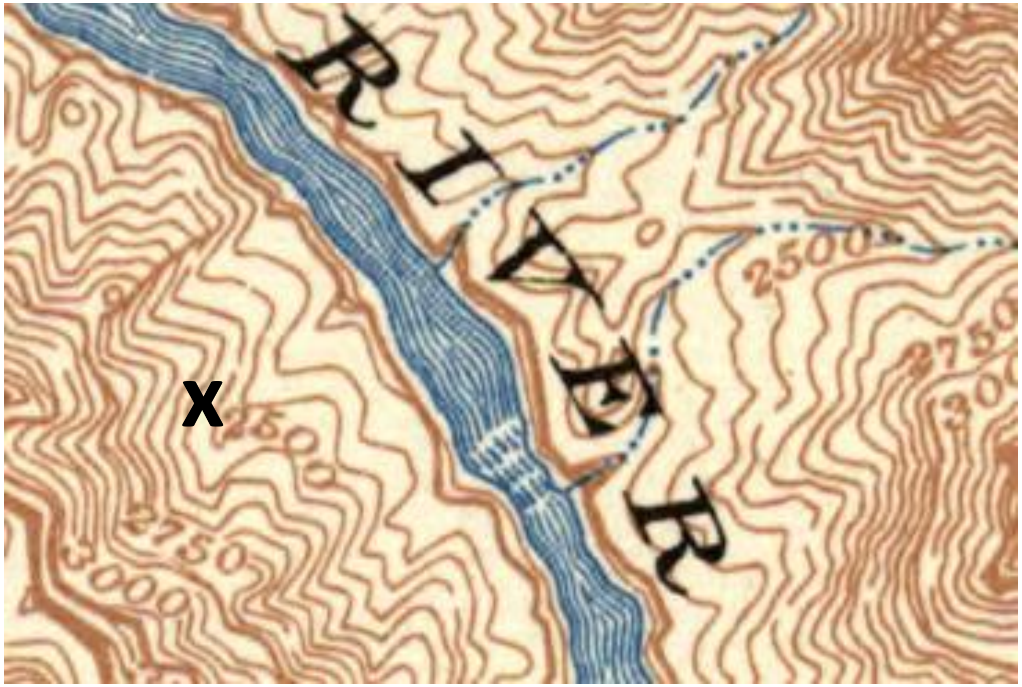
- A function may have multiple variables, e.g.,
  - $f(x, y) = x^2 y$ and $\mathrm{g}(x_1, x_2, x_3) = x_1 x_2 x_3$
- A **partial derivative** of a function of several variables is its derivative with respect to one of those variables, with the others held constant
- Usually denoted by:
  - $\frac{\partial f}{\partial x}$ or simply $\frac{df}{dx}$

Gradient

- Given a function $f(x_1, x_2, \ldots, x_n)$ with multiple variables $x_1, x_2, \ldots, x_n$
- Measure the partial derivatives for each variable
- A **gradient** is a *vector* holding all partial derivatives:
  - $\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})$
- ∇f points in the direction of greatest rate of change (or "steepest ascent")
- **Application**: *Gradient Descent Algorithm* for the training of most machine learning models.

Intuition of gradient descent

- How do I get to the bottom of this river canyon?



Look around me 360°

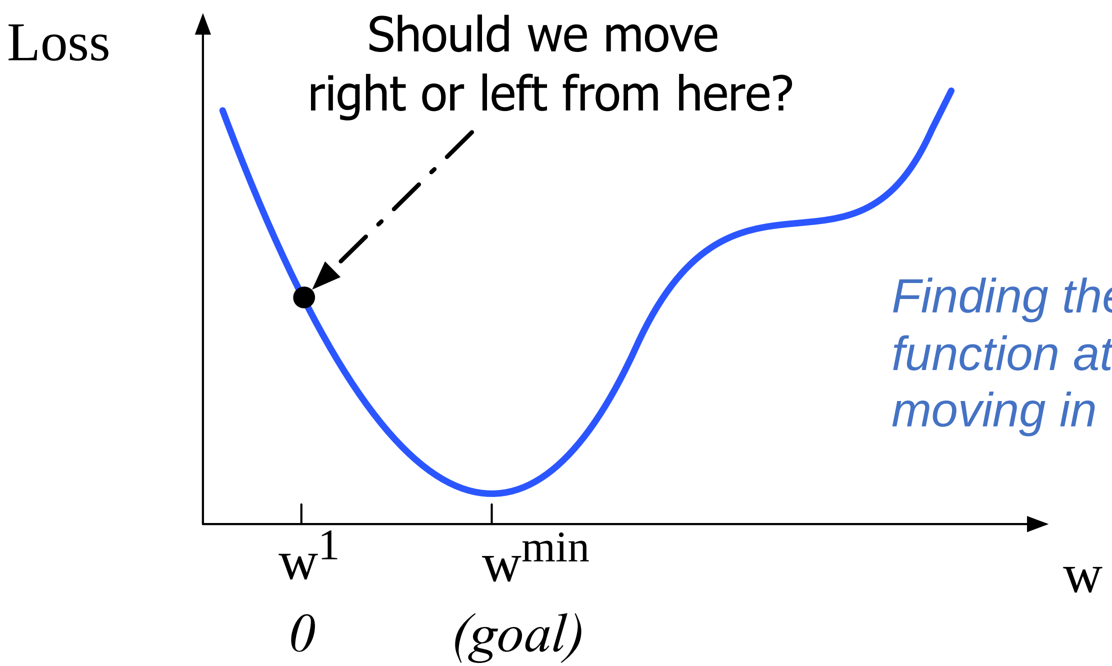
Find the direction of steepest slope down

Go that way

Let's first visualize for a single scalar w

Q: Given current w, should we make it bigger or smaller?

A: Move *w* in the reverse direction from the slope of the function

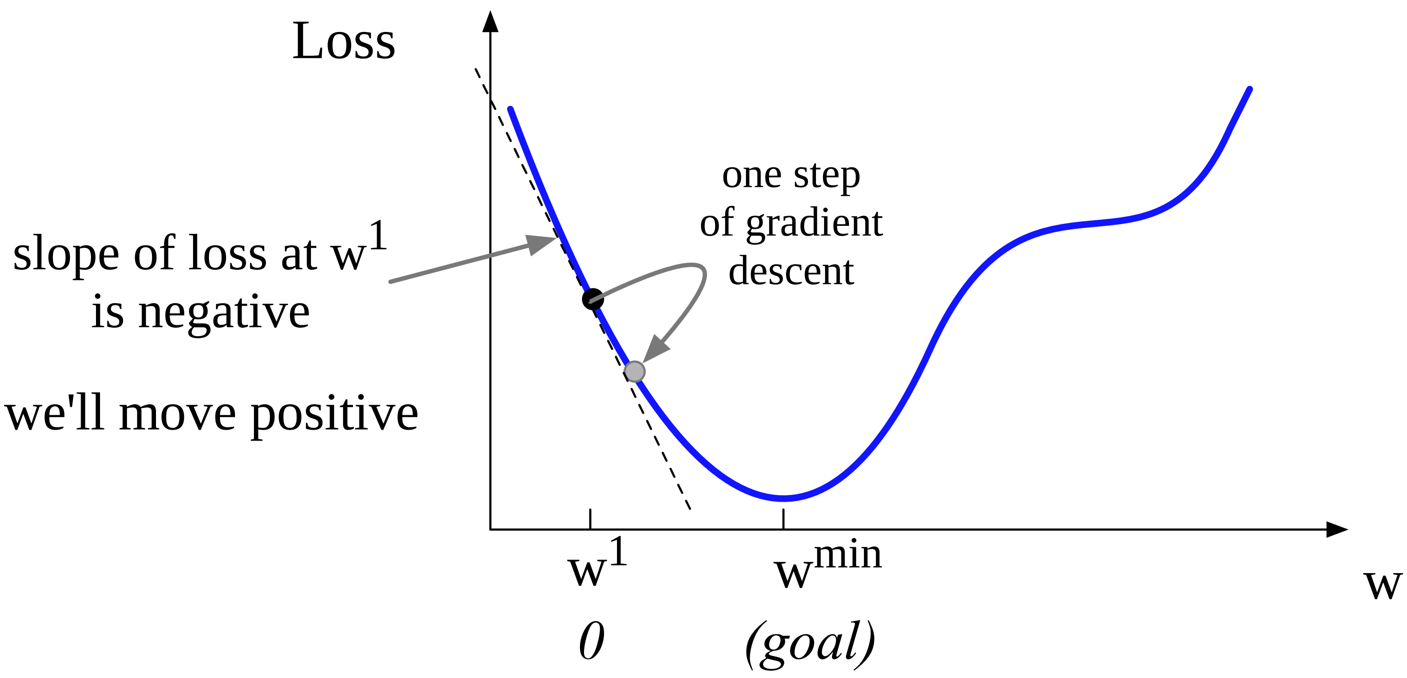


Finding the gradient of the loss function at the current point and moving in the opposite direction

Let's first visualize for a single scalar w

Q: Given current w, should we make it bigger or smaller?

A: Move *w* in the reverse direction from the slope of the function



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Generative vs. Discriminative Classifiers

Suppose we're distinguishing cat from dog images

imagenet

imagenet

Generative Classifier:

- Build a model of what's in a cat image
  - Knows about whiskers, ears, eyes
  - Assigns a *probability* to any image:
    - how cat-y is this image?

Also build a model for dog images

Now given a new image:

Run both models and see which one fits better

Discriminative Classifier

Just try to *distinguish* dogs from cats

Oh look, dogs have collars!
Let's ignore everything else

Components of Discriminative Classifier

Given input/output pairs *($x^{(i)}$, $y^{(i)}$):*

- A **feature representation** of the input. For each input observation $x^{(i)}$, a vector of features [x_1, x_2, ... , x_n]. Feature j for input $x^{(i)}$ is x_j, more completely $x_j^{(i)}$, or sometimes $f_j(x)$.
- A **classification function** that computes $\hat{y}$, the estimated class, via $p(y|x)$, like the *sigmoid* or *softmax* functions.
- An objective function for learning, like **cross-entropy loss**.
- An algorithm for optimizing the objective function: **gradient descent**.

Example of Classification Features

- For feature x_i, weight θ_i tells is how important is x_i
  - x_i ="review contains '`awesome`'": θ_i = `+10`
  - x_j ="review contains '`abysmal`'": θ_j = `-10`
  - x_k ="review contains '`mediocre`'": θ_k= `-2`

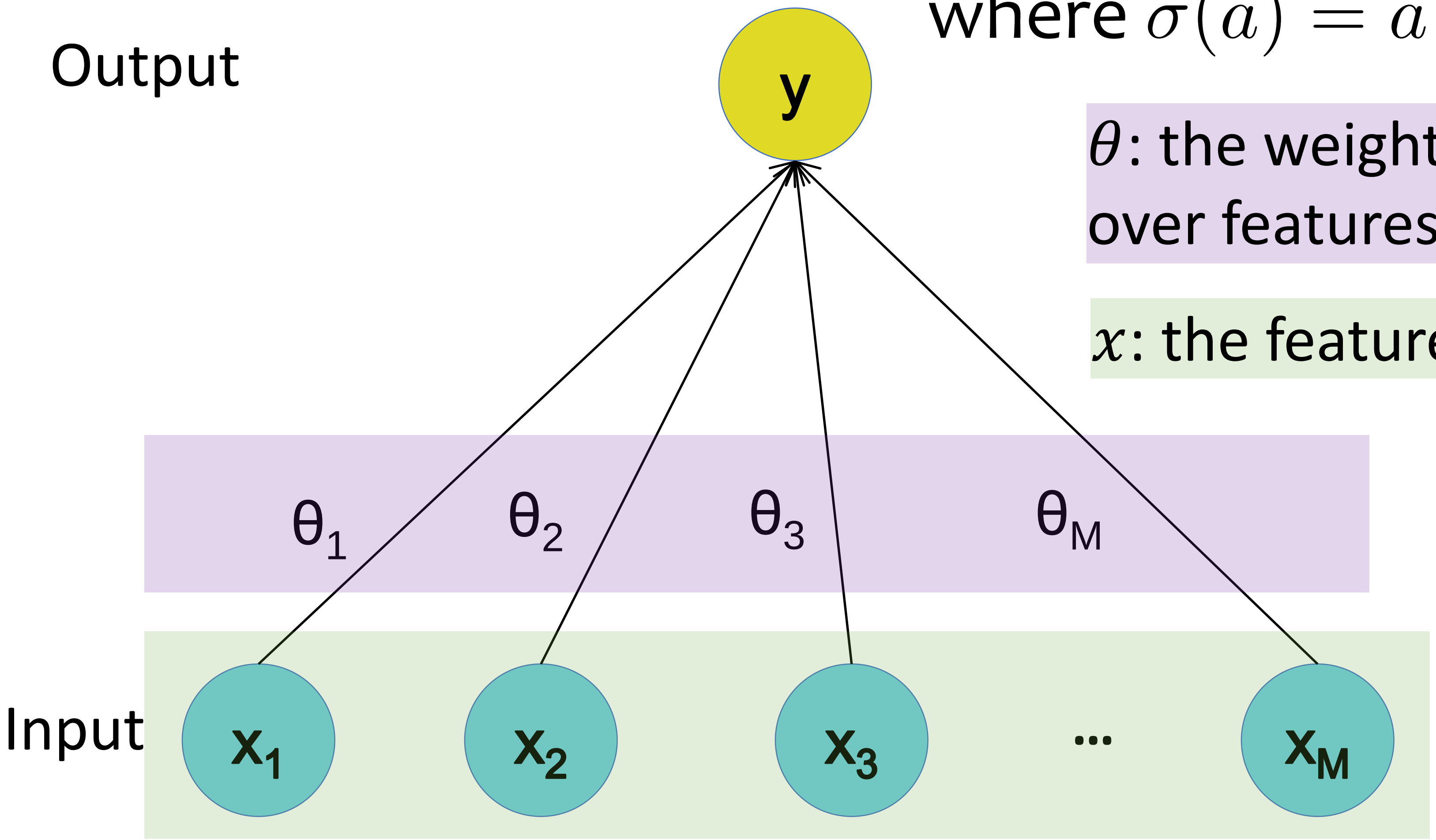
These weights are just assigned as examples. The real assignments may vary in the real training.

In the task to *predict the positive sentiment*

Linear Regression

$$y = h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \sigma(\boldsymbol{\theta}^T \boldsymbol{x})$$

where $\sigma(a) = a$



θ: the weights over features

x: the features

A probabilistic classifier?

$z = \theta^T x$ is a number, and we want to use a function of z that goes from 0 to 1

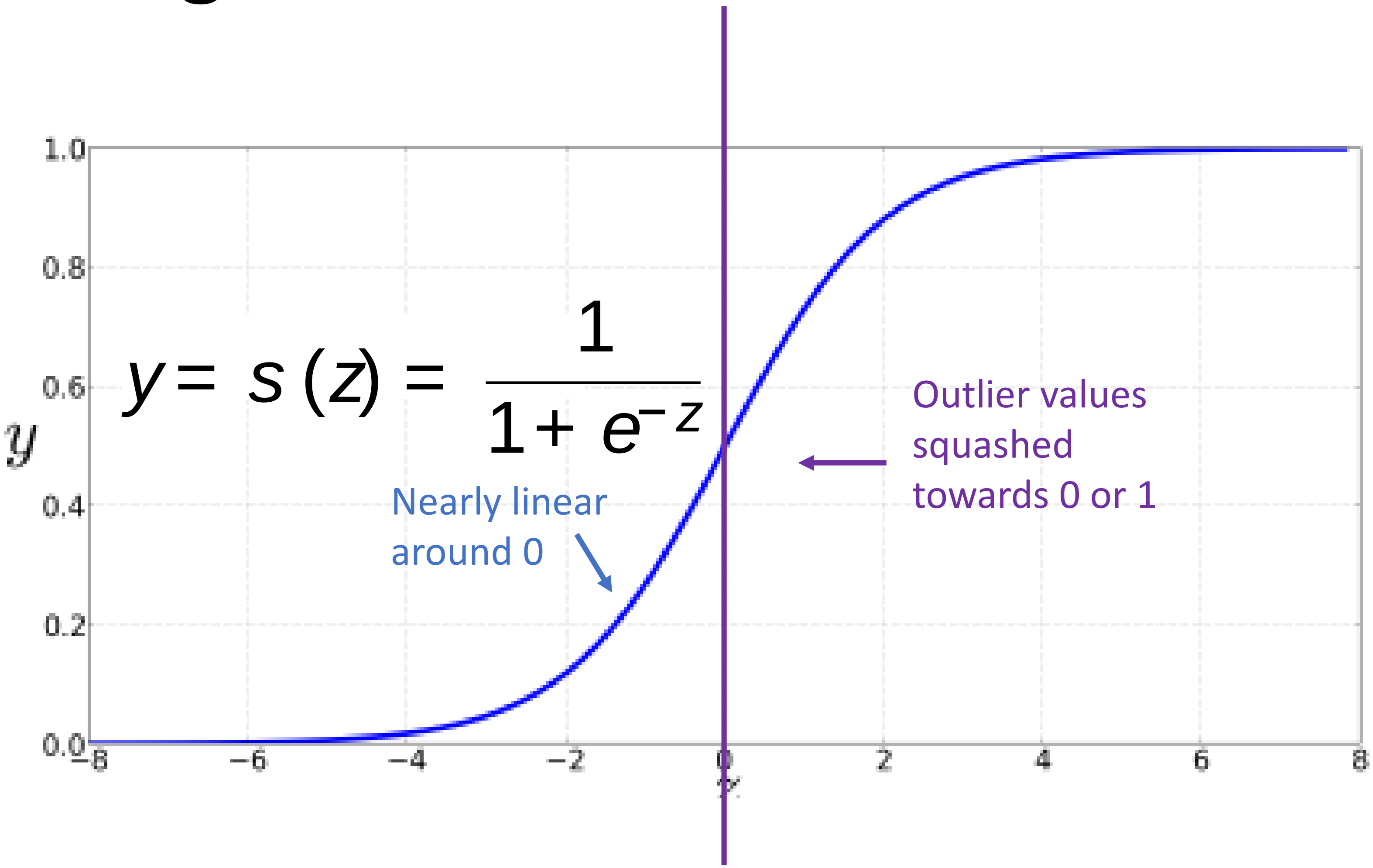
- We need to formalize "sum is high".
- We'd like a principled classifier that gives us a probability, just like Naive Bayes did
- We want a model that can tell us:

$$p(y = 1|x;\ \theta)$$

$$p(y = 0|x;\ \theta)$$

$$y = s(z) = \frac{1}{1 + e^{-z}}$$

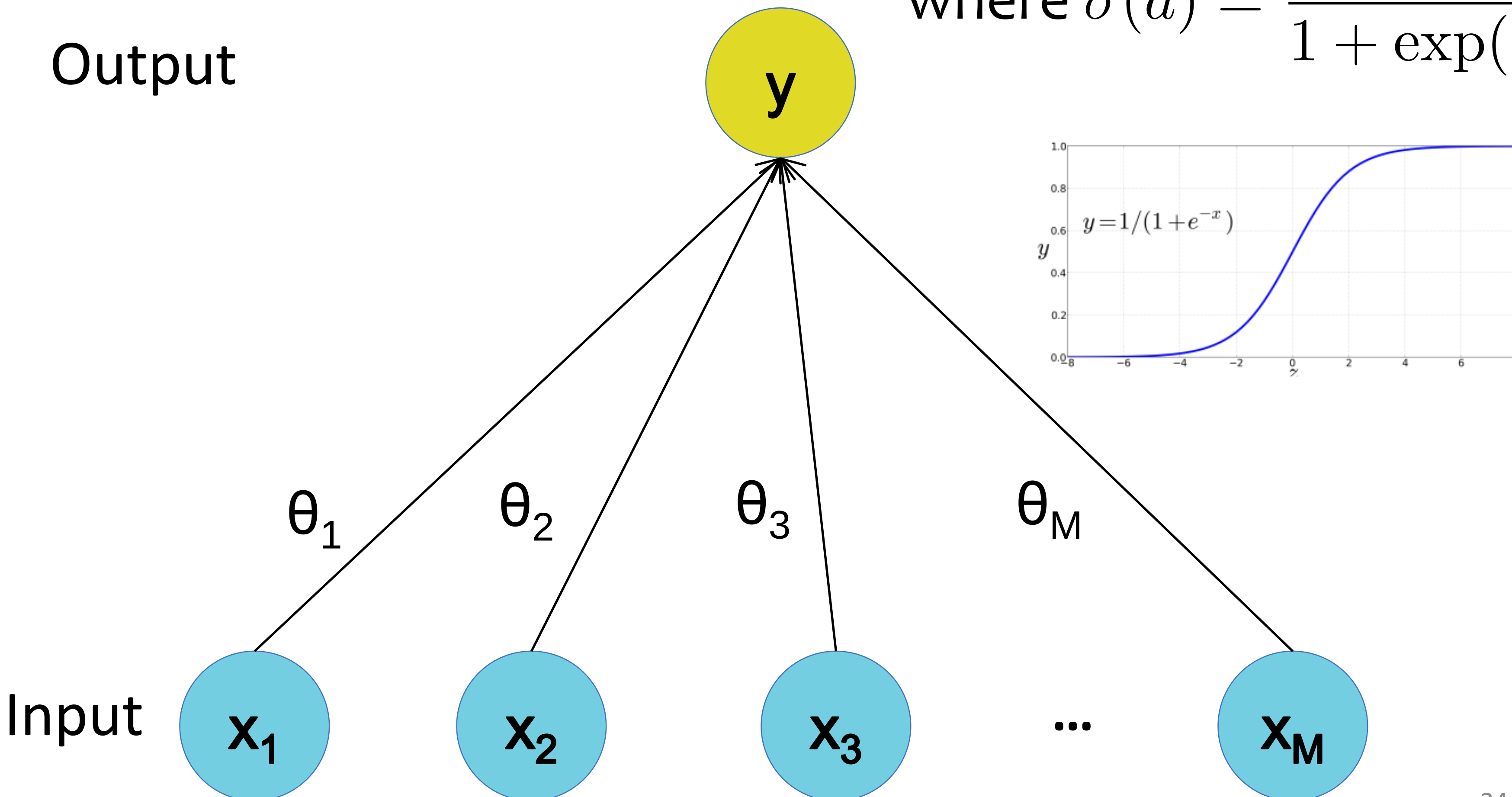
The very useful sigmoid or logistic function



Logistic Regression

$$y = h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \sigma(\boldsymbol{\theta}^T \boldsymbol{x})$$

where $\sigma(a) = \dfrac{1}{1 + \exp(-a)}$



A Recipe for Machine Learning

1. Given training data:

$$\{\boldsymbol{x}_i, \boldsymbol{y}_i\}_{i=1}^N$$

2. Choose each of these:
   - Decision function

$$\hat{\boldsymbol{y}} = f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)$$

   - Loss function

$$\ell(\hat{\boldsymbol{y}}, \boldsymbol{y}_i) \in \mathbb{R}$$

3. Define goal:

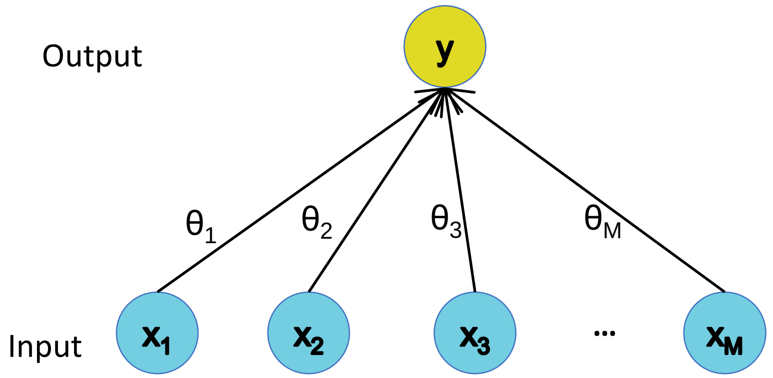
$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^N \ell(f_{\boldsymbol{\theta}}(\boldsymbol{x}_i), \boldsymbol{y}_i)$$

4. Train with SGD:

(take small steps opposite the gradient)

$$\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{(t)} - \eta_t \nabla \ell(f_{\boldsymbol{\theta}}(\boldsymbol{x}_i), \boldsymbol{y}_i)$$

Backpropagation for gradiant calculation



Forward

$$J = y^* \log y + (1 - y^*) \log(1 - y)$$

$$y = \frac{1}{1 + \exp(-a)}$$

$$a = \sum_{j=0}^{D} \theta_j x_j$$

Backward

$$\frac{dJ}{dy} = \frac{y^*}{y} + \frac{(1 - y^*)}{y - 1}$$

$$\frac{dJ}{da} = \frac{dJ}{dy}\frac{dy}{da}, \quad \frac{dy}{da} = \frac{\exp(-a)}{(\exp(-a) + 1)^2}$$

$$\frac{dJ}{d\theta_j} = \frac{dJ}{da}\frac{da}{d\theta_j}, \quad \frac{da}{d\theta_j} = x_j$$

$$\frac{dJ}{dx_j} = \frac{dJ}{da}\frac{da}{dx_j}, \quad \frac{da}{dx_j} = \theta_j$$

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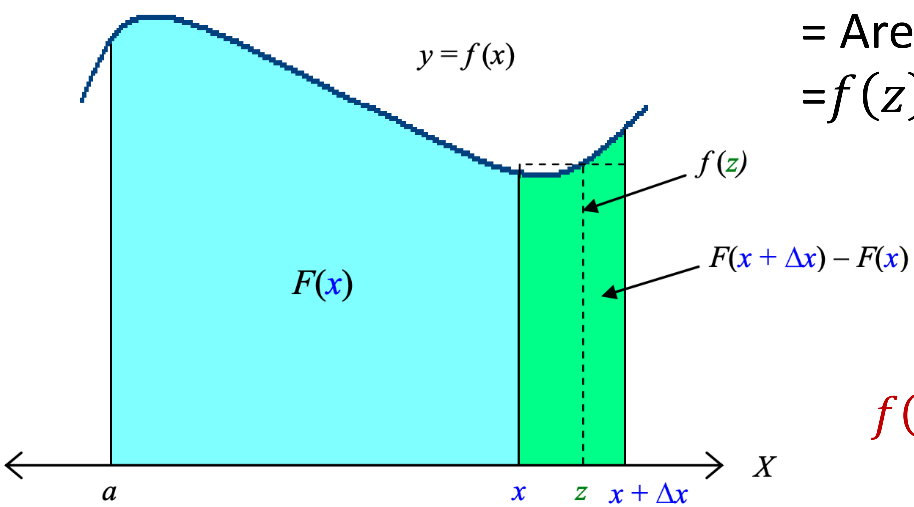
Areas Under Function Graph

- Given $y = f(x)$, *nonnegative* ($f(x) \geq 0$) and *continuous* (with no breaks or jumps).
- Function $F(x) = \int_a^x f(t)dt$ *Antiderivative*
  - The area under the graph of $f(x)$ in the interval $[a, x]$



$F(x + \Delta x) - F(x)$
= Area under the graph of $f(x)$ in $[x, x + \Delta x]$
$= f(z) \cdot \Delta x$

When $\Delta x \to 0$, $z \to x$

$$f(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = F'(x)$$

What are *Integrals*?

- *Definite Integrals*: $F(x) = \int_a^x f(t)dt$
  - Areas under $f(x)$ in the interval of $[a, x]$
  - In this context, $f(x)$ is called the *integrand*.
  - $F(x)$ is an *antiderivative* of $f(x)$.
- *Indefinite integrals*: $\int f(x)dx = F(x) + \boxed{C}$ *Arbitrary Constant*
  - **Example**. $F(x) = \frac{1}{10}x^{10} + C$ is the general antiderivative of $f(x) = x^9 = F'(x)$
  - $\int x^9 dx = \frac{1}{10}x^{10} + C$

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Properties of Integrals

- For any constant c and any integrable function $f(x)$
  - $\int [cf(x)]dx = c \int f(x)dx$
- For any integrable functions $f(x)$ and $g(x)$
  - **Sum and Difference Rules:**

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)\, dx$$

  - **Power Rule.**
  - **Exponential Rule.**
  - **Chain Rule.**

Properties of Integrals (cont.)

- **Power Rule:**
  - $\int u^p du = \begin{cases} \frac{u^{p+1}}{p+1} + C, p \neq -1 \\ \ln|u| + C, p = -1 \end{cases}$
- **Exponential Rule:**
  - $\int e^u du = e^u + C$
- **Chain Rule:**
  - **Example.** $\int \underbrace{(\mathrm{x}^5 + 2)^9}_{u^9} \underbrace{5x^4 dx}_{du} = \underbrace{\frac{(x^5+2)^{10}}{10} + C}_{\frac{u^{10}}{10} + C}$

More Examples of Chain Rules

- $\int \frac{x^2}{\sqrt{1+x^3}} dx$

$$u^{-\frac{1}{2}} \qquad du$$

$$= \frac{1}{3} \int (1+x^3)^{-\frac{1}{2}} \cdot 3x^2\, dx$$

$$= \frac{1}{3} \cdot 2(1+x^3)^{\frac{1}{2}} + C$$

$$\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

- $\int \frac{x^2}{1+x^3} dx$

$$u^{-1} \qquad du$$

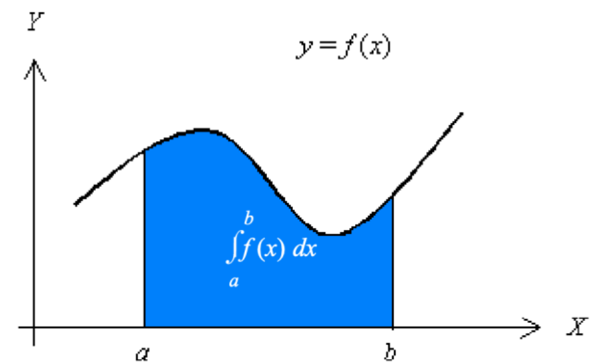
$$= \frac{1}{3} \int (1+x^3)^{-1} \cdot 3x^2\, dx$$

$$= \frac{1}{3} \cdot \ln|1+x^3| + C$$

$$\ln|u| + C$$

Fundamental Theorem of Calculus

- How to calculate *definite integrals*:
  - $\int_a^b f(x)dx = F(b) - F(a)$



- **Example.**
  - $\int_0^1 x^3(1-x^4)^2 dx$

$$= -\frac{1}{4}\int_{x=0}^{x=1} \underbrace{(1-x^4)^2}_{u^2}\underbrace{(-4)x^3\,dx}_{du}$$

$$= -\frac{1}{4}\int_{u=1}^{u=0} u^2\,du = \frac{1}{4}\int_0^1 u^2 du = \frac{1}{4}\left[\frac{u^3}{3}\right]_0^1 = \frac{1}{12}$$

A slide to take away

Chain Rule:
Super Useful!

- How to calculate derivatives?
  - **Tip1**: Remember the derivatives for the component functions, e.g., $y = x^r, y = e^x, y = lnx, y = c$, etc.
  - **Tip2**: Remember the properties for *sum*, *product*, *quotient*, and *chain rules*.
  - **Tip3**: Consider the function as the operation results of some component functions, e.g., $y = e^{-\frac{1}{2}x^2}$
- How to use a function to model a problem and how to use gradient descent to find optimal solutions.
- How to calculate integrals?
  - **Tip1**: Consider the antiderivatives first! (Remember the antiderivatives for the important components).
  - **Tip2**: Apply the integral properties for *sum*, *chain rule*, etc.