Question 3

3a

From the definition we know that $\varphi(s) = \sum_{r \in R} \sum_{i=1}^{n_r(s)} d_r(i)$. We will combine this definition with the following Lemma to prove the fact stated in Rosenthall's Theorem.

Lemma:

Let δ_p be an indicator over a predicate, such that $\delta_p = 1$ if p is true, otherwise $\delta_p = 0$. For a given resource r, we will show that $\sum_{i=1}^{n} \delta_{r \in S_i} d_r(n_r^{(i)}(s)) = \sum_{i=1}^{n} d_r(i)$

Proof (Lemma):

Since we can arbitrarily reorder the players, we can position all the players using resource r, at the beginning. In other words, arrange players in such a way that $\forall i . r \in S_i \Rightarrow i \leq n_r(s)$.

Now let's suppose that $1 \leq i \leq n_r(s)$, then in this reordered setup $\delta_{r \in S_i} = 1$, and $n_r^{(i)} = i$. Moreover, we know that when $i > n_r(s)$ then $\delta_{r \in S_i} = 0$. From here we can show:

$$\sum_{i=1}^{n} \delta_{r \in S_i} \cdot d_r(n_r^{(i)}(s)) = \sum_{i=1}^{n_r(s)} \delta_{r \in S_i} \cdot d_r(n_r^{(i)}(s)) + \sum_{i=n_r(s)+1}^{n} \delta_{r \in S_i} \cdot d_r(n_r^{(i)}(s)) =$$

$$= \sum_{i=1}^{n_r(s)} d_r(i) + \sum_{i=n_r(s)+1}^{n} 0 = \sum_{i=1}^{n_r(s)} d_r(i)$$

Fact from Rosenthall's Theorem:

$$\varphi(s) = \sum_{i=1}^{n} \sum_{r \in S_i} d_r(n_r^{(i)}(s))$$

Proof (Fact):

Using the Lemma stated and proved upstairs, we can show directly that the two expressions are equivelent.

$$\sum_{i=1}^{n} \sum_{r \in S_i} d_r\left(n_r^{(i)}(s)\right) = \sum_{i=1}^{n} \sum_{r \in R} \delta_{r \in R} \cdot d_r\left(n_r^{(i)}(s)\right) =$$

$$= \sum_{r \in R} \sum_{i=1}^{n} \delta_{r \in R} \cdot d_r\left(n_r^{(i)}(s)\right) = \sum_{r \in R} \sum_{i=1}^{n_r(s)} d_r(i)$$

Quod Erat Demonstrandum ■

3b

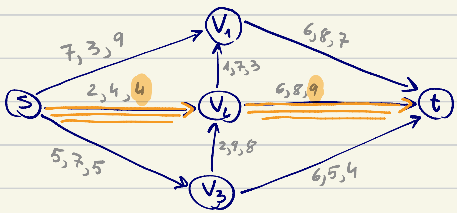
(i) By utilising the Rosenthal's Theorem we can compute a pure Nash Equilibrium for an atomic network congestion game by finding new improvments to the paths of the players, until no more improvements can be made.

As per the first step, let's define the initial strategy for each of the three players: $\pi = (\pi_1, \pi_2, \pi_3)$

STEP 1:

Let all players initially choose path: $s \to v_2 \to t$

PATH:	COST:
$\pi_1: s \to v_2 \to t$	$4+9=13$
$\pi_2: s \to v_2 \to t$	$4+9=13$
$\pi_3: s \to v_2 \to t$	$4+9=13$

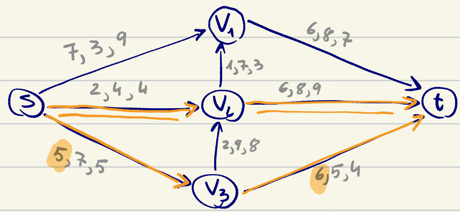


Now let's try to improve strategy π by finding another strategy $\pi' = (\pi'_1, \pi'_2, \pi'_3)$ with an improvement to π.

STEP 2:

Player 1 switches its path to go via node V_3 instead.

PATH:	COST:
π_1': $s \rightarrow V_3 \rightarrow t$	$5+6=11$
π_2': $s \rightarrow V_2 \rightarrow t$	$4+8=12$
π_3': $s \rightarrow V_2 \rightarrow t$	$4+8=12$

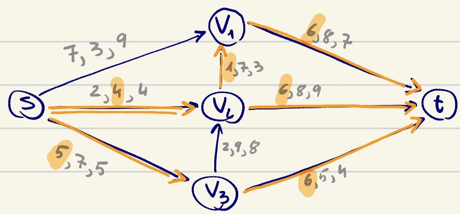


This change brings positive change!

STEP 3:

Player 2 changes his path to go via V_1.

PATH:	COST:
π_1': $s \rightarrow V_3 \rightarrow t$	$4+6=10$
π_2': $s \rightarrow V_2 \rightarrow V_1 \rightarrow t$	$4+1+6=11$
π_3': $s \rightarrow V_2 \rightarrow t$	$5+6=11$



This change brings positive change!

$\Rightarrow$ Now we don't have any alternative paths, that would improve any of the players' position. Thus by the Rosenthal's Theorem $\pi' = (\pi_1', \pi_2', \pi_3')$ listed upstairs, is a pure Nash Equilibrium.

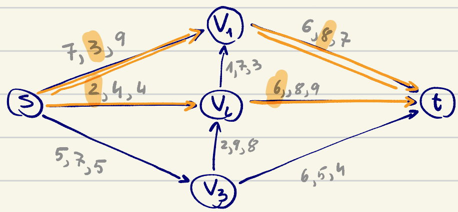
3b

ii

The above derived strategy is not Pareto optimal, even with pure strategies. We can prove this by finding a different strategy, which is at least as good for every player, $\pi^* = (\pi_1^*, \pi_2^*, \pi_3^*)$

A strategy where two players go via node V_1 and the other player goes via V_2 is one such strategy.

PATH:	COST:
π_1^* : $s \to V_2 \to t$	$2+6=8$
π_2^* : $s \to V_1 \to t$	$3+8=11$
π_3^* : $s \to V_1 \to t$	$3+8=11$



We see that $\forall i$. player : $Cost(\pi_i^*) \leq Cost(\pi_i')$ $\Rightarrow$ no player is worse off!

Since such strategy π^* exists, we have proven that strategy π' from upstairs is not Pareto optimal