Asset Pricing

Two major equation:

$$P_t = E(m_{t+1} x_{t+1})$$

$$m_{t+1} = f(\text{data, parameters})$$

P_t : asset price , x_{t+1} : asset payout , m_{t+1} : stochastic discounted factor

1.1 Basic Pricing Model

An investor's first-order conditions give basic comsumption-based model

$$P_t = E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)} \cdot x_{t+1}\right]$$

the value of payoff $x_{t+1} = P_{t+1} + d_{t+1}$ ⇒ dividend

We model investors by a utility function defined current and future value of consumptions.

$$U(C_t, C_{t+1}) = u(C_t) + \beta E_t[u(C_{t+1})]$$

$$u(C_t) = \frac{1}{1-r} C_t^{1-r}$$

The limit as $r \to 1$, $u(c) = \ln(c)$

How much will he buy or sell

$$\max_{\{\xi\}} u(C_t) + E_t[\beta u(C_{t+1})] \text{ s.t.}$$

original consumption level

$$\begin{cases} C_t = e_t - P_t \xi \\ C_{t+1} = e_{t+1} + x_{t+1}\xi \end{cases}$$

ξ ⇒ the amount of asset he choose to buy

delay the current consumption (C_t) into next period (C_{t+1})

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$$\max_{\{\xi\}} u(c_t) + E_t[\beta u(c_{t+1})]$$

代入 c_t, c_{t+1}

$$\max u(e_t - p_t\xi) + E_t[\beta u(e_{t+1} + x_{t+1}\xi)]$$

setting the derivative with respect to ξ

$$\underbrace{-p_t u'(e_t - p_t\xi)}_{\text{loss in utility at } t} + \beta x_{t+1} E_t[\underbrace{u'(e_{t+1} + x_{t+1}\xi)}_{\text{increase in utility at } t+1}] = 0$$

$$\Rightarrow p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right]$$

the central asset pricing model

1.2 Marginal Rate of Substitution / Stochastic Discount Factors

We break up the basic consumption-based pricing equation into

$$p = E(mx)$$
$$m = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

where m_{t+1} is the stochastic discount factor

Define stochastic discount factor $m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$

$$p_t = E_t(m_{t+1} x_{t+1})$$

If there is no uncertainty

$$p_t = \frac{1}{R^f} x_{t+1}$$

gross risk ~~factor~~-free rate

Risky assets:

$$p_t^i = \frac{1}{R^i} E_t(x_{t+1}^i)$$

$m_{t+1} \Rightarrow$ pricing kernel

1.3 Prices, Payoffs, and Notations

For stocks $x_{t+1} = P_{t+1} + d_{t+1}$

$$R_{t+1} = \frac{x_{t+1}}{P_t}.$$

If p and x denote nominal values, then we can create real price and payoff

$$\frac{P_t}{\Pi_t} = E_t\left[(\beta \frac{u'(C_{t+1})}{u'(C_t)} \cdot \frac{x_{t+1}}{\Pi_{t+1}}\right]$$

$\Downarrow$

Π denote the price level (CPI)

$$P_t = E_t\left[(\beta \frac{u'(C_{t+1})}{u'(C_t)} \cdot \frac{\Pi_t}{\Pi_{t+1}}) x_{t+1}\right]$$

1.4 Classic issue in Finance

Risk - Free Rate

The risk-free rate is related to discount factor by

$$R^f = 1 / E(m)$$

With lognormal consumption growth and power utility

$$r_t^f = \delta + \gamma E_t(\Delta \ln C_{t+1}) - \frac{\gamma^2}{2} \sigma_t^2(\Delta \ln C_{t+1})$$

Real interest are high when
① People are impatient (δ)
② consumption growth is high
③ risk is low

use power utility $u'(c) = c^{-\gamma} \Rightarrow R^f = \frac{1}{\beta}(\frac{C_{t+1}}{C_t})^{\gamma}$

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Risk Corrections

Payoffs that are positively correlated with consumption growth have lower prices, to compensate investors for risk.

$$P = \frac{E(x)}{R^f} + Cov(m, x)$$

$$E(R^i) - R^f = -R^f cov(m, R^i)$$

Expected returns are propotional to the covariance of returns with discount factors.

Using the definition of covariance $cov(m,x) = E(mx) - E(m) \cdot E(x)$

$P = E(mx)$ can be rewrited as

$$P = E(m) \cdot E(x) + cov(m, x)$$

$\Downarrow$ $E(m) \cdot R_f = 1$, i.e. $R_f = \frac{1}{E(m)}$

$$P = \frac{E(x)}{R^f} + cov(m, x) \Rightarrow$$ risk adjustment

$\Downarrow$ standard discounted present-value formula

subsitute back $m = \beta \frac{u'(C_{t+1})}{u'(C_t)}$

$$P = \frac{E(x)}{R^f} + \frac{cov[\beta u'(C_{t+1}), x_{t+1}]}{u'(C_t)}$$

chain: $u'(t) \downarrow \Leftrightarrow c \uparrow \Leftrightarrow cov \downarrow$

Consider then what happens to the volatility of consumption if the investors buys a little more ξ of payoff x, $\sigma^2(c)$ becomes

$$\sigma^2(c + \xi x) = \sigma^2(c) + 2\xi cov(c, x) + \xi^2 \sigma^2(x)$$

$$1 = E(mR^i)$$

$$1 = E(m)E(R^i) + cov(m, R^i)$$

using $R^f = \frac{1}{E(m)}$

$$E(R^i) - R^f = -R^f cov(m, R^i)$$

$$E(R^i) - R^f = -\frac{cov[u'(C_{t+1}), R^i_{t+1}]}{E[u'(C_{t+1})]}$$

Idiosyncratic Risk Does Not Affect Price

If $cov(m, x) = 0$, then $P = \frac{E(x)}{R^f}$, not matter how large $\sigma^2(x)$

Expected Return-Beta Representation

We can write $p = E(mx)$ as $E(R^i) = R^f + \beta_{i,m}\lambda_m$

express $E(R^i) - R^f = -R^f cov(m, R^i)$ into

$$E(R^i) = R^f + \left(\frac{cov(R^i, m)}{Var(m)}\right)\left(-\frac{var(m)}{E(m)}\right)$$

or $$E(R^i) = R^f + \underbrace{\beta_{i,m}}_{\text{regression coefficient}} \lambda_m \Rightarrow \text{price of risk}$$

$\beta_{i,m}$ vary from asset to asset

(the quantity of risk in each asset)

Mean-Variance Frontier

$$|E(R^i) - R^f| \le \frac{\sigma(m)}{E(m)} \cdot \sigma(R^i) \quad (1.17)$$

To derive (1.17) write for a given asset return R^i

$$1 = E(mR^i) = E(m) \cdot E(R^i) + \rho_{m,R^i}\sigma(R^i)\sigma(m)$$

$$\Rightarrow R^{(i)} = R^f - \rho_{m,R^i}\frac{\sigma(m)}{E(m)} \cdot \sigma(R^i)$$

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Random Walks and Time-Varying Expected Returns

First-order condition

$$p_t u'(c_t) = E_t[\beta u'(c_{t+1})(p_{t+1}+d_{t+1})]$$

$$\xRightarrow{\text{in a short time}} \quad p_t = E_t(p_{t+1})$$

prices follow a time-series process of the form

$$p_{t+1} = p_t + \varepsilon_{t+1}$$

If the variance $\sigma_t(\varepsilon_{t+1})$ is constant, prices follow a random walk. i.e. price follow a martingale.

$$E_t(R_{t+1}) - R_t^f = -\frac{cov_t(m_{t+1}, R_{t+1})}{E_t(m_{t+1})}$$

$$= -\frac{\sigma_t(m_{t+1})}{E_t(m_{t+1})} \cdot \sigma_t(R_{t+1}) \cdot \rho_t(m_{t+1}, R_{t+1})$$

$$\approx \gamma_t \sigma_t(\Delta c_{t+1}) \cdot \sigma_t(R_{t+1}) \rho_t(m_{t+1}, R_{t+1})$$

Continuous Time

A.1 Brownian Motion

z_t, dz_t are defined by $z_{t+\Delta} - z_t \sim N(0, \Delta)$

ε_t (discrete time) $\Leftrightarrow$ dz_t (continue time)

a random walk in discrete time $\quad z_t - z_{t-1} = \varepsilon_t$

the variance scales with time $\quad var(z_{t+2} - z_t) = 2Var(z_{t+1} - z_t)$

Define a Brownian motion as a process z_t for which

$$z_{t+\Delta} - z_t \sim N(0, \Delta)$$

use the notation dz_t to represent $z_{t+\Delta} - z_t$ for arbitary small time interval Δ