Trigonométrie

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MP2I

Exercice 1

Énoncé Calculer $\sin\left(\dfrac{45\pi}{12}\right)$ et $\cos\left(\dfrac{45\pi}{12}\right)$

Solution proposée On a

$$\begin{aligned}\sin\left(\frac{45\pi}{12}\right) &= \sin\left(\frac{3^2\times 5\times \pi}{2^2\times 3}\right)\\ &= \sin\left(\frac{15\pi}{4}\right)\\ &= \sin\left(4\pi-\frac{\pi}{4}\right)\\ &= -\sin\left(\frac{\pi}{4}\right)\\ &= -\frac{\sqrt{2}}{2}\end{aligned}$$

et

$$\begin{aligned}\cos\left(\frac{45\pi}{12}\right) &= \cos\left(4\pi-\frac{\pi}{4}\right)\\ &= \cos\left(\frac{\pi}{4}\right)\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Exercice 2

Énoncé Résoudre dans $\mathbb{R}$ l'équation $\sin\left(x+\dfrac{3\pi}{4}\right)=\cos\left(2x-\dfrac{\pi}{3}\right)$.

Solution proposée Soit $x\in\mathbb{R}$,

$$\sin\left(x+\frac{3\pi}{4}\right)=\cos\left(2x-\frac{\pi}{3}\right)\Leftrightarrow \sin\left(x+\frac{3\pi}{4}\right)=\sin\left(2x-\frac{\pi}{3}+\frac{\pi}{2}\right)$$

Donc

$$\begin{aligned}&\begin{cases} x+\dfrac{3\pi}{4}\equiv 2x+\dfrac{\pi}{6}[2\pi]\\ \text{ou}\\ x+\dfrac{3\pi}{4}\equiv \pi-2x-\dfrac{\pi}{6}[2\pi]\end{cases}\\ \Longleftrightarrow &\begin{cases} -x\equiv \dfrac{2\pi}{12}-\dfrac{9\pi}{12}[2\pi]\\ \text{ou}\\ 3x\equiv \dfrac{12\pi}{1}2-\dfrac{9\pi}{12}-\dfrac{2\pi}{12}[2\pi]\end{cases}\end{aligned}$$

$$\iff \begin{cases} x \equiv \frac{7\pi}{12} [2\pi] \\ \text{ou} \\ x \equiv \frac{\pi}{36} \left[\frac{2\pi}{3}\right] \end{cases}$$

Exercice 3

Énoncé Résoudre dans $\mathbb{R}$ l'équation $\sin\left(5x + \frac{\pi}{2}\right) = \sin(2x)$.

Solution proposée Soit $x \in \mathbb{R}$, Donc

$$\begin{aligned} &\sin\left(5x + \frac{\pi}{2}\right) = \sin(2x) \\ \iff &\begin{cases} 5x + \frac{\pi}{2} \equiv 2x[2\pi] \\ \text{ou} \\ 5x + \frac{\pi}{2} \equiv \pi - 2x[2\pi] \end{cases} \\ \iff &\begin{cases} 3x \equiv -\frac{\pi}{2}[2\pi] \\ \text{ou} \\ 7x \equiv \pi - \frac{\pi}{2}[2\pi] \end{cases} \\ \iff &\begin{cases} x \equiv -\frac{\pi}{6} \left[\frac{2\pi}{3}\right] \\ \text{ou} \\ x \equiv \frac{\pi}{14} \left[\frac{2\pi}{7}\right] \end{cases} \end{aligned}$$

Exercice 10

Énoncé

1. Exprimer $\tan(a+b+c)$ en fonction de $\tan(a),\ \tan(b)$ et $\tan(c)$.
2. Généraliser à $\tan\left(\sum_{k=1}^{n} a_k\right)$.

Solution proposée

1. Soit $(a,b,c) \in \mathbb{R}^3$ tel que $\tan(a),\ \tan(b)$ et $\tan(c)$ soient définis.

$$\begin{aligned} \tan(a+b+c) &= \frac{\tan(a+b) + \tan(c)}{1 - \tan(a+b)\tan(c)} \\ &= \frac{\frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)} + \tan(c)}{1 - \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}\tan(c)} \\ &= \frac{\tan(a) + \tan(b) + \tan(c)(1 - \tan(a)\tan(b))}{1 - \tan(a)\tan(b) - \tan(c)(\tan(a) + \tan(b))} \\ &= \frac{\tan(a) + \tan(b) + \tan(c) - \tan(a)\tan(b)\tan(c)}{1 - \tan(a)\tan(b) - \tan(a)\tan(c) - \tan(c)\tan(b)} \end{aligned}$$

2. Soit $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ tel que $\forall i \in \{1,2,3,4\},\ \tan(a_i)$ est défini.

$$\begin{aligned}
\tan\left(\sum_{k=1}^{4} a_k\right) &= \frac{\tan\left(\sum_{k=1}^{3} a_k\right) + \tan(a_4)}{1 - \tan\left(\sum_{k=1}^{3} a_k\right)\tan(a_4)} \\
&= \frac{\frac{\sum_{k=1}^{3}\tan(a_k) - \prod_{k=1}^{3}\tan(a_k)}{1-\sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{3} a_k} + \tan(a_4)}{1 - \left(\frac{\sum_{k=1}^{3}\tan(a_k) - \prod_{k=1}^{3}\tan(a_k)}{1-\sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{3} a_k}\right)\tan(a_4)} \\
&= \frac{\sum_{k=1}^{3}\tan(a_k) - \prod_{k=1}^{3}\tan(a_k) + \tan(a_4)\left(1 - \sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{3}\tan(a_k)\right)}{1 - \sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{3}\tan(a_k) - \left(\sum_{k=1}^{3}\tan(a_k) - \prod_{k=1}^{3}\tan(a_k)\right)\tan(a_4)} \\
&= \frac{\sum_{k=1}^{4}\tan(a_k) - \sum_{k=1}^{4}\prod_{i=1;\ i\neq k}^{4}\tan(a_k)}{1 - \sum_{k=1}^{3}\prod_{i=1;\ i\neq k}^{3}\tan(a_k) - \tan(a_4)\sum_{k=1}^{3}\tan(a_k) + \prod_{k=1}^{4}\tan(a_k)}
\end{aligned}$$

Pour tout $n \in \mathbb{N}$, posons $P(n)$ la propriété que

$$\forall (a_1, a_2, \dots, a_n) \in \mathbb{R}, \quad \tan\left(\sum_{k=1}^{n} a_k\right) = \frac{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \sum_{\substack{A\in\mathcal{P}(\{1,2,\dots,n\}) \\ |A|=2k+1}} \prod_{i\in A}\tan(a_i)}{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \sum_{\substack{A\in\mathcal{P}(\{1,2,\dots,n\}) \\ |A|=2k}} \prod_{i\in A}\tan(a_i)}$$