Information Sciences 312 (2015) 40–54

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Some minimal axiom sets of rough sets

Zhouming Ma [a,b,c,*], Jinjin Li [a], Jusheng Mi [b]

[a] School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian 363000, PR China
[b] College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050024, PR China
[c] Lab of Granular Computing, Minnan Normal University, Zhangzhou, Fujian 363000, PR China

ARTICLE INFO

Article history:
Received 27 April 2012
Received in revised form 13 March 2015
Accepted 22 March 2015
Available online 28 March 2015

Keywords:
Rough sets
The minimal axiom sets
Equivalence relations
Binary relations

ABSTRACT

The axiomatic approach is an important one to the development of rough set theory. This paper proposes some independent and minimal axiom sets for characterizing the classical rough set. First, we investigate the correlations among some arbitrary binary relations in rough sets. Then, we provide several forms of equivalence relations which are different from the original ones. Finally, to well characterize the classical rough set, we propose fifteen new minimal axiom sets of rough sets based on the new equivalence relations. By employing relative tables, the independence of axiom sets for characterizing the approximation operators is also examined.

1. Introduction

Rough set theory, proposed by Pawlak [15] in 1982, is a useful mathematical tool for dealing with imprecise, vague and uncertain data in information systems. This theory has been successfully applied in machine learning, pattern recognition, decision support and data mining [14,32,31,46]. The lower and upper approximations are two basic concepts in rough set theory. By using them, the knowledge hidden in database can be unraveled and expressed in the form of decision rules. Therefore, it is important to explore the two operators in rough set theory.

The classical rough set theory is based on equivalence relations. However, in many situations, equivalence relations are not suitable for dealing with the granularity. As a result, one of the main directions of research on this theory is to extend definitions of approximation operators [16–18]. Unavoidably, it leads to a series of studies on the generalized rough set from different angles [22]. The role of equivalence relations corresponds to that of partitions on a universe. As a result, the classical rough set is extended to the covering generalized rough set [1,7,11,12,29,37,42–44]. On the other hand, equivalence relations are extended directly to a variety of binary relations, such as being reflexive, symmetric, transitive, serial, etc [9,10,19,26,28,30].

There are two important methods for the development of rough set theory, the constructive and axiomatic approaches. In the constructive approach, lower and upper approximations are constructed from the primitive notions, such as binary relations on the universe of discourse [2,15,35,38], partitions (or coverings) of the universe of discourse [18,20,21,23,36,44,45]. Compared with the constructive approach, the axiomatic approach focuses on algebraic systems, it regards the lower and upper approximation operators as primitive notions. In this approach, a set of axioms is used to characterize approximation operators that are produced using the constructive approach [4,5,13,27,33]. Many authors explored and developed the axiomatic approach for the classical rough set, which were mostly concentrated on the axiomatizations of generalized rough sets

* Corresponding author at: School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian 363000, PR China. Tel.: +86 5962527012.
E-mail addresses: mazhouming@sina.com (Z. Ma), jinjinli@mnnu.edu.cn (J. Li).

http://dx.doi.org/10.1016/j.ins.2015.03.052
0020-0255/

based on arbitrary binary relations, including serial, reflexive, symmetric, transitive, positive alliance, and negative alliance relations. The works were mainly solved by Yao [33–35] and Zhu [38–41]. What's more, a more advanced definition of minimal axiom sets on rough sets was proposed by Sun [25]. In this approach, the approximation operators were characterized by the least axiom sets.

The lower and upper approximation operators of the generalized rough set based on arbitrary binary relations, own all the properties appearing in the corresponding operators of the classical rough set, because these binary relations are constructed from equivalence relations. In other words, a variety of corresponding generalized rough sets are the refinement of the classical rough set. However, the above mentioned studies have not explored the correlation in these arbitrary binary relations. Therefore, to further reveal the essence of the classical rough set, a series of works should be performed. This paper is devoted to the discussion of the above problems. Firstly, we construct several new different forms of equivalence relations in considering the correlation of these arbitrary binary relations. Secondly, some new minimal axiom sets of rough sets are proposed based on the new forms of equivalence relations. Finally, to justify these new minimal axiom sets, we investigate their independent and minimal qualities.

The structure of this paper is organized as follows: In Section 2, we give a brief introduction to the fundamental concepts and properties of rough sets. Section 3 studies correlation of some arbitrary binary relations in rough set theory, figures out some useful equivalent forms and builds different forms of equivalence relations. In Section 4, with the help of a brief review on the existing axiomatizations of rough set theory, fifteen new minimal axiom sets are proposed based on the new equivalence relations. Meanwhile, we also validate their independent and minimal qualities. We then conclude the paper with a summary and outlook for further research in Section 5.

2. The rough sets based on binary relations

In this section, we review some concepts and properties of the classical rough sets and those of the generalized rough sets based on binary relations.

2.1. The classical rough sets

Let U be a finite and nonempty set called the universe of discourse and R be an equivalence relation on U. R will generate a partition $U/R = \{Y_1, Y_2, \ldots, Y_m\}$ on U, where $Y_1, Y_2, \ldots, Y_m$ are called the equivalence classes generated by R. For any $X \subseteq U$, we define two operators as follows:

$$\underline{R}(X) = \bigcup\{Y_i \in U/R | Y_i \subseteq X\},$$
$$\overline{R}(X) = \bigcup\{Y_i \in U/R | Y_i \cap X \neq \emptyset\}.$$

They are called the lower and upper approximations of X, respectively.

Proposition 1. [15,42]. *The classical rough sets have the following properties:*

(1L) $\underline{R}(U) = U$	(Co-normality)
(1H) $\overline{R}(U) = U$	(Co-normality)
(2L) $\underline{R}(\emptyset) = \emptyset$	(Normality)
(2H) $\overline{R}(\emptyset) = \emptyset$	(Normality)
(3L) $\underline{R}(X) \subseteq X$	(Contraction)
(3H) $X \subseteq \overline{R}(X)$	(Extension)
(4L) $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$	(Multiplication)
(4H) $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$	(Addition)
(5L) $\underline{R}(\underline{R}(X)) = \underline{R}(X)$	(Idempotency)
(5H) $\overline{R}(\overline{R}(X)) = \overline{R}(X)$	(Idempotency)
(6L) $X \subseteq Y \Rightarrow \underline{R}(X) \subseteq \underline{R}(Y)$	(Monotone)
(6H) $X \subseteq Y \Rightarrow \overline{R}(X) \subseteq \overline{R}(Y)$	(Monotone)
(7L) $\underline{R}(-\underline{R}(X)) = -\underline{R}(X)$	(Lower complement relation)
(7H) $\overline{R}(-\overline{R}(X)) = -\overline{R}(X)$	(Upper complement relation)
(8L) $-X \subseteq \underline{R}(-\underline{R}(X))$	(Upper approximation's contraction)
(8H) $\overline{R}(-\overline{R}(X)) \subseteq -X$	(Lower approximation's contraction)
(9LH) $\underline{R}(-X) = -\overline{R}(X)$	(Duality)
(10LH) $\underline{R}(X) \subseteq \overline{R}(X)$	(Appropriateness)

Here, $-X$ be the complement of X in U. The (3L), (4L), (7L) and (3H), (4H), (7H) are characteristic properties for the lower and upper approximation operations of the classical rough sets [39], and all other properties can be deduced from the six properties.

2.2. The generalized rough sets based on binary relations

In the subsection, we define some important types of binary relations.

Definition 1 ([3,38]). Let R be a binary relation on U. xRy indicates $(x,y) \in R$ and $x\bar{R}y$ indicates $(x,y) \notin R$, then

(1) R is a serial relation $\iff \forall x \in U, \exists y \in U$, such that xRy;
(2) R is a reflexive relation $\iff \forall x \in U, xRx$;
(3) R is a symmetric relation $\iff \forall x,y \in U, xRy \Rightarrow yRx$;
(4) R is a transitive relation $\iff \forall x,y,z \in U, xRy, yRz \Rightarrow xRz$;
(5) R is an Euclidean relation $\iff \forall x,y,z \in U, xRy, xRz \Rightarrow yRz$;
(6) R is a positive alliance relation $\iff \forall x,y \in U, x\bar{R}y \Rightarrow \exists z \in U$, such that xRz and $z\bar{R}y$;
(7) R is a negative alliance relation $\iff \forall x,y,z \in U, xRz, z\bar{R}y \Rightarrow x\bar{R}y$;
(8) R is an alliance relation $\iff$ R is both a positive and a negative alliance relation.

Similar to the definitions of the classical lower and upper approximation operators, we give the definitions of the corresponding generalized lower and upper approximation operators based on binary relations.

Definition 2 ([23,24]). Let R be a binary relation on the universe U. $RN(x) = \{y \in U \mid xRy\}$ is called a successor neighborhood of x. The lower and upper approximation operators on R are defined as follows:

$$L(R)(X) = \{x | RN(x) \subseteq X\},$$
$$H(R)(X) = \{x | RN(x) \cap X \neq \emptyset\}.$$

We need to point out that $RN_p(x) = \{y \in U \mid yRx\}$ is called a predecessor neighborhood of x, and the lower and upper approximation operators determined by the predecessor neighborhood are similar to those determined by the successor one. Now we consider the axiomatization of the generalized rough set based on binary relations.

Theorem 1 [33]. *Let U be a finite set, L, H be unary operators on $\mathcal{P}(U) \to \mathcal{P}(U)$. If they satisfy (9LH), then*

L satisfies (1L) $L(U) = U$, (4L) $L(X \cap Y) = L(X) \cap L(Y)$ if and only if there exists one and only one binary relation R on U, such that $L = L(R)$;

H satisfies (2H) $H(\emptyset) = \emptyset$, (4H) $H(X \cup Y) = H(X) \cup H(Y)$ if and only if there exists one and only one binary relation R on U, such that $H = H(R)$.

Theorem 2 ([34,38]). *Let U be a finite set, L, H be unary operator on $\mathcal{P}(U) \to \mathcal{P}(U)$. If they satisfy (9LH), L satisfies (1L) $L(U) = U$, (4L) $L(X \cap Y) = L(X) \cap L(Y)$, H satisfies (2H) $H(\emptyset) = \emptyset$, (4H) $H(X \cup Y) = H(X) \cup H(Y)$, then*

(1) L satisfies (2L) $L(\emptyset) = \emptyset$ if and only if there exists one and only one serial relation R on U, such that $L = L(R)$;
H satisfies (1H) $H(U) = U$ if and only if there exists one and only one serial relation R on U, such that $H = H(R)$.
(2) L satisfies (3L) $L(X) \subseteq X$ if and only if there exists one and only one reflexive relation R on U, such that $L = L(R)$;
H satisfies (3H) $X \subseteq H(X)$ if and only if there exists one and only one reflexive relation R on U, such that $H = H(R)$.
(3) L satisfies (8L) $-X \subseteq L(-L(X))$ if and only if there exists one and only one symmetric relation R on U, such that $L = L(R)$;
H satisfies (8H) $H(-H(X)) \subseteq -X$ if and only if there exists one and only one symmetric relation R on U, such that $H = H(R)$.
(4) L satisfies (5L″) $L(X) \subseteq L(L(X))$ if and only if there exists one and only one transitive relation R on U, such that $L = L(R)$;
H satisfies (5H″) $H(H(X) \subseteq H(X))$ if and only if there exists one and only one transitive relation R on U, such that $H = H(R)$.
(5) L satisfies (7L′) $L(-L(X)) \subseteq -L(X)$ if and only if there exists one and only one positive alliance relation R on U, such that $L = L(R)$;
H satisfies (7H′) $-H(X) \subseteq H(-H(X))$ if and only if there exists one and only one positive alliance relation R on U, such that $H = H(R)$.
(6) L satisfies (7L″) $-L(X) \subseteq L(-L(X))$ if and only if there exists one and only one negative alliance relation R on U, such that $L = L(R)$;
H satisfies (7H″) $H(-H(X)) \subseteq -H(X)$ if and only if there exists one and only one negative alliance relation R on U, such that $H = H(R)$.

3. The correlations among some arbitrary binary relations

Now we consider the correlations among above binary relations.

3.1. Serial binary relations

For the sake of convenience, "Reflexive $\Rightarrow$ serial" stands for "R is a reflexive binary relation $\Rightarrow$ R is a serial binary relation", etc. Then, we consider the connection between the serial binary relations and the other binary relations to construct the other forms of equivalence relations.

First, we give the following connection between serial relations and equivalence relations.

Proposition 2 ([3,8]). *Equivalence relation $\iff$ serial, symmetric, and transitive. Next, we consider the connection between serial relations and positive alliance relations.*

Lemma 1.1.

(1) *Serial and transitive $\Rightarrow$ positive alliance;*
(2) *positive alliance $\Rightarrow$ serial.*

Proof.

(1) If R is a serial and transitive relation, suppose $x\overline{R}y$, then $\exists z \in U$ such that xRz. If zRy, then xRy holds by the transitive relation. This is contradictory to the hypothesis. Thus the conclusion holds.
(2) Suppose R is a positive alliance relation, $\forall x \in U$, if $\exists y \in U$ such that xRy, then the conclusion holds. Otherwise, $\forall z \in U, x\overline{R}z$ holds. Consequently, $\forall x\overline{R}z$, there does not exist $h \in U$ such that xRh and $h\overline{R}z$. This is contradictory to the definition of the positive alliance. Thus the conclusion holds. □

According to Lemma 1.1, serial and transitive relations hold if and only if positive alliances and transitive relations hold. In other words, serial relation can be interchanged with positive alliance relations in Proposition 2. Furthermore, we obtain the following new form of equivalence relations.

Proposition 3. *Equivalence relation $\iff$ positive alliance, symmetric, and transitive.*

3.2. Positive alliance relations

Starting from the positive alliance relation, we try to create the other forms of equivalence relations via the connection between positive alliance relations and the other binary relations. We now investigate the connection between positive alliance relations and the other relations.

Lemma 2.1.

(1) *Reflexive $\Rightarrow$ positive alliance;*
(2) *symmetric and positive alliance $\Rightarrow$ reflexive.*

Proof.

(1) Since R is a reflexive relation, if $x\overline{R}y$, then there is x such that xRx and $x\overline{R}y$. Thus R is a positive alliance relation from Definition 1.
(2) Suppose that R does not satisfy reflexive, then $\exists x \in U$ such that $x\overline{R}x$. Since R is a positive alliance relation, we have $\exists z, xRz, z\overline{R}x$. It is contradictory to the symmetry of the relation. Thus the conclusion holds. □

In accordance with Lemma 2.1, a relation is reflexive and symmetric if and only if it is a positive alliance and symmetric relation. In other words, reflexive can be interchanged with positive alliance by the definition of equivalence relations.

3.3. Negative alliance relations

Starting from the negative alliance relation, we try to create the other forms of equivalence relations via the connection between negative alliance relations and the other binary relations.

Now we investigate the connection between negative alliance relations and the other relations. First, the connection between negative alliance relation and reflexive, symmetric or transitive relation is given.

Lemma 3.1.

(1) *Reflexive and negative alliance* $\Rightarrow$ *symmetric;*
(2) *symmetric and transitive* $\Rightarrow$ *negative alliance;*
(3) *symmetric and negative alliance* $\Rightarrow$ *transitive.*

Proof.

(1) If R is a reflexive and negative alliance relation, then $xRy \Rightarrow yRx$. Otherwise, there are xRy, and $y\overline{R}x$, namely, $x\overline{R}x$ holds. It is contradictory to the reflexivity of the relation. Thus the conclusion holds.
(2) If R is a symmetric and transitive relation, then xRy and $y\overline{R}z \Rightarrow x\overline{R}z$. Otherwise, if xRz holds, whereas yRx holds from xRy and symmetric relation, thus yRx and $xRz \Rightarrow yRz$ from transitive relation. It is contradictory to $y\overline{R}z$. So the conclusion holds.
(3) If R is a symmetric and negative alliance relation, then xRy and $yRz \Rightarrow xRz$. Otherwise, if $x\overline{R}z$ holds, whereas yRx holds from xRy and symmetric relation, thus yRx and $x\overline{R}z \Rightarrow y\overline{R}z$ from negative alliance relation. It is contradictory to yRz. So the conclusion holds. □

According to Lemma 3.1 and Proposition 3, a relation is symmetric and transitive if and only if it is a symmetric and negative alliance relation. Now we have the following new form of equivalence relations.

Proposition 4. *Equivalence relation* $\iff$ *positive alliance, symmetric, and negative alliance* $\iff$ *alliance and symmetric.*

Moreover, by Proposition 4, a relation is a reflexive and negative alliance relation if and only if it is a reflexive, symmetric and transitive relation. Thus, we obtain the following new form of equivalence relations.

Proposition 5. *Equivalence relation* $\iff$ *reflexive and negative alliance.*

3.4. Euclidean binary relations

Starting from the Euclidean relation, we try to create the other forms of equivalence relations via the connection between Euclidean relations and the other binary relations.

Euclidean relation is inverse negative proposition of negative alliance relation by Definition 1. In fact, the two kinds of binary relations are equivalent.

Proposition 6. *Euclidean* $\iff$ *negative alliance.*

Thus, we replace negative alliance relation with Euclidean relation in above all of the lemmas and theorems. The corresponding conclusions hold.

So far, we get the following some important equivalence forms of these binary relations.

Proposition 7.

(1) *Reflexive and symmetric* $\iff$ *positive alliance and symmetric;*
(2) *serial and transitive* $\iff$ *positive alliance and transitive;*
(3) *symmetric and transitive* $\iff$ *symmetric and negative alliance;*
(4) *Euclidean* $\iff$ *negative alliance.*

In summary, we get the following forms of equivalence relations.

Proposition 8.

(1) *The equivalence relation* $\Longleftrightarrow$ *reflexive, symmetric, and transitive;*
(2) *the equivalence relation* $\Longleftrightarrow$ *serial, symmetric, and transitive;*
(3) *the equivalence relation* $\Longleftrightarrow$ *positive alliance, symmetric, and transitive;*
(4) *the equivalence relation* $\Longleftrightarrow$ *serial, symmetric, and Euclidean;*
(5) *the equivalence relation* $\Longleftrightarrow$ *serial, symmetric, and negative alliance;*
(6) *the equivalence relation* $\Longleftrightarrow$ *reflexive and negative alliance;*
(7) *the equivalence relation* $\Longleftrightarrow$ *reflexive and Euclidean;*
(8) *the equivalence relation* $\Longleftrightarrow$ *alliance and symmetric;*
(9) *the equivalence relation* $\Longleftrightarrow$ *positive alliance, symmetric, and Euclidean.*

Proposition 8 puts up different forms of equivalence relations. In other words, we have a lot of different characterizations of the definition of equivalence relations. Among them, (1), (2), (4), (7) and related conclusions in Proposition 8 have also been discussed in [3,8] from the modal logic viewpoint.

4. The minimal axiom sets of the classical rough set

In this section, we recall the existing axiomatizations of the classical rough set and present the new minimal axiom sets of rough sets.

4.1. The existing axiom sets of rough sets

Many scholars have focused on the research of axiomatization of rough sets to grasp their more profound essence.
Lin et al. [13] proposed the following axiom set of lower and upper approximation operators.

(A1) $H(\emptyset) = \emptyset$;
(A2) $L(X) \subseteq X$;
(A3) $H(X \cup Y) = H(X) \cup H(Y)$;
(A4) $L(X) = L(L(X))$;
(A5) $H(X) = -L(-X)$;
(A6) $L(X) = H(L(X))$.

Zhu [39] got a simpler axiom set (B1–B3) by removing the redundancy from the axiom set (A1–A6) with the aid of his further investigation on the axiomatization of rough set as follows:

(B1) $L(X) \subseteq X$;
(B2) $L(X \cap Y) = L(X) \cap L(Y)$;
(B3) $-L(X) \subseteq L(-L(X))$.

He also presented the following axiom set (C1–C3) of the classical rough set. The axiom set (A1–A6) is mutually independent, as well as the axiom sets (B1–B3) and (C1–C3).

(C1) $L(U) = U$;
(C2) $L(X) \subseteq X$;
(C3) $L(L(X) \cup Y) = L(X) \cup L(Y)$.

Sun [25] had the more reasonable definition of minimal axiom set for rough sets because the definition can remove more redundancies for axiom sets.

Definition 3. Rough set axiom formulas are defined as follows:

(1) for any set $X \subseteq U, X$ is a rough set axiom formula;
(2) if α is a rough set axiom formula, so are $-\alpha, L(\alpha)$ and $H(\alpha)$;
(3) if α, β are rough set axiom formulas, so are $\alpha \cup \beta$ and $\alpha \cap \beta$;
(4) the only rough set axiom formulas are those obtainable by finite application of (1)–(3) in the above.

Definition 4. If α, β are rough set axiom formulas, then $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$ are rough set inclusions.

Definition 5. The rough set axiom set satisfying the following conditions is called minimal rough set axiom set:

(1) each axiom in the axiom set is a rough set inclusion;
(2) each axiom in the axiom set is independent of others.

We will investigate the axiomatic characterizations of rough sets by means of these rough set inclusions.

Sun got a simpler axiom set (D1-D3) from axiom set (B1-B3) by the above definitions, and the axioms are mutually independent.

(D1) $L(X) \subseteq X$;
(D2) $L(X \cap Y) \subseteq L(X) \cap L(Y)$;
(D3) $-L(X) \subseteq L(-L(X))$.

Sun also got a minimal axiom set (E1-E3) by adjusting and simplifying the axiom set (C1-C3). The axioms are mutually independent.

(E1) $L(X) \subseteq X$;
(E2) $L(X) \cup L(Y) \subseteq L(L(X) \cup Y)$;
(E3) $-L(X) \subseteq L(-L(X))$.

Sun have presented the following axiom set (F1-F3), too. The axioms are mutually independent.

(F1) $L(X) \subseteq X$;
(F2) $L(-X \cup Y) \subseteq -L(X) \cup L(Y)$;
(F3) $-L(X) \subseteq L(-L(X))$.

Dai [6] have presented the following minimal axiom set (G1-G4) of the classical rough set, and the axioms are mutually independent.

(G1) $L(-X \cup Y) \subseteq -L(X) \cup L(Y)$;
(G2) $L(X) \subseteq -L(-X)$;
(G3) $-X \subseteq L(-L(X))$;
(G4) $L(X) \subseteq L(L(X))$.

Yao [33,34] offered the axiom sets of generalized rough set based on a variety of binary relations (serial, symmetric, transitive, and Euclidean binary relation) from the perspective of rough set algebras. Zhu [38] proposed the definitions of positive alliance, negative alliance binary relation, and discussed corresponding axiomatic characterizations of rough sets. In addition, Zhu [41] pointed out that the binary relation bounded by corresponding axiom sets is the only one by displaying the proof.

4.2. Minimal axiom sets of the classic rough set

From Propositions 7 and 8, Euclidean $\iff$ negative alliance, we can replace Euclidean with negative alliance. Thus we have got the six different forms of equivalence relations:

(1) reflexive, symmetric, and transitive;
(2) serial, symmetric, and transitive;
(3) positive alliance, symmetric, and transitive;
(5) serial, symmetric, and negative alliance;
(6) reflexive and negative alliance;
(8) alliance and symmetric.

Obviously, we can obtain the axiom sets (D1–D3), (E1–E3), and (F1–F3) from the form (6). Therefore, we will only take account of the forms (1), (2), (3), (5), and (8). Due to the duality property of the lower and upper approximation operators, this paper only discusses the axiomatic characterizations of the lower approximation operator.

Above all, we conclude the following important results.

Proposition 9. *Let U be a finite set, L be a unary operator on* $\mathcal{P}(U) \to \mathcal{P}(U)$. *If (8L)* $-X \subseteq L(-L(X))$, *(4L′)* $L(X \cap Y) \subseteq L(X) \cap L(Y)$ *hold, then (4L″)* $L(X) \cap L(Y) \subseteq L(X \cap Y)$ *holds, too.*

Proof. Suppose (8L) and (4L′) hold, but (4L″) does not hold, then $\exists Z \subseteq U$ such that $Z \subseteq L(X) \cap L(Y)$ and $Z \neg \subseteq L(X \cap Y)$. This will lead to a contradictory conclusion.

Because $Z \subseteq L(X) \cap L(Y), Z \subseteq L(X)$, namely, $-L(X) \subseteq -Z$, thus $L(-L(X)) \subseteq L(-Z)$ holds according to the monotonicity from (4L′). Therefore, $-X \subseteq L(-Z)$ holds by (8L)$-X \subseteq L(-L(X))$. In the same way, we have $-Y \subseteq L(-Z)$. Finally, both $-X \subseteq L(-Z)$ and $-Y \subseteq L(-Z)$ hold. Thus $-X \cup -Y \subseteq L(-Z), -(X \cap Y) \subseteq L(-Z)$. As a result, $X \cap Y \supseteq -L(-Z)$ and $L(X \cap Y) \supseteq L(-L(-Z))$, according to the monotonicity. By (8L), we have $Z \subseteq L(X \cap Y)$. This is contradictory to the hypothesis, thus the conclusion holds. □

Theorem 3. *The following axiom set (called RST) is a minimal axiom set of the classical rough set.*

(4L′) $L(X \cap Y) \subseteq L(X) \cap L(Y)$;
(3L) $L(X) \subseteq X$;
(5L″) $L(X) \subseteq L(L(X))$;
(8L) $-X \subseteq L(-L(X))$.

Proof. First, we prove its dependability, namely, the above inclusions are the axiom sets of the classical rough set.

From Theorems 1 and 2, and Proposition 8, we get following conclusions: (1L) and (4L) in Proposition 1 correspond to an arbitrary binary relation; (3L) in Proposition 1 corresponds to a reflexive binary relation; (8L) in Proposition 1 corresponds to a symmetric binary relation and (5L″) in Proposition 1 corresponds to a transitive binary relation. They form an axiom set of the classical rough set. We use (4L′) in place of (4L) from Proposition 9 (the below is same). Therefore, we just show that the above four axioms imply (1L).

$L(\emptyset) = \emptyset$ obviously holds from (3L). So we replace X with U in (8L), which implies $U \subseteq L(U)$, namely, (1L) holds.

Then, we prove the minimality of axiom set (RST). Definition 5(1) is evident, thereafter, we illustrate (2) that the axioms are mutually independent. (4L′), (5L″), (8L) $\nRightarrow$ (3L) is given by (a) of Table 5 in Appendix; (3L), (4L′), (8L) $\nRightarrow$ (5L″) is given by (a) of Table 7 in Appendix; (3L), (4L′), (5L″) $\nRightarrow$ (8L) is given by (a) of Table 4 in Appendix; (3L), (5L″), (8L) $\nRightarrow$ (4L′) is given by (a) of Table 8 in Appendix. □

We have shown that the above axiom set induces all properties of the classical lower approximation operator, which is also one of important characterizations of the axiom set. For instance, (3L) and (5L″) in Theorem 3 imply $L(X) = L(L(X))$, because (3L) $L(X) \subseteq X$ implies $L(L(X)) \subseteq L(X)$, together with (5L″) $L(X) \subseteq L(L(X))$, thus $L(X) \subseteq L(L(X)) \subseteq L(X)$. Namely, $L(X) = L(L(X))$.

Theorem 4. *The following axiom set (called SST) is a minimal axiom set of the classical rough set.*

(2L) $L(\emptyset) = \emptyset$;
(4L′) $L(X \cap Y) \subseteq L(X) \cap L(Y)$;
(5L″) $L(X) \subseteq L(L(X))$;
(8L) $-X \subseteq L(-L(X))$.

Proof. First, we prove its dependability.

From Theorems 1 and 2, and Proposition 8, we get following conclusions: (1L) and (4L) in Proposition 1 correspond to an arbitrary binary relation, (2L) in Proposition 1 corresponds to a serial binary relation, (8L) in Proposition 1 corresponds to a symmetric binary relation and (5L″) in Proposition 1 corresponds to a transitive binary relation. They make an axiom set of the classical rough set. We use (4L′) in place of (4L). Therefore, we just show the above four axioms imply (1L).

From (2L) $L(\emptyset) = \emptyset$, we replace X with U in (8L). This implies $U \subseteq L(U)$, namely, (1L) holds.

Then, we prove the minimality of axiom set (SST). Since $L(\emptyset) = \emptyset \Longleftrightarrow L(\emptyset) \subseteq \emptyset$, Definition 5(1) is evident(the below is same), then we illustrate (2) that the axioms are mutually independent. (4L′), (5L″), (8L) $\nRightarrow$ (2L) is given by (b) of Table 5 in Appendix; (2L), (4L′), (8L) $\nRightarrow$ (5L″) is given by (a) of Table 2 in Appendix; (2L), (4L′), (5L″) $\nRightarrow$ (8L) is given by (b) of Table 4 in Appendix; (2L), (5L″), (8L) $\nRightarrow$ (4L′) is given by (b) of Table 8 in Appendix. □

Theorem 5. *The following axiom set (called PST) is a minimal axiom set of the classical rough set.*

(4L′) $L(X \cap Y) \subseteq L(X) \cap L(Y)$;
(5L″) $L(X) \subseteq L(L(X))$;
(7L′) $L(-L(X)) \subseteq -L(X)$;
(8L) $-X \subseteq L(-L(X))$.

Proof. First, we prove its dependability.

From Theorems 1 and 2, and Proposition 8, we get following conclusions: (1L) and (4L) in Proposition 1 correspond to an arbitrary binary relation, (7L′) in Proposition 1 corresponds to a positive alliance binary relation, (8L) in Proposition 1 corresponds to a symmetric binary relation and (5L″) in Proposition 1 corresponds to a transitive binary relation. They create an axiom set of the classical rough set. We use (4L′) in place of (4L). Therefore, we just show the above four axioms imply (1L).(7L′) $L(-L(X)) \subseteq -L(X)$ and (8L) $-X \subseteq L(-L(X))$ imply $-X \subseteq -L(X)$, namely, $L(X) \subseteq X$. Therefore, (2L) $L(\emptyset) = \emptyset$ holds.

Then, we prove the minimality of axiom set (PST). Definition 5(1) is evident, then we illustrate (2) that the axioms are mutually independent. (4L′), (5L″), (8L) $\not\Rightarrow$ (7L′) is given by (a) of Table 3 in Appendix; (4L′), (7L′), (8L) $\not\Rightarrow$ (5L″) is given by (b) of Table 7 in Appendix; (4L′), (5L″), (7L′) $\not\Rightarrow$ (8L) is given by (c) of Table 4 in Appendix; (5L″), (7L′), (8L) $\not\Rightarrow$ (4L′) is given by (a) of Table 6 in Appendix. □

Theorem 6. *The following axiom set (called SSN) is a minimal axiom set of the classical rough set.*

(2L) $L(\emptyset) = \emptyset$;
(4L′) $L(X \cap Y) \subseteq L(X) \cap L(Y)$;
(7L″) $-L(X) \subseteq L(-L(X))$;
(8L) $-X \subseteq L(-L(X))$.

Proof. First, we prove its dependability.

From Theorems 1 and 2, and Proposition 8, we get following conclusions: (1L) and (4L) in Proposition 1 correspond to an arbitrary binary relation, (2L) in Proposition 1 corresponds to a serial binary relation, (8L) in Proposition 1 corresponds to a symmetric binary relation and (7L″) in Proposition 1 corresponds to a negative alliance binary relation. They shape an axiom set of the classical rough set. We use (4L′) in place of (4L). Therefore, we just show the above four axioms imply (1L).

From (2L) $L(\emptyset) = \emptyset$, we replace X with U in (8L). This implies $U \subseteq L(U)$, namely, (1L) holds.

Then, we prove the minimality of axiom set (SSN). Definition 5(1) is evident, so we illustrate (2) that the axioms are mutually independent. (4L′), (7L″), (8L) $\not\Rightarrow$ (2L) is given by (c) of Table 5 in Appendix; (2L), (4L′), (8L) $\not\Rightarrow$ (7L″) is given by (b) of Table 2 in Appendix; (2L), (4L′), (7L″) $\not\Rightarrow$ (8L) is given by (a) of Table 1 in Appendix; (2L), (7L″), (8L) $\not\Rightarrow$ (4L′) is given by (c) of Table 8 in Appendix. □

Theorem 7. *The following axiom set (called SPN) is a minimal axiom set of the classical rough set.*

(4L′) $L(X \cap Y) \subseteq L(X) \cap L(Y)$;
(7L′) $L(-L(X)) \subseteq -L(X)$;
(7L″) $-L(X) \subseteq L(-L(X))$;
(8L) $-X \subseteq L(-L(X))$.

Proof. First, we prove its dependability.

From Theorems 1 and 2, and Proposition 8, we get following conclusions: (1L) and (4L) in Proposition 1 correspond to an arbitrary binary relation, (7L′) in Proposition 1 corresponds to a positive alliance binary relation, (7L″) in Proposition 1 corresponds to a negative alliance binary relation and (8L) in Proposition 1 corresponds to a symmetric binary relation. They create an axiom set of the classical rough set. We use (4L′) in place of (4L). Therefore, we just show the above four axioms imply (1L).(7L′) $L(-L(X)) \subseteq -L(X)$ and (7L″) $-L(X) \subseteq L(-L(X))$ imply $L(-L(X)) = -L(X)$, thus $-X \subseteq -L(X)$ holds from (8L), namely $L(X) \subseteq X$, therefore (2L) $L(\emptyset) = \emptyset$ holds. We replace X with U in (8L), which implies $U \subseteq L(U)$, namely, (1L) holds.

Then, we prove the minimality of axiom set (SPN). Definition 5(1) is evident, thereafter, we illustrate (2) that the axioms are mutually independent. (4L′), (7L″), (8L) $\not\Rightarrow$ (7L′) is given by (b) of Table 3 in Appendix; (4L′), (7L′), (8L) $\not\Rightarrow$ (7L″) is given by (c) of Table 7 in Appendix; (4L′), (7L′), (7L″) $\not\Rightarrow$ (8L) is given by (b) of Table 1 in Appendix; (7L′), (7L″), (8L) $\not\Rightarrow$ (4L′) is given by (b) of Table 6 in Appendix. □

Proposition 10. [34]. *Let U be a finite set, L be a unary operator on* $\mathcal{P}(U) \to \mathcal{P}(U)$. *Then,*

(1) *(4L′)* $L(X \cap Y) \subseteq L(X) \cap L(Y) \iff$ *(9L)* $L(X) \cup L(Y)) \subseteq L(X \cup Y)$;
(2) *If (1L)* $L(U) = U$ *holds, then (4L′)* $L(X \cap Y) \subseteq L(X) \cap L(Y) \iff$ *(10L)* $L(-X \cup Y) \subseteq -L(X) \cup L(Y)$.

Proposition 10 also shows:

(1) *(4L′)* $L(X\cap Y)\neg\subseteq L(X)\cap L(Y) \iff$ *(9L)* $L(X)\cup L(Y))\neg\subseteq L(X\cup Y)$;
(2) *If (1L)* $L(U)=U$ *holds, then (4L′)* $L(X\cap Y)\neg\subseteq L(X)\cap L(Y) \iff$ *(10L)* $L(-X\cup Y)\neg\subseteq -L(X)\cup L(Y)$.

Obviously, (1L) $L(U)=U$ *always holds in the above Theorems 3–7. So, we can replace (4L′) with (9L) or (10L), thus the other ten minimal axiom sets of the classical rough set will be obtained.*

Theorem 8. *The following axiom sets are all minimal axiom set of the classical rough set.*

(RST9L) (9L) $L(X)\cup L(Y))\subseteq L(X\cup Y)$;
(3L) $L(X)\subseteq X$;
(5L″) $L(X)\subseteq L(L(X))$;
(8L) $-X\subseteq L(-L(X))$.
(SST9L) (9L) $L(X)\cup L(Y))\subseteq L(X\cup Y)$;
(2L) $L(\emptyset)=\emptyset$;
(5L″) $L(X)\subseteq L(L(X))$;
(8L) $-X\subseteq L(-L(X))$.
(PST9L) (9L) $L(X)\cup L(Y))\subseteq L(X\cup Y)$;
(7L′) $L(-L(X))\subseteq -L(X)$;
(5L″) $L(X)\subseteq L(L(X))$;
(8L) $-X\subseteq L(-L(X))$.
(SSN9L) (9L) $L(X)\cup L(Y))\subseteq L(X\cup Y)$;
(2L) $L(\emptyset)=\emptyset$;
(7L″) $-L(X)\subseteq L(-L(X))$;
(8L) $-X\subseteq L(-L(X))$.
(SPN9L) (9L) $L(X)\cup L(Y))\subseteq L(X\cup Y)$;
(7L′) $L(-L(X))\subseteq -L(X)$;
(7L″) $-L(X)\subseteq L(-L(X))$;
(8L)$-X\subseteq L(-L(X))$.
(RST10L) (10L) $L(-X\cup Y)\subseteq -L(X)\cup L(Y)$;
(3L) $L(X)\subseteq X$;
(5L″) $L(X)\subseteq L(L(X))$;
(8L) $-X\subseteq L(-L(X))$.
(SST10L) (10L) $L(-X\cup Y)\subseteq -L(X)\cup L(Y)$;
(2L) $L(\emptyset)=\emptyset$;
(5L″) $L(X)\subseteq L(L(X))$;
(8L) $-X\subseteq L(-L(X))$.
(PST10L) (10L) $L(-X\cup Y)\subseteq -L(X)\cup L(Y)$;
(7L′) $L(-L(X))\subseteq -L(X)$;
(5L″) $L(X)\subseteq L(L(X))$;
(8L) $-X\subseteq L(-L(X))$.
(SSN10L) (10L) $L(-X\cup Y)\subseteq -L(X)\cup L(Y)$;
(2L) $L(\emptyset)=\emptyset$;
(7L″) $-L(X)\subseteq L(-L(X))$;
(8L) $-X\subseteq L(-L(X))$.
(SPN10L) (10L) $L(-X\cup Y)\subseteq -L(X)\cup L(Y)$;
(7L′) $L(-L(X))\subseteq -L(X)$;
(7L″) $-L(X)\subseteq L(-L(X))$;
(8L)$-X\subseteq L(-L(X))$.

Table 1
(a) (2L)(4L′)(7L″) $\not\Rightarrow$ (8L); (b) (4L′)(7L′)(7L″) $\not\Rightarrow$ (8L).

		(7L″)	(8L)	(2L)	Y (4L′)$L(X\cap Y)\subseteq L(X)\cap L(Y)$			
X	L(X)	$-L(X)\subseteq L(-L(X))$	$-X\not\subseteq L(-L(X))$	$L(\emptyset)=\emptyset$	U	$\{a\}$	$\{b\}$	$\emptyset$
U	U	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	*	$U=U$	$U=U$	$\emptyset=\emptyset$	$\emptyset=\emptyset$
$\{a\}$	U	$\emptyset\subseteq\emptyset$	$\{b\}\not\subseteq\emptyset$	*	$U=U$	$U=U$	$\emptyset=\emptyset$	$\emptyset=\emptyset$
$\{b\}$	$\emptyset$	$U\subseteq U$	$\{a\}\subseteq U$	*	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$
$\emptyset$	$\emptyset$	$U\subseteq U$	$U\subseteq U$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$

Table 2
(a) (2L)(4L)(8L) $\nRightarrow$ (5L″); (b) (2L)(4L′)(8L) $\nRightarrow$ (7L″).

	(5L″)	(7L″)	(8L)	(2L)	Y (4L′)$L(X \cap Y) \subseteq L(X) \cap L(Y)$			
X	$L(X) \not\subseteq L(L(X))$	$-L(X) \not\subseteq L(-L(X))$	$-X \subseteq L(-L(X))$	$L(\emptyset) = \emptyset$	U	$\{a\}$	$\{b\}$	$\emptyset$
U	$U \subseteq U$	$\emptyset \subseteq \emptyset$	$\emptyset \subseteq \emptyset$	$*$	$U = U$	$\{b\} = \{b\}$	$\{a\} = \{a\}$	$\emptyset = \emptyset$
$\{a\}$	$\{b\} \not\subseteq \{a\}$	$\{a\} \not\subseteq \{b\}$	$\{b\} \subseteq \{b\}$	$*$	$\{b\} = \{b\}$	$\{b\} = \{b\}$	$\emptyset = \emptyset$	$\emptyset = \emptyset$
$\{b\}$	$\{a\} \not\subseteq \{b\}$	$\{b\} \not\subseteq \{a\}$	$\{a\} \subseteq \{a\}$	$*$	$\{a\} = \{a\}$	$\emptyset = \emptyset$	$\{a\} = \{a\}$	$\emptyset = \emptyset$
$\emptyset$	$\emptyset \subseteq \emptyset$	$U \subseteq U$	$U \subseteq U$	$\emptyset = \emptyset$	$\emptyset = \emptyset$	$\emptyset = \emptyset$	$\emptyset = \emptyset$	$\emptyset = \emptyset$

Table 3
(a) (4L′)(5L″)(8L) $\nRightarrow$ (7L′); (b) (4L′)(7L″)(8L) $\nRightarrow$ (7L′).

	(5L″)	(7L′)	(7L″)	(8L)	Y (4L′)$L(X \cap Y) \subseteq L(X) \cap L(Y)$			
X	$L(X) \subseteq L(L(X))$	$L(-L(X)) \not\subseteq -L(X)$	$-L(X) \subseteq L(-L(X))$	$-X \subseteq L(-L(X))$	U	$\{a\}$	$\{b\}$	$\emptyset$
U	$U \subseteq U$	$U \not\subseteq \emptyset$	$\emptyset \subseteq U$	$\emptyset \subseteq U$	$U = U$	$U = U$	$U = U$	$U = U$
$\{a\}$	$U \subseteq U$	$U \not\subseteq \emptyset$	$\emptyset \subseteq U$	$\{b\} \subseteq U$	$U = U$	$U = U$	$U = U$	$U = U$
$\{b\}$	$U \subseteq U$	$U \not\subseteq \emptyset$	$\emptyset \subseteq U$	$\{a\} \subseteq U$	$U = U$	$U = U$	$U = U$	$U = U$
$\emptyset$	$U \subseteq U$	$U \not\subseteq \emptyset$	$\emptyset \subseteq U$	$U \subseteq U$	$U = U$	$U = U$	$U = U$	$U = U$

Table 4
(a) (3L)(4L′)(5L″) $\nRightarrow$ (8L); (b) (2L)(4L′)(5L″) $\nRightarrow$ (8L); (c) (4L′)(5L″)(7L′) $\nRightarrow$ (8L).

	(3L)	(5L″)	(7L′)	(8L)	(2L)	Y (4L′)$L(X \cap Y) \subseteq L(X) \cap L(Y)$			
X	$L(X) \subseteq X$	$L(X) \subseteq L(L(X))$	$L(-L(X)) \subseteq -L(X)$	$-X \not\subseteq L(-L(X))$	$L(\emptyset) = \emptyset$	U	$\{a\}$	$\{b\}$	$\emptyset$
U	$U \subseteq U$	$U \subseteq U$	$\emptyset \subseteq \emptyset$	$\emptyset \subseteq \emptyset$	$*$	$U = U$	$\{a\} = \{a\}$	$\emptyset = \emptyset$	$\emptyset = \emptyset$
$\{a\}$	$\{a\} \subseteq \{a\}$	$\{a\} \subseteq \{a\}$	$\emptyset \subseteq \{b\}$	$\{b\} \not\subseteq \emptyset$	$*$	$\{a\} = \{a\}$	$\{a\} = \{a\}$	$\emptyset = \emptyset$	$\emptyset = \emptyset$
$\{b\}$	$\emptyset \subseteq \{b\}$	$\emptyset \subseteq \emptyset$	$U \subseteq U$	$\{a\} \subseteq U$	$*$	$\emptyset = \emptyset$	$\emptyset = \emptyset$	$\emptyset = \emptyset$	$\emptyset = \emptyset$
$\emptyset$	$\emptyset \subseteq \emptyset$	$\emptyset \subseteq \emptyset$	$U \subseteq U$	$U \subseteq U$	$\emptyset = \emptyset$	$\emptyset = \emptyset$	$\emptyset = \emptyset$	$\emptyset = \emptyset$	$\emptyset = \emptyset$

Table 5
(a) (4L′)(5L″)(8L) $\nRightarrow$ (3L); (b) (4L′)(5L″)(8L) $\nRightarrow$ (2L); (c) (4L′)(7L″)(8L) $\nRightarrow$ (2L).

	(5L″)	(7L″)	(8L)	(3L)	(2L)	Y (4L′)$L(X \cap Y) \subseteq L(X) \cap L(Y)$			
X	$L(X) \subseteq L(L(X))$	$-L(X) \subseteq L(-L(X))$	$-X \subseteq L(-L(X))$	$L(X) \not\subseteq X$	$L(\emptyset) \neq \emptyset$	U	$\{a\}$	$\{b\}$	$\emptyset$
U	$U \subseteq U$	$\emptyset \subseteq \{a\}$	$\emptyset \subseteq \{a\}$	$U \subseteq U$	$*$	$U = U$	$\{a\} = \{a\}$	$U = U$	$\{a\} = \{a\}$
$\{a\}$	$\{a\} \subseteq \{a\}$	$\{b\} \subseteq U$	$\{b\} \subseteq U$	$\{a\} \subseteq \{a\}$	$*$	$\{a\} = \{a\}$	$\{a\} = \{a\}$	$\{a\} = \{a\}$	$\{a\} = \{a\}$
$\{b\}$	$U \subseteq U$	$\emptyset \subseteq \{a\}$	$\{a\} \subseteq \{a\}$	$U \not\subseteq \{b\}$	$*$	$U = U$	$\{a\} = \{a\}$	$U = U$	$\{a\} = \{a\}$
$\emptyset$	$\{a\} \subseteq \{a\}$	$\{b\} \subseteq U$	$U \subseteq U$	$\{a\} \not\subseteq \emptyset$	$\{a\} \neq \emptyset$	$\{a\} = \{a\}$	$\{a\} = \{a\}$	$\{a\} = \{a\}$	$\{a\} = \{a\}$

Theorem 8 also suggests that (4L′) can be changed with (9L) or (10L). In other words, we verify the unary operator L via the calculation on union of the sets. As far as Theorems 3–7 are concerned, those minimal axiom sets express the different operator properties of equivalence relations in the rough set. But (9L) and (10L) in Theorem 8 express the different operator properties of an arbitrary binary relation in the rough set.

So far, in the framework of rough set theory, some properties of an equivalence relation and unary operator on a universe are well established with each other. The meaning of Theorem 8 also lies in acquiring some relatively compact properties (axiom sets) of unary operator established by equivalence relations.

5. Conclusion

The axiomatization of rough set, based on binary relations and coverings of a universe, is one important issue. In this paper, we have investigated the correlation of arbitrary binary relations in the rough set theory. Consequently, we have acquired some new equivalent forms of equivalence relations. Furthermore, we have presented five new minimal axiom sets of the classical rough set. Combining some conclusions of Yao [34] with the five new axiom sets, we have also put forward another ten new minimal axiom sets. Finally, fifteen new minimal axiom sets were displayed. In our future work, we will further study correlation between properties of binary relations and operators. Some properties of approximation operators will be sought for generalized rough set based on a new type of binary relations. Furthermore, the connections between generalized rough sets based on binary relations and coverings will be studied.

Table 6
(a) (5L″)(7L′)(8L) $\nRightarrow$ (4L′); (b) (7L′)(7L″)(8L) $\nRightarrow$ (4L′).

	(5L″)$L(X)$	(7L′)(7L″)$-L(X)$	(8L)$-X$	Y (4L′)$L(X\cap Y)\ \not\subseteq\ L(X)\cap L(Y)$							
X	$\subseteq L(L(X))$	$= L(-L(X))$	$\subseteq L(-L(X))$	U	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\emptyset$
U	$U\subseteq U$	$\emptyset=\emptyset$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$\{a,b\}\subseteq\{a,b\}$	$\{b,c\}\subseteq\{b,c\}$	$\emptyset\subseteq\emptyset$	$\{a\}\subseteq\{a\}$	$\emptyset\subseteq\emptyset$	$\{c\}\subseteq\{c\}$	$\emptyset\subseteq\emptyset$
$\{a,b\}$	$\{a,b\}\subseteq\{a,b\}$	$\{c\}=\{c\}$	$\{c\}\subseteq\{c\}$	$\{a,b\}\subseteq\{a,b\}$	$\{a,b\}\subseteq\{a,b\}$	$\emptyset\subseteq\{b\}$	$\{a\}\ \not\subseteq\ \emptyset$	$\{a\}\subseteq\{a\}$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$
$\{b,c\}$	$\{b,c\}\subseteq\{b,c\}$	$\{a\}=\{a\}$	$\{a\}\subseteq\{a\}$	$\{b,c\}\subseteq\{b,c\}$	$\emptyset\subseteq\{b\}$	$\{b,c\}\subseteq\{b,c\}$	$\{c\}\ \not\subseteq\ \emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\{c\}\subseteq\{c\}$	$\emptyset\subseteq\emptyset$
$\{a,c\}$	$\emptyset\subseteq\emptyset$	$U=U$	$\{b\}\subseteq U$	$\emptyset\subseteq\emptyset$	$\{a\}\ \not\subseteq\ \emptyset$	$\{c\}\ \not\subseteq\ \emptyset$	$\emptyset\subseteq\emptyset$	$\{a\}\ \not\subseteq\ \emptyset$	$\emptyset\subseteq\emptyset$	$\{c\}\ \not\subseteq\ \emptyset$	$\emptyset\subseteq\emptyset$
$\{a\}$	$\{a\}\subseteq\{a\}$	$\{b,c\}=\{b,c\}$	$\{b,c\}\subseteq\{b,c\}$	$\{a\}\subseteq\{a\}$	$\{a\}\subseteq\{a\}$	$\emptyset\subseteq\emptyset$	$\{a\}\ \not\subseteq\ \emptyset$	$\{a\}\subseteq\{a\}$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$
$\{b\}$	$\emptyset\subseteq\emptyset$	$U=U$	$\{a,c\}\subseteq U$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$
$\{c\}$	$\{c\}\subseteq\{c\}$	$\{a,b\}=\{a,b\}$	$\{a,b\}\subseteq\{a,b\}$	$\{c\}\subseteq\{c\}$	$\emptyset\subseteq\emptyset$	$\{c\}\subseteq\{c\}$	$\{c\}\ \not\subseteq\ \emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\{c\}\subseteq\{c\}$	$\emptyset\subseteq\emptyset$
$\emptyset$	$\emptyset\subseteq\emptyset$	$U=U$	$U\subseteq U$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$

Table 7
(a) (3L)(4L′)(8L) $\nRightarrow$ (5L″); (b) (4L′)(7L′)(8L) $\nRightarrow$ (5L″); (c) (4L′)(7L″)(8L) $\nRightarrow$ (7L″).

	(3L)	(5L″)$L(X)$	(7L′) $L(-L(X))$	(7L″)$-L(X)$	(8L)$-X$	Y (4L′)$L(X\cap Y)\subseteq L(X)\cap L(Y)$							
X	$L(X)\nsubseteq X$	$\nsubseteq L(L(X))$	$\subseteq -L(X)$	$\nsubseteq L(-L(X))$	$\subseteq L(-L(X))$	U	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\emptyset$
U	$U\subseteq U$	$U\subseteq U$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$U=U$	$\{a\}=\{a\}$	$\{c\}=\{c\}$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$
$\{a,b\}$	$\{a\}\subseteq\{a,b\}$	$\{a\}\nsubseteq\emptyset$	$\{c\}\subseteq\{b,c\}$	$\{b,c\}\nsubseteq\{c\}$	$\{c\}\subseteq\{c\}$	$\{a\}=\{a\}$	$\{a\}=\{a\}$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$
$\{b,c\}$	$\{c\}\subseteq\{b,c\}$	$\{c\}\nsubseteq\emptyset$	$\{a\}\subseteq\{a,b\}$	$\{a,b\}\nsubseteq\{a\}$	$\{a\}\subseteq\{a\}$	$\{c\}=\{c\}$	$\emptyset=\emptyset$	$\{c\}=\{c\}$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$
$\{a,c\}$	$\emptyset\subseteq\{a,c\}$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$U\subseteq U$	$\{b\}\subseteq U$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$
$\{a\}$	$\emptyset\subseteq\{a\}$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$U\subseteq U$	$\{b,c\}\subseteq U$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$
$\{b\}$	$\emptyset\subseteq\{b\}$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$U\subseteq U$	$\{a,c\}\subseteq U$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$
$\{c\}$	$\emptyset\subseteq\{c\}$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$U\subseteq U$	$\{a,b\}\subseteq U$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$
$\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$U\subseteq U$	$U\subseteq U$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$	$\emptyset=\emptyset$

Table 8
(a) (3L)(5L″)(8L) $\nRightarrow$ (4L′); (b) (2L)(5L″)(8L) $\nRightarrow$ (4L′); (c) (2L)(7L″)(8L) $\nRightarrow$ (4L′).

	(3L)	(5L″)$L(X)$	(7L″)$-L(X)$	(8L)$-X$	(2L)	Y (4L′)$L(X\cap Y)\not\subseteq L(X)\cap L(Y)$							
X	$L(X)\subseteq X$	$\subseteq L(L(X))$	$\subseteq L(-L(X))$	$\subseteq L(-L(X))$	$L(\emptyset)$=$\emptyset$	U	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\emptyset$
U	$U\subseteq U$	$U\subseteq U$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$*$	$U\subseteq U$	$\{a,b\}\subseteq\{a,b\}$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\{c\}\subseteq\{c\}$	$\emptyset\subseteq\emptyset$
$\{a,b\}$	$\{a,b\}\subseteq\{a,b\}$	$\{a,b\}\subseteq\{a,b\}$	$\{c\}\subseteq\{c\}$	$\{c\}\subseteq\{c\}$	$*$	$\{a,b\}\subseteq\{a,b\}$	$\{a,b\}\subseteq\{a,b\}$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$
$\{b,c\}$	$\emptyset\subseteq\{b,c\}$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$\{a\}\subseteq U$	$*$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\{c\}\not\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\{c\}\not\subseteq\emptyset$	$\emptyset\subseteq\emptyset$
$\{a,c\}$	$\emptyset\subseteq\{a,c\}$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$\{b\}\subseteq U$	$*$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\{c\}\not\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\{c\}\not\subseteq\emptyset$	$\emptyset\subseteq\emptyset$
$\{a\}$	$\emptyset\subseteq\{a\}$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$\{b,c\}\subseteq U$	$*$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$
$\{b\}$	$\emptyset\subseteq\{b\}$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$\{a,c\}\subseteq U$	$*$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$
$\{c\}$	$\{c\}\subseteq\{c\}$	$\{c\}\subseteq\{c\}$	$\{a,b\}\subseteq\{a,b\}$	$\{a,b\}\subseteq\{a,b\}$	$*$	$\{c\}\subseteq\{c\}$	$\emptyset\subseteq\emptyset$	$\{c\}\not\subseteq\emptyset$	$\{c\}\not\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\{c\}\subseteq\{c\}$	$\emptyset\subseteq\emptyset$
$\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$U\subseteq U$	$U\subseteq U$	$\emptyset=\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$	$\emptyset\subseteq\emptyset$

Acknowledgements

We are grateful to the Editors and the anonymous reviewers for their valuable comments and remarks which led to significant improvements on the paper. This work was supported by grants from the National Natural Science Foundation of China (Nos. 61170107, 61379021, 61303131 and 61300121), the Foundation of Fujian Province in China (Nos. 2013J01028, 2013J01029, 2013J01265, 2012D141, JK2013027, JA13202 and JA12223), the Foundation of Hebei Province in China (Nos. A2013208175, A2014205157, QN20131159 and Q2012093).

Appendix A

1. Because (2L)$L(\emptyset) = \emptyset \iff L(\emptyset) \subseteq \emptyset$, we regard (2L) as a rough set inequality. To replace $L(\emptyset) \subseteq \emptyset$ with (2L)$L(\emptyset) = \emptyset$ is solely for custom;
2. Form Theorem 16, when (1L)$L(U) = U$ always holds, (4L)$L(X \cap Y) = L(X) \cap L(Y) \iff$ (4L')$L(X \cap Y) \subseteq L(X) \cap L(Y)$. So in the following tables, we take (4L) as (4L′), which is solely for custom, too (Tables 1–8).

References

[1] Z. Bonikowski, E. Bryniarski, U. Wybraniec-Skardowska, Extensions and intentions in the rough set theory, Inform. Sci. 107 (1998) 149–167.
[2] G. Cattaneo, Generalized rough sets: preclusivity fuzzy-intuitionistic (BZ) lattices, Stud. Logica 58 (1997) 47–77.
[3] B.F. Chellas, Modal Logic: An Introduction, Cambridge University Press, 1980.
[4] S. Comer, An algebraic approach to the approximation of information, Fundam. Inform. 14 (1991) 492–502.
[5] S. Comer, On connections between information systems, rough sets, and algebraic logic, Algebraic Methods in Logic and Computer Science, vol. 28, Banach Center Publications, 1993, pp. 117–124.
[6] J.H. Dai, A new minimal rough set axiom set, Wuhan Univ. J. Natural Sci. 9 (6) (2004) 875–878 (in Chinese).
[7] J.H. Dai, Logic for rough sets with rough double stone algebraic semantics, in: RSFDGrC 2005, LNCS, vol. 3641, 2005, pp. 141–147.
[8] J.Y. Halpern, Y. Moses, A guide to the modal logics of knowledge and belief: preliminary draft, in: 9th Int. Joint Conf. on Artificial Intelligence, 1985, pp. 480–490.
[9] M. Kryszkiewicz, Rough set approach to incomplete information systems, Inform. Sci. 112 (1998) 39–49.
[10] M. Kryszkiewicz, Rule in incomplete information systems, Inform. Sci. 113 (1998) 271–292.
[11] E.F. Lashin, A.M. Kozae, A.A. Abo Khadra, T. Medhat, Rough set theory for topological spaces, Int. J. Approx. Reason. 40 (2005) 35–43.
[12] J.J. Li, Topological methods on the theory of covering generalized rough sets, Pattern Recogn. Artif. Intell. 17 (1) (2004) 7–10 (in Chinese).
[13] T.Y. Lin, Q. Liu, Rough approximate operators: axiomatic rough set theory, in: W. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer, Berlin, 1994, pp. 256–260.
[14] F. Min, Q.H. Liu, C.L. Fang, Rough sets approach to symbolic value partition, Int. J. Appr. Reason. 49 (2008) 689–700.
[15] Z. Pawlak, Rough sets, Int. J. Comput. Inform. Sci. 11 (1982) 341–356.
[16] Z. Pawlak, A. Skowron, Rudiments of rough sets, Inform. Sci. 177 (1) (2007) 3–27.
[17] Z. Pawlak, A. Skowron, Rough sets and Boolean reasoning, Inform. Sci. 177 (1) (2007) 41–73.
[18] Z. Pawlak, A. Skowron, Rough sets: some extensions, Inform. Sci. 177 (1) (2007) 28–40.
[19] Z. Pei, D. Pei, L. Zheng, Topology vs generalized rough sets, Int. J. Approx. Reason. 52 (2) (2011) 231–239.
[20] J.A. Pomykala, Approximation operations in approximation space, Bull. Polish Acad. Sci. Math. 35 (1987) 653–662.
[21] J.A. Pomykala, On definability in the nondeterministic information system, Bull. Polish Acad. Sci. Math. 36 (1988) 193–210.
[22] A. Skowron, J. Stepaniuk, R. Swiniarski, Approximation spaces in rough-granular computing, Fundam. Inform. 100 (2010) 141–157.
[23] A. Skowron, J. Stepaniuk, Tolerance approximation spaces, Fundam. Inform. 27 (1996) 245–253.
[24] A. Skowron, J. Stepaniuk, Generalized approximation spaces, in: Proceedings of the Third International Workshop on Rough Sets and Knowledge Discovery RSSC '94, San Jose, USA, 1994, pp. 156–163.
[25] H. Sun, D.Y. Liu, W. Li, The minimization of axiom sets of rough set, Chinese J. Comput. 25 (2) (2002) 202–209.
[26] W.Z. Wu, Y. Leung, J.S. Mi, On characterizations of (I, T)-fuzzy rough approximation operators, Fuzzy Sets Syst. 154 (2005) 76–102.
[27] U. Wybraniec-Skardowska, On a generalization of approximation space, Bull. Polish Acad. Sci. Math. 37 (1989) 51–61.
[28] L.Y. Yang, L.S. Xu, Algebraic aspects of generalized approximation spaces, Int. J. Approx. Reason. 51 (1) (2009) 151–161.
[29] T. Yang, Q.G. Li, Reduction about approximation spaces of covering generalized rough sets, Int. J. Approx. Reason. 51 (3) (2010) 335–345.
[30] X.P. Yang, T.J. Li, The minimization of axiom sets characterizing generalized approximation operators, Inform. Sci. 176 (2006) 887–889.
[31] Y.J. Yang, R.I. John, Generalisation of roughness bounds in rough set operations, Int. J. Approx. Reason. 48 (2008) 868–878.
[32] Y.Y. Yao, Probabilistic rough set approximations, Int. J. Approx. Reason. 49 (2) (2008) 255–271.
[33] Y.Y. Yao, Two views of the theory of rough sets in finite universes, Int. J. Approx. Reason. 15 (1996) 291–317.
[34] Y.Y. Yao, Constructive and algebraic methods of theory of rough sets, Inform. Sci. 109 (1998) 21–47.
[35] Y.Y. Yao, T.Y. Lin, Generalization of rough sets using modal logic, Intell. Autom. Soft Comput. Int. J. 2 (1996) 103–120.
[36] W. Zakowski, Approximations in the space (U, Π), Demonstratio Math. 16 (1983) 761–769.
[37] P. Zhu, Covering rough sets based on neighborhoods: an approach without using neighborhoods, Int. J. Approx. Reason. 52 (3) (2011) 461–472.
[38] W. Zhu, Generalized rough sets based on relations, Inform. Sci. 177 (2007) 4997–5011.
[39] F. Zhu, H.C. He, The axiomization of the rough set, Chinese J. Comput. 23 (3) (2000) 330–333.
[40] W. Zhu, F.Y. Wang, Axiomatic systems of generalized rough sets, in: Proc. First Int'l Conf. Rough Sets and Knowledge Technology, 2006, pp. 216–221.
[41] W. Zhu, F.Y. Wang, Binary relation based rough set, in: Proc. Third Int'l Conf. Fuzzy Systems and Knowledge Discovery, 2006, pp. 276–285.
[42] W. Zhu, F.Y. Wang, Reduction and axiomization of covering generalized rough sets, Inform. Sci. 152 (2003) 217–230.
[43] W. Zhu, F.Y. Wang, Relationships among three types of covering rough sets, in: IEEE GrC 2006, 2006, pp. 43–48.
[44] W. Zhu, F.Y. Wang, Properties of the first type of covering-based rough sets, in: Proceedings of DM Workshop 06, ICDM 06, Hong Kong, China, December 18, 2006, pp. 407–411.
[45] W. Zhu, F.Y. Wang, On three types of covering rough sets, IEEE Trans. Knowl. Data Eng. 19 (8) (2007) 1131–1144.
[46] W. Ziarko, Probabilistic approach to rough sets, Int. J. Approx. Reason. 49 (2008) 272–284.