Exercice 9.4

1. Soit $a \in \mathbb{R}_+$. On a, en posant le changement de variable $t = -x$

$$\int_{-a}^{a} f(x)\mathrm{d}x = \int_{a}^{-a} f(-t) - \mathrm{d}t = \int_{a}^{-a} f(t)\mathrm{d}t = -\int_{-a}^{a} \mathrm{d}t$$

D'où

$$\int_{-a}^{a} f(x)\mathrm{d}x = 0$$

2. On pose $f : x \mapsto \ln\left(\dfrac{1+e^{\arctan(x)}}{1+e^{-\arctan(x)}}\right)$. On a

$$\begin{aligned}\mathcal{D}_f &= \{x \in \mathbb{R} | f(x) \text{ est défini}\} \\ &= \left\{x \in \mathbb{R} | \frac{1+e^{\arctan(x)}}{1+e^{-\arctan(x)}} > 0\right\} \\ &= \mathbb{R}\end{aligned}$$

On montre que f est impaire. Soit $x \in \mathbb{R}$, on a

$$\begin{aligned}f(-x) &= \ln\left(\frac{1+e^{\arctan(x)}}{1+e^{-\arctan(x)}}\right) \\ &= \ln\left(\frac{1+e^{\arctan(-x)}}{1+e^{-\arctan(-x)}}\right) \\ &= \ln\left(\frac{1+e^{-\arctan(x)}}{1+e^{\arctan(x)}}\right) \\ &= -\ln\left(\frac{1+e^{\arctan(x)}}{1+e^{-\arctan(x)}}\right) \\ &= -f(x)\end{aligned}$$

f étant continue, $\int_{-666}^{666} f(x)\mathrm{d}x$ est définie et, d'après la question 1, on a $a = 666$ donc $\int_{-a}^{a} f(x)\mathrm{d}x = 0$

Exercice 9.7

1. $\alpha = \ln\left(1+\sqrt{2}\right)$.
2. On pose le changement de variable $x = \sinh(t)$, d'où $\mathrm{d}x = \cosh(t)\mathrm{d}t$. On a alors

$$\begin{aligned}J = \int_0^1 \frac{1}{\sqrt{1+x^2}}\mathrm{d}x &= \int_0^{\alpha} \frac{1}{\sqrt{1+\sinh^2(t)}}\cosh(t)\mathrm{d}t \\ &= \int_0^{\alpha} \frac{1}{\cosh(t)}\cosh(t)\mathrm{d}t \\ &= \int_0^{\alpha} \mathrm{d}t \\ &= \alpha\end{aligned}$$

3. On a

$$I = \int_0^1 \sqrt{1+x^2}\mathrm{d}x = [x\sqrt{1+x^2}]_0^1 - \int_0^1 x\frac{2x}{2\sqrt{1+x^2}}$$

Exercice 9.9

En posant $x = \dfrac{\pi}{4} - u$, d'où $\mathrm{d}x = -\mathrm{d}u$.

$$\begin{aligned}I &= \int_0^{\frac{\pi}{4}} \ln(1+\tan(x))\mathrm{d}x \\ &= \int_{\frac{\pi}{4}}^{0} -\ln\left(1+\tan\left(\frac{\pi}{4}-u\right)\right)\mathrm{d}u \\ &= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan(u)}{1+\tan(u)}\right)\mathrm{d}u \\ &= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan(u)}\right)\mathrm{d}u \\ &= \int_0^{\frac{\pi}{4}} \ln 2\mathrm{d}u - I \\ &= \ln 2\frac{\pi}{4} - I\end{aligned}$$

D'où $2I = \ln 2\dfrac{\pi}{4} \iff I = \ln 2\dfrac{\pi}{8}$.