Trigonométrie

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MP2I

Exercice 10

Énoncé

1. Exprimer $\tan(a+b+c)$ en fonction de $\tan(a),\ \tan(b)$ et $\tan(c)$.
2. Généraliser à $\tan\left(\sum_{k=1}^{n} a_k\right)$.

Solution proposée

1. Soit $(a,b,c) \in \mathbb{R}^3$ tel que $\tan(a),\ \tan(b)$ et $\tan(c)$ soient définis.

$$\begin{aligned}
\tan(a+b+c) &= \frac{\tan(a+b)+\tan(c)}{1-\tan(a+b)\tan(c)} \\
&= \frac{\frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}+\tan(c)}{1-\frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}\tan(c)} \\
&= \frac{\tan(a)+\tan(b)+\tan(c)(1-\tan(a)\tan(b))}{1-\tan(a)\tan(b)-\tan(c)(\tan(a)+\tan(b))} \\
&= \frac{\tan(a)+\tan(b)+\tan(c)-\tan(a)\tan(b)\tan(c)}{1-\tan(a)\tan(b)-\tan(a)\tan(c)-\tan(c)\tan(b)}
\end{aligned}$$

2. Soit $(a_1,a_2,a_3,a_4) \in \mathbb{R}^4$ tel que $\forall i \in \{1,2,3,4\},\ \tan(a_i)$ est défini.

$$\begin{aligned}
\tan\left(\sum_{k=1}^{4} a_k\right) &= \frac{\tan\left(\sum_{k=1}^{3} a_k\right)+\tan(a_4)}{1-\tan\left(\sum_{k=1}^{3} a_k\right)\tan(a_4)} \\
&= \frac{\frac{\sum_{k=1}^{3}\tan(a_k)-\prod_{k=1}^{3}\tan(a_k)}{1-\sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{3} a_k}+\tan(a_4)}{1-\left(\frac{\sum_{k=1}^{3}\tan(a_k)-\prod_{k=1}^{3}\tan(a_k)}{1-\sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{3} a_k}\right)\tan(a_4)} \\
&= \frac{\sum_{k=1}^{3}\tan(a_k)-\prod_{k=1}^{3}\tan(a_k)+\tan(a_4)\left(1-\sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{3} a_k\right)}{1-\sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{3} a_k-\left(\sum_{k=1}^{3}\tan(a_k)-\prod_{k=1}^{3}\tan(a_k)\right)\tan(a_4)} \\
&= \frac{\sum_{k=1}^{4}\tan(a_k)-\prod_{k=1}^{3}\tan(a_k)-\sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{4} a_k}{1-\sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{3} a_k-\tan(a_4)\sum_{k=1}^{3}\tan(a_k)+\prod_{k=1}^{4}\tan(a_k)}
\end{aligned}$$