Implementation of a linear Drucker Prager material model with linear isotropic hardening and additive visco-plasticity for thermoplastics

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Abstract

Motivated by the pressure sensitivity and visco-plastic behavior of thermoplastics, a constitutive model based on a Drucker Prager yield surface and additive visco-plasticity is formulated. The return mapping was performed by means of a backward Euler scheme. The constitutive model conforms to corresponding available models found in Abaqus.

1. Introduction

The most common material models used for engineering purposes are typically based on J_2 plasticity for metals or on a purely elastic strain energy potentials such as Neo-Hookean or Ogden for rubbery materials. Thermo plastics does typically exhibit a approximately linear elastic loading up to a yield point followed by a visco plastic yielding, often influenced triaxiality ratio. This behavior does not conform to the response of a purely J_2 based yield criterion nor a commonly used hyper-elastic potential. Motivated by this, a commonly used pressure sensitive yield surface developed by Drucker and Prager in 1952 [1] is combined with a additive visco-plastic hardening law.

2. Derivation of a viscoplastic model

The constitutive model is implemented as a VUMAT fortran subroutine, with input defined by the Abaqus Explicit solver. Abaqus explicit provides a strain increment from the co-rotated rate of deformation tensor and the time increment. The stress state is rotated by using a Green Naghdi rate formulation:

$$\sigma^{\nabla J} = \dot{\sigma} - \mathbf{\Omega} \cdot \sigma + \sigma \cdot \mathbf{\Omega} \tag{1}$$

where $\mathbf{\Omega}$ is the material spin tensor and σ is a stress tensor referring to the current orientation of the material. The material subroutine must then provide the co-rotated Cauchy stress tensor along with any relevant state variables for later iterations. For the sake of compact notation, all stresses presented later refers to the co-rotated Cauchy stress tensor. The constitutive model uses a Drucker Prager yield surface [1] , additive logarithmic visco-plasticity and linear isotropic work hardening using associated plastic flow. The total strain is decomposed as:

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p \tag{2}$$

where ϵ_{ij}^e and ϵ_{ij}^p are the elastic and plastic portion of the total strain ϵ_{ij} respectively. The components of the Cauchy stress tensor σ_{ij} is then determined from the elastic strain as:

$$\sigma_{ij} = C_{ijkl}(\epsilon_{kl} - \epsilon_{kl}^p) \tag{3}$$

A linear Hookean Isotropic material behavior is here assumed, giving the following material stiffness tensor:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{4}$$

where λ and ν are the Lam coefficients.

Whether the material is deformed in its elastic or plastic regime is determined by the Drucker Prager yield criterion defined as:

$$f = \phi - (\sigma^0 + R(P)) = \frac{\sqrt{\frac{3}{2}\tilde{\sigma}_{ij}\tilde{\sigma}_{ij}} - \alpha\sigma_{kk}}{1 + \alpha} - (\sigma^0 + R(P)) \tag{5}$$

where ϕ denotes the equivalent stress, $\tilde{\sigma}_{ij}$ is the deviatoric part of the Cauchy stress tensor, α is a material constant determining the pressure sensitivity of the material, σ_0 is the nominal yield stress and $R(P)$ is the material hardening law. The partial derivatives of the yield criterion is then:

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{1}{1 + \alpha}\left(\frac{3\tilde{\sigma}_{ij}}{2\sqrt{\tilde{\sigma}_{ij}\tilde{\sigma}_{ij}}} + \alpha\delta_{ij}\right) \tag{6}$$

$$\frac{\partial f}{\partial P} = -H_r \qquad (7)$$

Isotropic material hardening is given by :

$$R(P) = H_r P \qquad (8)$$

where H_r and P are the hardening modulus and accumulated plastic strain respectively. Additive visco-plastic behavior is modeled by introducing the following relation

$$\dot{P} = \begin{cases} 0 & \text{if } f \leq 0 \\ \dot{p_0}(e^{(\frac{f(\sigma,P)}{S}-1)}) & \text{if } f > 0 \end{cases} \qquad (9)$$

where $\dot{p_0}$ and S are positive scalar coefficients. By manipulation of equation 9 an augmented flow criteria is then defined as:

$$\hat{f}(\sigma, P) = \frac{\sqrt{\frac{3}{2}\tilde{\sigma}_{ij}\tilde{\sigma}_{ij}} - \alpha\sigma_{kk}}{1+\alpha} - (\sigma^0 + R(P)) - S\ln(1 + \frac{\dot{p}}{\dot{p_0}}) = 0 \qquad (10)$$

The partial derivatives of $\hat{f}$ with respect to its arguments are determined as:

$$\frac{\partial \hat{f}}{\partial \sigma_{ij}} = \frac{1}{1+\alpha}\left(\frac{3\tilde{\sigma}_{ij}}{2\sqrt{\tilde{\sigma}_{ij}\tilde{\sigma}_{ij}}} + \alpha\delta_{ij}\right) \qquad (11)$$

$$\frac{\partial \hat{f}}{\partial P} = -H_r - \frac{S}{1 + \frac{\dot{p}}{\dot{p_0}}}\frac{\partial(\partial P)}{\partial P \partial t} \qquad (12)$$

The evolution of plastic flow is determined by an associated flow rule:

$$\dot{\epsilon}_{ij}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} \qquad (13)$$

where λ is the plastic multiplier, being a positive scalar quantity. In order to ensure that the plastic dissipation D_p is positive for all states, it is observed that:

$$\frac{\partial f}{\partial \sigma_{ij}}\sigma_{ij} = \frac{1}{1+\alpha}\left(\frac{3\hat{\sigma}_{ij}}{2\sqrt{\hat{\sigma}_{ij}\hat{\sigma}_{ij}}} + \alpha\delta_{ij}\right)\sigma_{ij} = f \qquad (14)$$

hence the yield function f is found to a positive homogeneous function of order one and Eulers's theorem is therefore applicable. The plastic dissipation is found as:

$$D_p = \sigma_{ij}\dot{\epsilon}_{ij}^P = \sigma_{ij}\dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}f \geq 0 \qquad (15)$$

This inequality holds as the plastic multiplier and the augmented yield function are positive or zero for all

states by definition. The plastic dissipation here includes the visco-plastic dissipation, being defined to have the same sign as the time rate of change of the accumulated plastic strain. Hence, positive plastic dissipation is ensured. If now an equivalent plastic strain P is defined as being power conjugate to the equivalent stress ϕ as:

$$D_p = \sigma_{ij}\dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \sigma_{ij}\dot{\lambda}\frac{\partial \phi}{\partial \sigma_{ij}} = \phi\dot{\lambda} = \sigma_{eq}\dot{P} \qquad (16)$$

giving $\dot{P} = \dot{\lambda}$ and therefore

$$P = \int_0^t \dot{P}dt = \int_0^t \dot{\lambda}dt \qquad (17)$$

3. Numerical scheme

For the sake of generality, a full backward Euler scheme is used, which enforces the augmented flow criterion at the end of the step and calculates all updated variables at the end of the step as:

$$\hat{f}(\sigma_{n+1}, P_{n+1}) = 0 \qquad (18)$$

$$\epsilon_{n+1} = \epsilon_n + \Delta\epsilon \qquad (19)$$

$$\epsilon_{n+1}^P = \epsilon_n^P + \Delta\epsilon^P \qquad (20)$$

$$\sigma_{n+1} = C : (\epsilon_{n+1} - \epsilon_{n+1}^p) \qquad (21)$$

$$P_{n+1} = P_n + \Delta\lambda_{n+1} \qquad (22)$$

where the subscript n denotes increment counter.

The material state is determined for a given strain increment $\Delta\epsilon$ by means of an elastic prediction followed by a plastic corrector scheme. The full derivation of this scheme is found in [2] and only the most relevant details here included here adopting notation from the authors. The general form of the return mapping scheme is first presented and then adapted to the constitutive model.

By assuming that the strain increment is elastic, a trial stress σ^{trial} is determined

$$\sigma^{trial} = C : \Delta\epsilon \qquad (23)$$

if now the stress in step $n + 1$ is written as:

$$\sigma_{n+1} = C : (\epsilon_{n+1} - \epsilon_{n+1}^p) = \sigma_{n+1}^{trial} - \Delta\lambda_{n+1}C : r_{n+1} \qquad (24)$$

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where r_{n+1} is the gradient of the plastic potential, and the plastic strain tensor ϵ^p and the state variables q at time $n + 1$ is written on residual form:

$$a = -e_{n+1}^p + e_n^p - \Delta\lambda r = 0 \qquad (25)$$

$$b = -q_{n+1} + q_n - \Delta\lambda h = 0 \qquad (26)$$

$$f = f(\sigma_{n+1}, q_{n+1}) = 0 \qquad (27)$$

and linearized with respect to σ and q, insertion of (25) and (26) into (27) results in the following expression:

$$\delta\lambda^k = \frac{f^k - \delta f^k A^k \tilde{a}^k}{\delta f^k A^k \tilde{r}^k} \qquad (28)$$

where k denotes the iteration counter and:

$$\delta f^k = [f_\sigma^k, f_q^k] \qquad (29)$$

$$\tilde{r}^k = \begin{bmatrix} f_\sigma^k \\ h^k \end{bmatrix}$$

$$\tilde{a}^k = \begin{bmatrix} a^k \\ b^k \end{bmatrix}$$

$$[A^k]^{-1} = \begin{bmatrix} (C^{-1} + \Delta\lambda^k r_\sigma) & \Delta\lambda^k r_q \\ \Delta\lambda^k h_\sigma & -I + \Delta\lambda^k h_q \end{bmatrix}$$

By insertion of the presented constitutive model into (28 the following expressions are found:

$$\delta f^k = [\frac{\partial \hat{f}^k}{\partial \sigma_{ij}}, \frac{\partial \hat{f}^k}{\partial P}] \qquad (30)$$

$$\tilde{r}^k = \begin{bmatrix} \frac{\partial \hat{f}^k}{\partial \sigma_{ij}} \\ 1 \end{bmatrix}$$

As the evolution equation of the plastic strain and the accumulated plastic strain are linear functions, their residual at iteration k is zero:

$$\tilde{a}^k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrix $[A^k]^{-1}$ is then reduced as:

$$[A^k]^{-1} = \begin{bmatrix} (C^{-1} + \Delta\lambda^k \frac{\partial^2 f}{\partial\sigma\partial\sigma}) & 0 \\ 0 & -I \end{bmatrix}$$

where:

$$\frac{\partial^2 f}{\partial\sigma\partial\sigma} = \frac{3\Delta\lambda}{2\sqrt{\frac{3}{2}\sigma : \sigma}}\hat{I} = a\hat{I} \qquad (31)$$

and $\hat{I}$ is a fourth order deviatoric symmetric tensor further described in [2]. The matrix A^k can now be determined by inversion as:

$$[A^k] = \begin{bmatrix} C + 2\mu b\hat{I} & 0 \\ 0 & -I \end{bmatrix}$$

where:

$$b = \frac{2\mu a}{1 + 2\mu a} \qquad (32)$$

by approximating $\dot{p}$ as $\frac{\Delta p}{\Delta t}$

$$f_q = \frac{\partial \hat{f}}{\partial P} = -H_r - \frac{S}{\dot{p}_0 \Delta t + \Delta p} \qquad (33)$$

by insertion, and recognizing that $\tilde{a} = 0$ the following expression is found for $\delta\lambda$:

$$\delta\lambda^k = \frac{\hat{f}^k}{\delta f^k A^k \tilde{r}^k} \qquad (34)$$

expanding all terms gives:

$$f_\sigma : (C + 2vb\hat{I}) : f_\sigma = \frac{\partial \hat{f}}{\sigma_{ij}}(C_{ijkl} - 2vb\hat{I}_{ijkl})\frac{\partial \hat{f}}{\partial \sigma_{kl}} \qquad (35)$$

$$= \frac{\partial \hat{f}}{\sigma_{ij}}(2v\frac{\partial \hat{f}}{\partial \sigma_{ij}} + \alpha C_{ijkl})\frac{1}{1 + \alpha} \qquad (36)$$

$$= \frac{3v + \alpha^2 C_{iikk}}{(1 + \alpha)^2} \qquad (37)$$

$$\qquad (38)$$

where:

$$C_{iikk} = \frac{3E}{1 - 2v} \qquad (39)$$

and

$$f_q : (-I) : h = -H_r - \frac{S}{\dot{p}_0 \Delta t + \Delta p} \qquad (40)$$

and

$$\hat{f} = \frac{\sqrt{\frac{3}{2}\tilde{\sigma}_{n+1}\tilde{\sigma}_{n+1}} - \alpha\bar{\sigma}_{n+1}}{1 + \alpha} - (\sigma^0 + R(P)) - S\,ln(1 + \frac{\Delta p}{\dot{p}_0 \Delta t})) = 0 \qquad (41)$$

where $\bar{\sigma}$ is equal σ_{kk}.

The complete numerical scheme is then summarized as:

1. Initialize variables:

$$k = 0 \qquad \epsilon_{n+1}^{p(0)} = \epsilon_n^p \qquad P_{n+1}^{(0)} = P_n \qquad (42)$$

$$\Delta\lambda^{(0)} = 0 \qquad \sigma^{(0)} = \sigma^{trial} = C : (\epsilon_{n+1} - \epsilon^{p(0)}) \qquad (43)$$

2. Evaluate the flow criterion:

$$\hat{f}^k = \hat{f}(\sigma^{(k)}, P^{(k)}) \qquad (44)$$

3. if $\hat{f}^k$ is within a prescribed tolerance, the scheme has converged, else:

$$\delta\lambda = \frac{\hat{f}^k}{-H_r - \frac{S}{\dot{p}_0\Delta t + \Delta\lambda^{(k)}}} \qquad (45)$$

$$\sigma^{(k+1)} = \sigma^0 - C\epsilon^{p(k+1)} \qquad (46)$$

$$\Delta\lambda^{(n+1)} = \Delta\lambda^{(n)} + \delta\lambda \qquad (47)$$

$$\epsilon^{p(k+1)} = \epsilon^{p(k)} + \delta\lambda\frac{\partial f^{(k)}}{\partial\sigma} \qquad (48)$$

$$P^{(k+1)} = P^{(k)} + \delta\lambda \qquad (49)$$

go to (2.)

It should here be noted that the viscous back stress can become undefined in an iteration where $\frac{\Delta\lambda}{\dot{p}_0\Delta t} \leq -1$. This case is unfeasible as $\Delta\lambda$ is a positive scalar, but can occur during iterations due to the non-linear nature of the function describing the viscous back stress. This is handled explicitly in the numerical implementation found in the appendix.

4. Model verification

4.1. Method

During the model development, the model was incrementally developed as follows:

1. J2-plasticity with linear isotropic hardening using a radial return alghorithm
2. J2-plasticity with linear isotropic hardening using a backward Euler scheme
3. J2-plasticity with linear isotropic hardening and additive visco plasticity using a backward Euler scheme

4. Drucker Prager yield surface with linear isotropic hardening and additive visco plasticity using a backward Euler scheme

During each increment, the model was compared to a corresponding Abaqus implementation for the following cases:

1. Uniaxial tension
2. Uniaxial compression
3. Simple shear
4. Biaxial tension
5. Biaxial compression
6. Rigid rotation

Here, only the results obtained for the final Drucker Prager model is discussed for two cases being:

1. linear hardening
2. linear hardening and additive visco-plasticity

The final implementation of the material subroutine is found in the appendix.

4.2. Results and discussion

The linear Drucker-Prager model with isotropic hardening implemented in Abaqus is defined as:

$$F = t - p\tan(\beta) - d = 0 \qquad (50)$$

where d is the cohesion of the material and

$$p = -\frac{1}{3}\sigma_{kk} \qquad (51)$$

$$q = \sqrt{\frac{3}{2}(S_{ij}S_{ij})} \qquad (52)$$

$$t = \frac{q}{2}[1 + \frac{1}{K} - (1 - \frac{1}{K})(\frac{r}{q})^3] \qquad (53)$$

$$r = \sqrt[3]{\frac{9}{2}S_{ij}S_{jk}S_{ik}} \qquad (54)$$

here K is the ratio between the yield stress in tri-axial tension and yield stress in tri-axial compression. β is defined as the slope of the yield surface in the p-t plane corresponding to the $(-\frac{I_1}{3}, \sqrt{J_2})$ plane in the case of K=1. For comparison of the models, the relation between β and α is:

$$\beta = \tan^{-1}(3\alpha) \qquad (55)$$

The Abaqus implementation has the possibility to use a non-associated flow rule where a parameter ψ defines the plastic dilation of the material. In order to directly
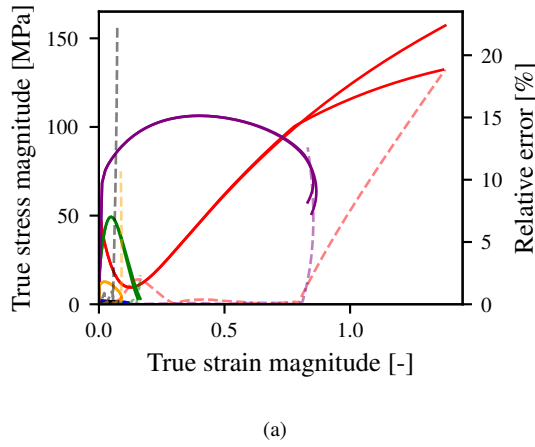
(a)

Figure 1: Simple shear to large strains. Relative error denoted by dotted lines and the stress components are colored as: σ_{xx} is red, σ_{yy} is blue, σ_{zz} is green, σ_{xy} is purple, σ_{xz} is black and σ_{yz} is orange

compare the models $\psi = \beta$ is used, resulting in a associated plastic flow rule. When comparing the Abaqus implementation and the VUMAT subroutine for linear isotropic hardening with a hardening modulus of 50 Mpa and no visco-plasticity, close matching between the obtained stress and strain values are found for all cases. The stress values for a given strain value deviated with less than one percent for all loading cases being the larges for simple shear where a peak relative error of approx. 0.5% is observed, see the appendix.

No direct comparison to the visco-plastic was possible as only the possibility to tabulate hardening as function of strain rate was possible in Abaqus, and was considered out of scope for this study. However, as the stress state at an strain increment is determined by an elastic prediction followed by a plastic corrector step using the gradient of the yield surface and the plastic multiplier $\Delta\lambda$, only the stress magnitude should be influenced by the visco-plastic back stress corresponding to the findings shown in the appendix.

Rigid rotation was imposed on a unstrained element using a second order accurate formulation, showing no self straining due to rotation within two revolutions.

The choice of stress rate formulation has been observed to influence the material response at large shear strains [3], here investigated by a simple shear test.

It is here observed that the rate formulations conform well up to longitudinal strain ϵ_{xx} of approx 0.7 after which the Jaumann formulation shows a decrease in σ_{xx} component of the stress tensor. It is interesting to see that the deviation is observed in σ_{xx} and not τ_{xy}, moti-

vating further investigation.

All test cases without visco-plasticity did converge during one iteration, as could be expected due to the closed form of the expression for the increment in the plastic multiplier.
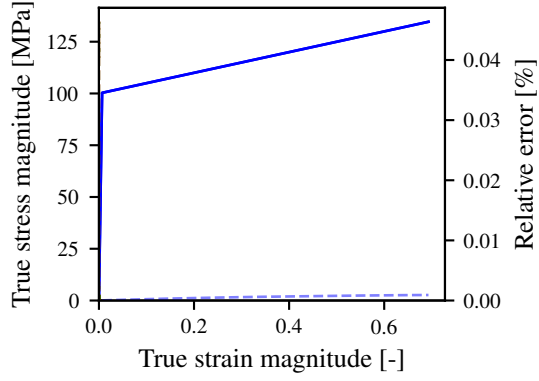
5. Conclusion

The derivation of a constitutive model based on a Drucker-Prager yield surface, linear isotropic hardening and additive visco-plastisity has been presented and verified against an available commercial implementation. The match between the VUMAT routine and the commercial implementation was considered close for all loading cases.

5.1. References

[1] D. Drucker, W. Prager, Soil Mechanics and Plastic Analysis or Limit Design, Quarterly of Applied Mathematics 10 (2) (1952) 157–165. doi:10.1090/qam/48291.

[2] T. Belytschko, W. K. W. K. Liu, B. B. Moran, K. I. Elkhodary, Nonlinear finite elements for continua and structures.

[3] E. d. Souza Neto, D. Peric, D. R. J. Owen, Computational Methods for Plasticity, Computational methods for plasticity-theory and applications 55 (2008) 816. doi:10.1002/9780470694626.
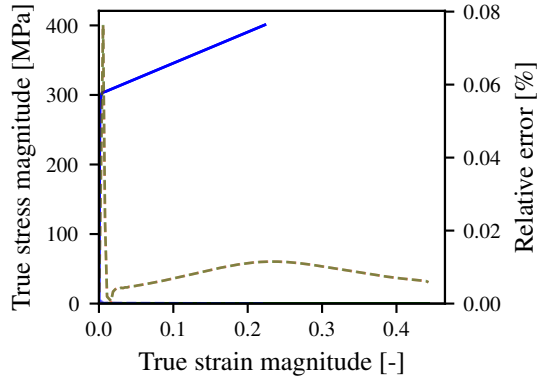URL http://www.sciencedirect.com/science/article/pii/S0266352X13001122

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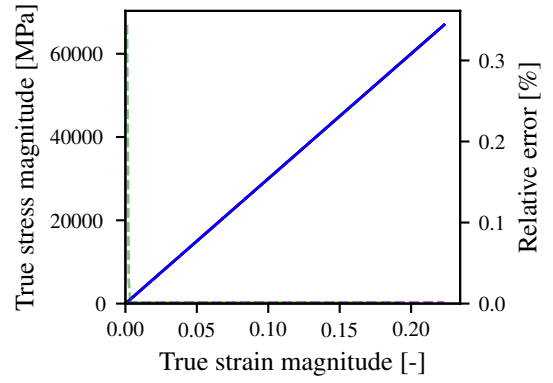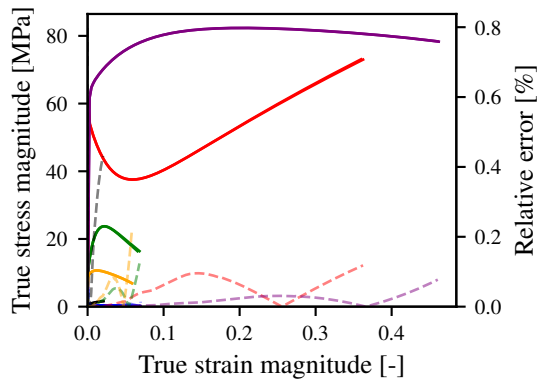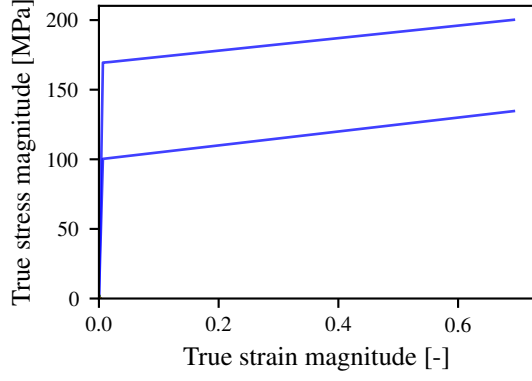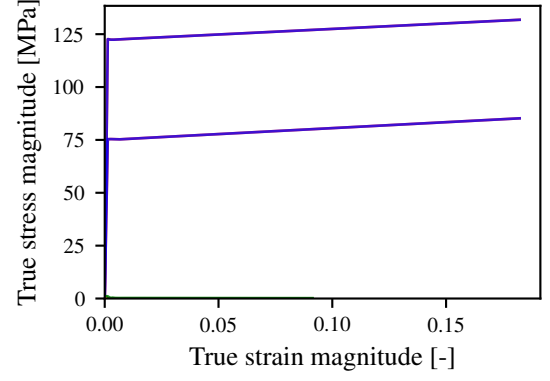# Linear isotropic hardening

(a)

(b)

(c)

(d)

(e)
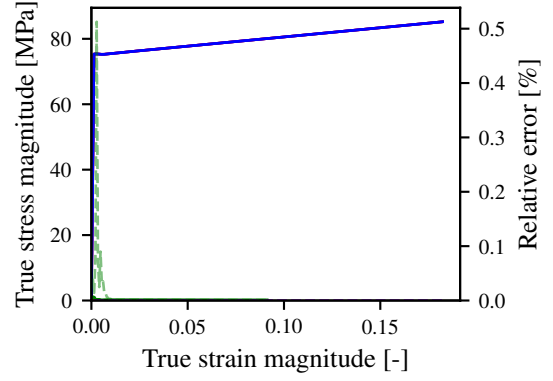
Figure .2: Stress components with corresponding strain components shown for uniaxial tension (a), uni-axial compression (b), simple shear(c), biaxial tension (d) and biaxial compression (e). The stress components are colored as: σ_{xx} is red, σ_{yy} is blue, σ_{zz} is green, σ_{xy} is purple, σ_{xz} is black and σ_{yz} is orange
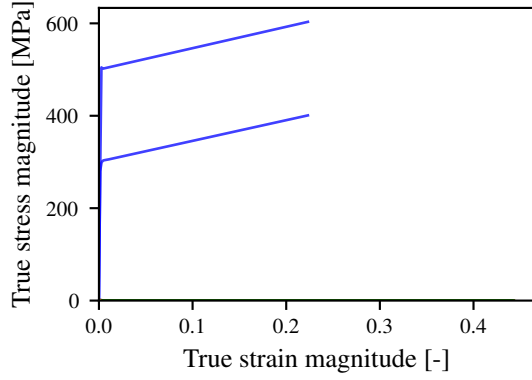
Linear isotropic hardening and additive visco-plasticity
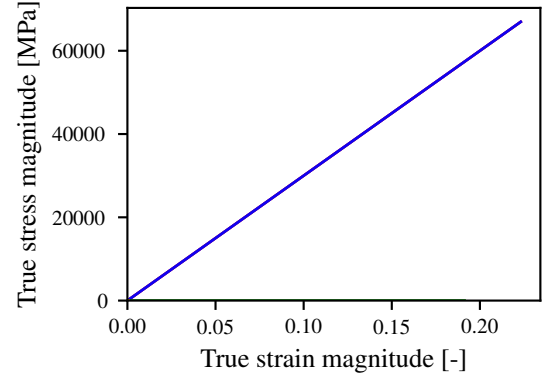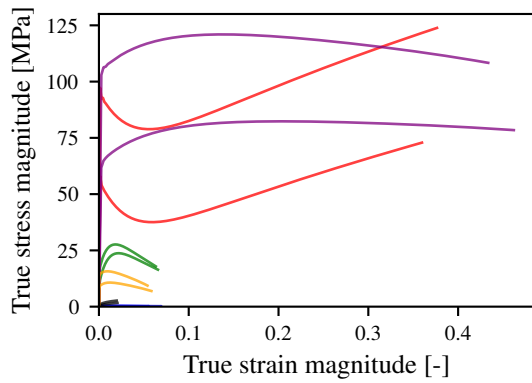

(a)

(d)

(b)

(e)

(c)

Figure .3: Stress components with corresponding strain components shown for uniaxial tension (a), uni-axial compression (b), simple shear(c), biaxial tension (d) and biaxial compression (e). The stress components are colored as: σ_{xx} is red, σ_{yy} is blue, σ_{zz} is green, σ_{xy} is purple, σ_{xz} is black and σ_{yz} is orange