HW 10

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Problem 1

> Consider a forward SDE
>
> $$dx_t = f(x_t, t)\, dt + g(x_t, t) dW_t$$
>
> show that the corresponding probability flow ODE is written as
>
> $$dx_t = \left[f(x_t, t) - \frac{1}{2} \frac{\partial}{\partial x} g^2(x_t, t) - \frac{g^2(x_t, t)}{2} \frac{\partial}{\partial x} \log p(x_t, t) \right] dt.$$

Solution

The evolution of the probability density $p(x, t)$ for the SDE is given by the Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}[f(x, t)p] + \frac{1}{2} \frac{\partial^2}{\partial x^2}[g^2(x, t)p]$$

The probability flow ODE, $dx_t = v(x_t, t)dt$, describes a deterministic velocity field $v(x, t)$ that transports the probability density according to the continuity equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}[v(x, t)p]$$

We need to find the velocity v. Note that the right-hand sides of the two equations above are equal. So we get

$$-\frac{\partial}{\partial x}(vp) = -\frac{\partial}{\partial x}(fp) + \frac{1}{2} \frac{\partial^2}{\partial x^2}(g^2 p)$$

We can write the right side under a single derivative:

$$-\frac{\partial}{\partial x}(vp) = -\frac{\partial}{\partial x}\left[fp - \frac{1}{2} \frac{\partial}{\partial x}(g^2 p) \right]$$

Comparing the terms inside the derivatives, we get:

$$vp = fp - \frac{1}{2} \frac{\partial}{\partial x}(g^2 p)$$

Solving for v by dividing by p:

$$v = f - \frac{1}{2p} \frac{\partial}{\partial x}(g^2 p)$$

Using the product rule, $\frac{\partial}{\partial x}(g^2 p) = p \frac{\partial g^2}{\partial x} + g^2 \frac{\partial p}{\partial x}$:

$$\begin{aligned} v &= f - \frac{1}{2p} \left(p \frac{\partial g^2}{\partial x} + g^2 \frac{\partial p}{\partial x} \right) \\ &= f - \frac{1}{2} \frac{\partial g^2}{\partial x} - \frac{g^2}{2} \frac{1}{p} \frac{\partial p}{\partial x} \end{aligned}$$

Using the identity $\frac{1}{p} \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \log p$:

$$v = f - \frac{1}{2} \frac{\partial g^2}{\partial x} - \frac{g^2}{2} \frac{\partial}{\partial x} \log p$$

Therefore, the probability flow ODE is:

$$dx_t = \left[f - \frac{1}{2} \frac{\partial g^2}{\partial x} - \frac{g^2}{2} \frac{\partial}{\partial x} \log p \right] dt$$