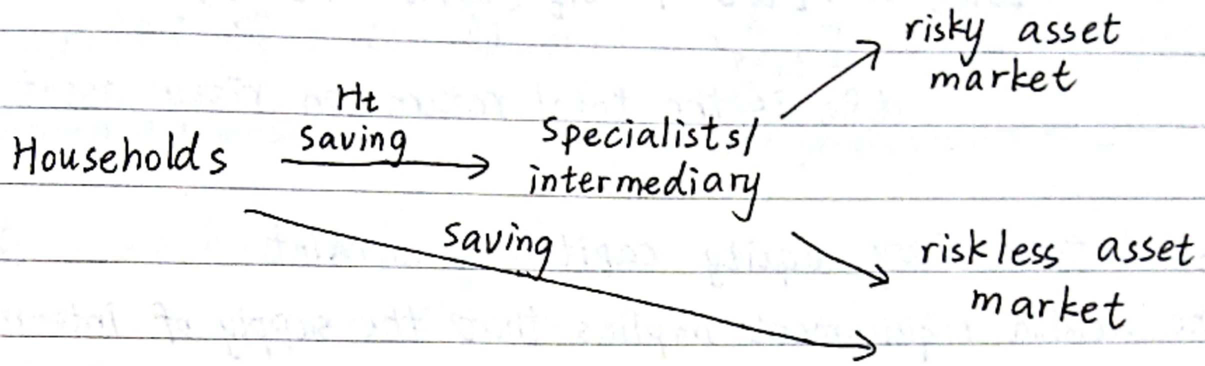
Some frontier theories

Intermediary Asset Pricing

2. The Model : Intermediation and Asset Prices



2.1 Asset Price

The risky asset pays a dividend of D_t per unit time, where $\{D_t\}$ follows a geometrics Brownian motion.

$$\frac{dD_t}{D_t} = g\,dt + \sigma\, dZ_t \qquad \text{given } D_0$$

$\{Z_t\}$ is a standard Brownian motion

We also define the total return on the risky asset as

$$dR_t = \frac{D_t dt + dP_t}{P_t}$$

2.2 Specialists and intermediation

The specialists are infinitely-lived and maximize an objective function

$$E\left[\int_0^\infty e^{-\rho t} u(C_t)\,dt\right], \quad \rho > 0$$

C_t is the date t consumption rate of specialist

CRRA instantaneous utility function $u(C_t) = \frac{1}{1-\gamma} C_t^{1-\gamma}$

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Suppose that the specialist chooses to invest a fraction α_t^I of the portfolio in risky asset and $1-\alpha_t^I$ in the riskless asset.

Then, the return delivered by the intermediary is

$$\widetilde{dR_t} = r_t\,dt + \alpha_t^I (dR_t - r_t\,dt)$$

dR_t is the total return on risky asset

2.3 Intermediary equity capital constraint

The wealth requirement implies that the supply of intermediation facing a household is at most,

$$H_t \le m W_t \qquad (5)$$

Debt and leverage constraint:

$$B_t \le m^b W_t$$

2.4 Specialist / intermediary decision

choose $\begin{cases} \text{his consumption rate} \\ \text{the portfolio decision} \end{cases}$

this term can be offset

$$\max_{\{C_t,\, \alpha_t^I\}} E\left[\int_0^\infty e^{-\rho t} u(C_t)\,dt\right] \qquad (6)$$

$$\text{s.t. } dW_t = -C_t\,dt + W_t r_t\,dt + W_t\left[\widetilde{dR_t}(\alpha_t^I) - r_t\,dt\right]$$

⇓ rewrite the budget constraint

$$dW_t = -C_t\,dt + W_t r_t\,dt + \alpha_t^I W_t (dR_t - r_t\,dt)$$

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2.5 Household : The demand for intermediation

A unit mass of generation t agents are born with wealth W_t^h and live in periods t and $t+\delta t$

They maximize utility :

$$\rho \cdot \delta t \ln C_t^h + (1-\rho\delta t) E_t[\ln W_{t+\delta t}^h]$$

the household's consumption — a bequest for generation $t+\delta t$

$\delta t \to dt$, we have $C_t^h = \rho W_t^h$

ρ: the household's rate of time preference

2.6 Household Decisions Markowitz model

$$\max_{\alpha_t^h \in [0,1]} \alpha_t^h E_t[\widetilde{dR_t}] - \frac{1}{2}(\alpha_t^h)^2 Var_t[\widetilde{dR_t}] \quad (9)$$

$$\text{s.t.} \quad \alpha_t^h (1-\lambda) W_t^h \equiv H_t \le m W_t$$

α_t^h: invest propotion (verus consumption)

$(1-\lambda)W_t^h$: in risky asset

H_t: the fund to intermediary

GIven the decisions by the debt household and the risky asset household, the evolution of W_t^h across generations is described by

$$dW_t^h = (lD_t - \rho W_t^h)dt + W_t^h r_t dt + \alpha_t^h(1-\lambda)W_t^h(\widetilde{dR_t} - r_t dt)$$

λ : a fraction of household invest riskless bond

$1-\lambda$: risky asset

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2.7 Equilibrium

Definition 1 An Equilibrium is a set of price & process $\{P_t\}$ and $\{r_t\}$, and decision $\{C_t, C_t^h, \alpha_t^I, \alpha_t^h\}$ such that

1. Given Price processes, decision solve the consumption-saving problems of the debt household, the risky asset household (9) and the specialist (6).
2. Decision satisfy the intermediation constraint of (5)
3. The risky asset market clear: $\frac{\alpha_t^I (W_t + \alpha_h^t (1-\lambda) W_t^h)}{P_t} = 1$ (11)

4. The good market clear: $C_t + C_t^h = D_t(1+l)$ (12)

bond are zero net supply: $W_t + W_t^h = P_t$

3.2 State Variables and Specialists' Euler Equation

Look for a stationary Markov equilibrium

where the ~~sa~~ state variables are (y_t, D_t)

$y_t \equiv \frac{W_t^h}{D_t}$ is the dividend scaled wealth of the household

conjecture the equilibrium risky asset price is

$$P_t = \underbrace{D_t}_{\text{dividends}} \underbrace{F(y_t)}_{\frac{\text{price}}{\text{dividend}}}$$

~~ratio of the risky asset~~

the household's optimal consumption given W_t^h is $C_t^h = \rho W_t^h$

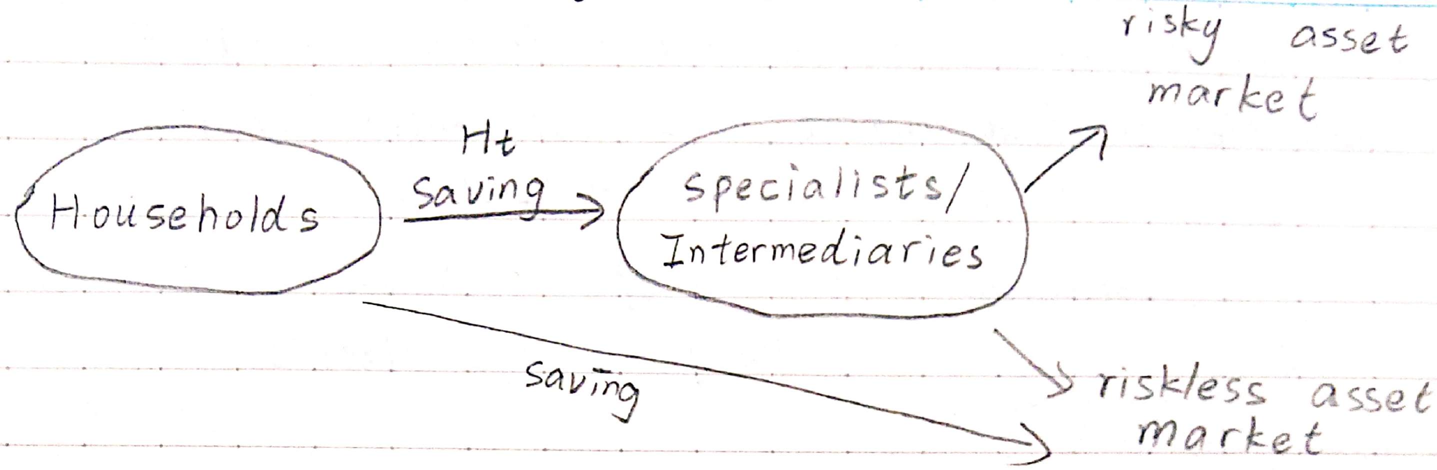
which we can rewrite as $C_t^h = \rho y_t D_t$

market clearing condition for goods is

$$\cancel{C_t = D}$$

$$C_t + \rho y_t D_t = D_t(1+l)$$

Intermediary Assets Price



The risky assets pay a dividend of D_t per unit time, where $\{D_t\}$ follows a $\boxed{\text{geometric Brownian motion}}$,

$$\frac{dD_t}{D_t} = g\,dt + \sigma\, dZ_t \quad \text{given } D_0$$

$g>0$ and $\sigma>0$ are constants. Throughout this paper $\{Z_t\}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$.

We denote the process $\{P_t\}$ and $\{r_t\}$ as the risky asset price and interest rate process, respectively.

We also define the total return on the risky asset as

$$dR_t = \frac{D_t\,dt + dP_t}{P_t}$$

Geometric Brownian motion (exponential Brownian motion)

GBM is a continuous time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (i.e. Wiener process) with drift.

Technical definition : the SDE

A stochastic process S_t is said to follow a GMB if it is satisfies the following stochastic differential equation (SDE)

$$dS_t = \underbrace{\mu}_{\text{percentage drift}} S_t\, dt + \underbrace{\sigma}_{\text{the percentage volatility}} S_t\, \underbrace{dW_t}_{\text{Wiener process}}$$

The specialists are infinitely-lived and maximize an objective function,

$$E\left[\int_0^\infty e^{-\rho t} u(C_t)\, dt\right] \qquad \rho > 0$$

where C_t is the date t consumption rate of the specialist. We consider a CRRA instantaneous utility function with parameter γ for the specialists, $u(C_t) = \frac{1}{1-\gamma} C_t^{1-\gamma}$

We denote the date t wealth of specialists as $\boxed{W_t}$

① W_t is wholly invested in the intermediary

② W_t as the specialist's "stake" in the intermediary

The household $\xrightarrow[\text{fund } \boxed{H_t}]{\text{allocates some}}$ the intermediary

At $t+dt$ the match is broken, and the intermediation market repeat itself.

Total capital of intermediary = specialist's wealth W_t + the household allocates to the intermediary, H_t

It is easy to verify that $\delta t \to dt$ in the continues time limit, the household's ~~con~~ consumption rule is:

$$C_t^h = \rho W_t^h \quad (8)$$

$\rho > 0$ as the household's rate of time preference

2.6 Household decisions

To summarize

- a debt and risky asset household are born at generation t with wealth of W_t^h
- The households receive labor income and choose a ~~con~~ consumption rate of ρW_t^h

Given the log objective function, this decision solves,

$$\max_{\alpha_t^h \in [0,1]} \alpha_t^h E_t[\tilde{dR}_t] - \frac{1}{2}(\alpha_t^h)^2 Var_t[\tilde{dR}_t]$$

$$\text{s.t.} \quad \alpha_t^h (1-\lambda) W_t^h = H_t \leq m W_t$$

Given the decisions by the debt household and the risky asset household, the evolution of W_t^h across generations is described by:

$$dW_t^h = (lD_t - \rho W_t^h)dt + W_t^h r_t dt + \alpha_t^h (1-\lambda) W_t^h (\tilde{dR}_t - r_t dt)$$

2.7 Equilibrium

Definition 1

An equilibrium is a set of price process $\{P_t\}$ and $\{r_t\}$, and decisions $\{C_t, C_t^h, \alpha_t^I, \alpha_t^h\}$

Intermediary Assets Prices

specialist → a fraction $\alpha_t^I \Rightarrow$ risky asset
specialist → a fraction $1-\alpha_t^I \Rightarrow$ riskless asset

Then, the total return delivered by the intermediary

$$\widetilde{dR_t} = r_t\,dt + \alpha_t^I (dR_t - r_t\,dt) \qquad (4)$$

$$= \underbrace{(1-\alpha_t^I) r_t\,dt}_{\text{riskless}} + \underbrace{\alpha_t^I\,dR_t}_{\text{risky asset return}}$$

2.3 Intermediary equity capital constraint

Key assumption: the household is unwilling to invest more than mW_t of funds in the intermediary $(m>0)$

constraint: $H_t \le m W_t$ (5)

In our model, the intermediaries raise equity capital from households as well as borrowing as selling (i.e. shorting) riskless bonds.

Denote such borrowing as B_t. We can imagine a constraint.

$$B_t \le m^b W_t$$ debt or leverage constraint

2.4 Specialist / intermediary decision

The specialist chooses his consumption rate and the portfolio decision of the intermediary to solve.

$$\max_{\{C_t, \alpha_t^I\}} E\left[\int_0^\infty e^{-\rho t} u(C_t) dt\right] \qquad (6)$$

s.t. $dw_t = -C_t dt + w_t r_t dt + w_t(d\tilde{R}_t(\alpha_t^I) - r_t dt)$

Notation. $d\tilde{R}_t = r_t dt + \alpha_t^I (dR_t - r_t dt)$

$W_t \rightarrow$ wealth of specialists at date t

rewrite the constraint

$$dw_t = -C_t dt + w_t r_t dt + \alpha_t^I w_t (dR_t - r_t dt)$$

2.5 Households : The demand for intermediation

We index time as $t, t+\delta t, t+2\delta t$, and consider the continuous time limit when δt is of order dt.

A unit mass of generation t agents are $\boxed{\text{born with wealth } W_t^h}$ and live in period t and $t+\delta t$.

They maximize utility :

$$\rho \delta t \ln C_t^h + (1-\rho\delta t) E_t[\ln W_{t+\delta t}^h] \qquad (7)$$

- C_t^h is the household's consumption in period t
- $W_{t+\delta t}^h$ is a bequest for generation $t+\delta t$

Both utility and bequest function are log

In addition to wealth of W_t^h, we assume that generation t households receive $\boxed{\text{labor income}}$ at date t of $l D_t \delta$

$l>0$ is a constant and D_t is the dividend on the risky asset at time t.

Intermediary Asset Prices

2.1 Assets

- We normalize the total supply of intermediated risky assets to be one unit. The riskless bond is zero net supply and can be invested in by both households and specialists.
- The risky asset pays a dividend of D_t per unit, where $\{D_t\}$ follows a gametric Brownian motion.

$$\frac{dD_t}{D_t} = g\,dt + \sigma\, dZ_t \quad \text{given } D_0 \qquad (1)$$

$\{Z_t\}$ is a standard Brownian motion

supplement: SDE (stochastic differential equation)

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$

μ: percentage drift

σ: the percentage volatility

W_t: winner process

We denote the process $\{P_t\}$ and $\{r_t\}$ as $\boxed{\text{the risky asset}}$ price and interest rate process.

- Total return on the risky asset:

$$dR_t = \frac{D_t\,dt + dP_t}{P_t} \qquad (2)$$

2.2 Specialists and intermediation

The specialists maximize an objective function.

$$E\left[\int_0^\infty e^{-\rho t} u(C_t)\,dt\right] \quad \rho > 0$$

- C_t is the date t consumption rate of specialist
- CRRA instantaneous utility function with parameter γ for specialists, $u(C_t) = \frac{1}{1-\gamma} C_t^{1-\gamma}$

- W_t the date t wealth of specialists
- The household allocates some funds H_t to ~~ime~~ intermediary.

The return delivered by the intermediary is

$$\widetilde{dR}_t = (1-\alpha_t^I) r_t dt + \alpha_t^I dR_t \quad (4)$$
$$= r_t dt + \alpha_t^I (dR_t - r_t dt)$$

where $dR_t = \frac{D_t dt + dP_t}{P_t}$ is the total return on the risky asset

2.3 Intermediary equity capital constraint

The wealth requirement implies that the supply of the intermediation facing a household is at most,

$$H_t \le m W_t \quad (5)$$

Debt or leverage (Not studied) : $B_t \le m^b W_t$

2.4 Specialist / intermediary decision

The specialist chooses his consumption rate and the portfolio decision of the intermediary to solve:

$$\max_{\{C_t, \alpha_t^I\}} E\left[\int_0^\infty e^{-\rho t} u(C_t)\, dt\right] \quad (6)$$

s.t. $dW_t = -C_t dt + W_t r_t dt + W_t(\widetilde{dR}_t(\alpha_t^I) - r_t dt)$

rewrite. $dW_t = -C_t dt + W_t r_t dt + \alpha_t^I W_t (dR_t - r_t dt)$

2.5 Households: The demand for intermediation

overlapping generation model

- A unit mass of generation t agents are born with wealth W_t^h and live in period t and $t+\delta t$
- They maximize utility:

$$\rho\delta t \ln C_t^h + (1-\rho\delta t) E_t[\ln W_{t+\delta t}^h] \qquad (7)$$

- C_t^h is the household's consumption in period t and $W_{t+\delta t}^h$ is bequest for generation $t+\delta t$
- In addition to wealth of W_t^h, we assume that generation t households receive labor income at date t of $\boxed{l D_t \delta}$ $l>0$ is a constant and D_t is the dividend on the risky asset at time t.

the household's consumption rule is

$$C_t^h = \underbrace{\rho}_{\text{household's rate of time perference}} W_h^t \qquad (8)$$

Two type of households $\Rightarrow$ introduce heterogeneity

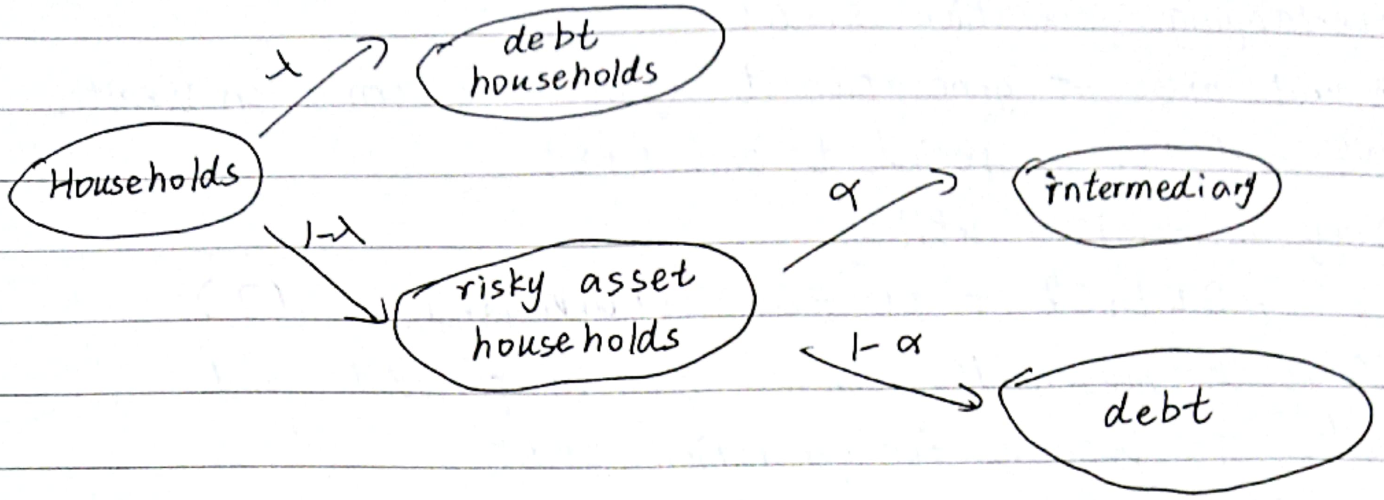
① debts households:

a fraction λ of the households can ever only invest in the riskless bond.

② risky asset households:

The remaining fraction, $1-\lambda$, may enter the intermediation market and save a fraction of their wealth with intermediation.

2.6 Household decisions



~~Given the decisions by the debt~~

Given the log objective function, risky asset households' decision solves:

$$\max_{\alpha_t^h \in [0,1]} \quad \alpha_t^h E_t[d\tilde{R}_t] - \frac{1}{2}(\alpha_t^h)^2 Var_t[d\tilde{R}_t] \qquad (9)$$

$$\text{s.t.} \quad \alpha_t^h (1-\lambda) W_t^h \equiv H_t \leq m W_t$$

the evolution of W_t^h across generation is described by

$$dW_t^h = (lD_t - \rho W_t^h)dt + \lambda W_t^h r_t dt + (1-\lambda)\alpha_t^h W_t^h d\tilde{R}_t + (1-\lambda)(1-\alpha_t^h) W_t^h r_t dt$$

simplify $\Rightarrow dW_t^h = (lD_t - \rho W_t^h)dt + W_t^h r_t dt + \alpha_t^h (1-\lambda) W_t^h (d\tilde{R}_t - r_t dt)$

(10)

2.7 Equilibrium

Definition 1:

An equilibrium is a set of price process $\{P_t\}$ and $\{r_t\}$, and decisions $\{c_t, C_t^h, \alpha_t^I, \alpha_t^h\}$ such that,

1. Given the price process, decisions solve the consumption optimal problem savings problems of the debt household (9) and the specialist (6).
2. Decisions satisfy the intermediation constraint of (5) $H_t \leq m W_t$
3. The risky asset market clears:

$$\frac{\alpha_t^I (W_t + \alpha_t^h (1-\lambda) W_t^h)}{P_t} = 1 \qquad (11)$$

4. The goods market clears:

$$C_t + C_t^h = D_t(1+l) \qquad (12)$$

The market clearing condition for the risky asset market is the only direct holder of risky assets and has total funds under management of $W_t + \alpha_t^h (1-\lambda) W_t^h$, and the total holding of risky asset by the intermediary must equal the supply of risky assets.

Market clearing:

market clearing is the process by which, in an economic market, the supply of whatever is traded is equated to the demand.

3.2 State Variables and Specialists' Euler Equation

• We look for a stationary Markov equilibrium where the state variables are (y_t, D_t), where $y_t = \frac{W_t^h}{D_t}$ is the dividend scaled wealth of the household.

We conjecture that the equilibrium risk asset price

$$P_t = D_t F(y_t) \qquad (13)$$

$\cancel{F(t)}$ $F(y)$ is the price / dividend ratio of the risky asset

From (8)

⇒ the household's optimal consumption given W_t^h is $C_t^h = \rho W_t^h$

rewrite ⇒ $C_t^h = \rho y_t D_t$

The market clearing condition for goods (from (12))

$$C_t + \rho y_t D_t = D_t(1+l)$$

Thus, in equilibrium, the specialist consumes:

$$C_t = D_t(1+l-\rho y_t) \qquad (14)$$

The Euler equation:

- Consider an infinitely-lived agent choosing a control variable (c) in each period (t) to maximize an intertemporal objective $\sum_{t=1}^{\infty} \beta^{t-1} u(C_t)$ where $u(C_t)$ represents the flow payoff in t, $u'>0$, $u''<0$, and β is the discount factor, $0<\beta<1$
- The agents face a present-value budget constraint:

$$\sum_{t=1}^{\infty} R^{1-t} C_t \leq W_1 \qquad (1)$$

where R is the gross interest rate ($R=1+r$ where r is the interest rate) and W_1 is given.

If a time-path of the control is optimal, a marginal increase in the control at any t, dC_t, must have benefits equal to the cost of the decrease in $t+1$ of the same present value amount, $-Rd_t$

$$\beta^{t-1} u'(C_t)\, dC_t - \beta^{t} u'(C_{t+1}) R\, dC_t = 0$$

Reorganization gives the Euler equation

$$u'(C_t) = \beta R\, u'(C_{t+1})$$

Optimality for the specialist gives us the standard consumption based asset pricing relation (Euler equation): (15)

$$-\rho dt - \gamma E_t\left[\frac{dC_t}{C_t}\right] + \frac{1}{2}\gamma(\gamma+1) Var_t\left[\frac{dC_t}{C_t}\right] + E_t[dR_t] = \gamma Cov_t\left[\frac{dC_t}{C_t}, dR_t\right]$$

and for interest rate, we have

$$r_t dt = \rho dt + \gamma E_t\left[\frac{dC_t}{C_t}\right] - \frac{\gamma(\gamma+1)}{2} Var_t\left[\frac{dC_t}{C_t}\right] \qquad (16)$$

3.3 Dynamics of Household Wealth

Given the wealth dynamics of household in (10)

$$dW_t^h = (lD_t - \rho W_t^h) dt + W_t^h r_t dt + \alpha_t^h (1-\lambda) W_t^h (\widetilde{dR}_t - r_t dt)$$

and the intermediary return $\widetilde{dR}_t - r_t dt = \alpha_t^I (dR_t - r_t dt)$

$$\overset{\text{we have}}{\Longrightarrow} dW_t^h = (lD_t - \rho W_t^h) dt + W_t^h r_t dt + (\alpha_t^h \alpha_t^I)(1-\lambda) W_t^h (dR_t - r_t dt)$$

- The household's exposure to the risky asset return $(\alpha_t^h \alpha_t^I)(1-\lambda)$
- when the ~~return~~ intermediation constraint of equation (5) (i.e. $H_t \leq mW_t$) blinds, the household choice must ~~st~~ satisfy

$$\alpha_t^{h,const}(1-\lambda)W_t^h = mW_t$$

which implies, $\alpha_t^{h,const} = \frac{m(F(y)-y)}{(1-\lambda)y} = \frac{mW_t}{(1-\lambda)W_t^h}$ (17)

The equilibrium market clearing condition (11) gives

$$\frac{\alpha_t^{I,const}(W_t + mW_t)}{P_t} = 1$$