Strain calulations in μDIC

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TL;DR
In μDIC, all strains are determined from the right stretch tensor $\mathbf{U}$, obtained from the deformation gradient $\mathbf{F} = \mathbf{RU}$ using spectral decomposition.
Operations such as determining the Hencky strain tensor is calculated principal values of $\mathbf{U}$ and the tensor components in image coordinates are obtained by rotating the strain tensor into the image coordinate system.

1 The deformation gradient

When a solid is deformed, a material point P which resides at $\mathbf{X}$ in the undeformed state, resides at $\mathbf{x}$ after deformation. The displacement between the point P before and after deformation is denoted $u(\mathbf{X})$ as:

$$\mathbf{x} = \mathbf{u}(\mathbf{X}) + \mathbf{X} \tag{1}$$

where $\mathbf{x}$ and $\mathbf{X}$ are the position vectors in the current and initial configurations, and the displacement vector $u(\mathbf{X})$ is a function of $\mathbf{X}$.

We now define the deformation gradient $\mathbf{F}$ as:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \tag{2}$$

The deformation gradient $\mathbf{F}$ is a non-symmetric second-order tensor.

Note that:

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} d\mathbf{X}$$

which means that the deformation gradient $\mathbf{F}$ maps an infinitesimal line segment in its undeformed configuration $d\mathbf{X}$ into the deformed configuration $d\mathbf{x}$.

2 Polar decomposition

The deformation gradient $\mathbf{F}$ can be multiplicatively decomposed into two parts:

$$\mathbf{F} = \mathbf{R}\mathbf{U} \tag{3}$$

where $\mathbf{R}$ is the orthogonal rotation tensor and $\mathbf{U}$ is the symmetric right stretch tensor. The operation of splitting $\mathbf{F}$ into $\mathbf{R}$ and $\mathbf{U}$ is known as polar decomposition and is illustrated in Figure 1.

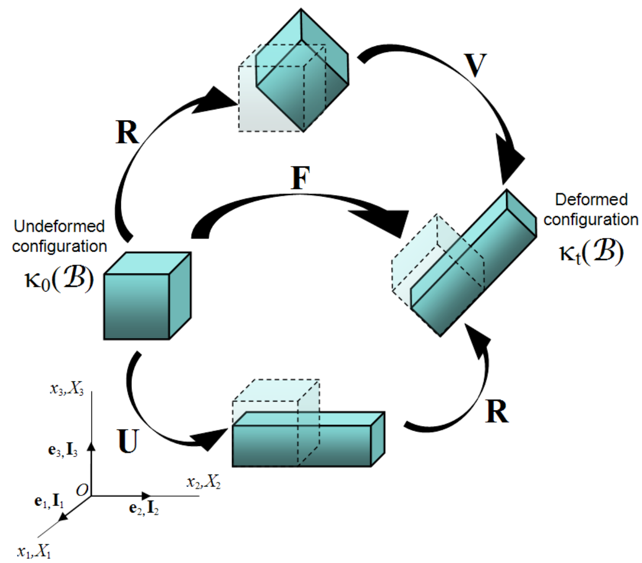

Figure 1: Illustration of the deformation of a solid by the deformation gradient. Taken from https://commons.wikimedia.org/wiki/File:Polar$_{d}ecomposition_{o}f_{F}.png$

Equation 3 implies that the deformation of a material fiber can be split into two sequential operations being first a deformation by $\mathbf{U}$ and a subsequent rotation by $\mathbf{R}$. As only $\mathbf{U}$ deforms the material and refers to the materials un-rotated state, the measures of strain used in μDIC are based on $\mathbf{U}$

Determining $\mathbf{U}$ from $\mathbf{F}$ is done by exploiting the symmetry of the right Cauchy-Green deformation tensor $\mathbf{C}$, namely:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{R}^T\mathbf{R}\mathbf{U} = \mathbf{U}^T\mathbf{U} \tag{4}$$

which allows $\mathbf{U}$ to be determined by calculating the square root of the eigenvalues of $\mathbf{U}^T\mathbf{U}$.

Since $\mathbf{C}$ is a symmetric second-order tensor, we can determine the orientation θ_p of the largest eigenvector directly as:

$$\tan(2\theta_p) = \frac{2C_{12}}{C_{22} - C_{11}} \tag{5}$$

where C_{ij} are the tensor components in the image coordinate system. We can now rotate $\mathbf{C}$ into the coordinate system defined by its principal directions. We do this by introducing a rotation matrix $[\mathbf{Q}]$ as:

$$[\mathbf{Q}] = \begin{bmatrix} \cos(\theta_p) & \sin(\theta_p) \\ -\sin(\theta_p) & \cos(\theta_p) \end{bmatrix} \tag{6}$$

The tensor $\mathbf{C}$ can be rotated into its principal orientation by:

$$[\bar{\mathbf{C}}] = [\mathbf{Q}][\mathbf{C}][\mathbf{Q}]^T \tag{7}$$

where $[\mathbf{C}]$ and $[\bar{\mathbf{C}}]$ represent the component matrices of the tensor $\mathbf{C}$ in the two coordinate systems. Since the matrix $[\bar{\mathbf{C}}]$ is diagonal, $\mathbf{U}$ can be calculated directly in the principal coordinate system as:

$$[\bar{\mathbf{U}}] = \begin{bmatrix} \sqrt{C_1} & 0 \\ 0 & \sqrt{C_2} \end{bmatrix} \tag{8}$$

where C_1 and C_2 are the eigenvalues of $\mathbf{C}$.

We can now rotate $[\bar{\mathbf{U}}]$ back into the image coordinate system by:

$$[\mathbf{U}] = [\mathbf{Q}]^T[\bar{\mathbf{U}}][\mathbf{Q}] \tag{9}$$

Note that $\mathbf{C}$ and $\mathbf{U}$ have the same principal directions (or eigenvectors).

The eigenvalues of $\mathbf{U}$ are often referred to as principal stretches.

3 Strain tensors

The Hencky strain tensor $\mathbf{E}_l$ (often referred to as the logarithmic strain tensor) is determined as:

$$[\bar{\mathbf{E}}_l] = \begin{bmatrix} \ln(\sqrt{C_1}) & 0 \\ 0 & \ln(\sqrt{C_2}) \end{bmatrix} \tag{10}$$

Note that the eigenvectors of $\boldsymbol{\varepsilon}_t$ and $\mathbf{U}$ are the same. The components of the Hencky strain tensor $\boldsymbol{\varepsilon}_t$ can be determined in image coordinate system as:

$$[\mathbf{E}_l] = [\mathbf{Q}]^T[\bar{\mathbf{E}}_l][\mathbf{Q}] \tag{11}$$

The principal values of the Hencky strain tensor are often referred to as the logarithmic (or true) strains.

Two other strain tensors that can be calculated from the right stretch tensor are the Biot strain tensor

$$\mathbf{E}_B = \mathbf{U} - \mathbf{I} \tag{12}$$

and the Green-Lagrange strain tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) \tag{13}$$

where $\mathbf{I}$ is the identity matrix.