Algorithmic Game Theory and Applications
Homework 1

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I have chosen Question 2 and Question 4 to be answered.

Question 2 a

In order to compute the minimax value for this game (for which we were given the payoff matrix), I decided to approach it from an optimization perspective. As we have learnt in Lecture 4, we can present minimax as an optimization problem, for which we can use the following simple algorithm:

Algorithm: Maximize v

Subject to constraints:

$$(x_1^T A)_j \geq v \text{ for } j = 1, \dots, m_2$$

$$x_1(A) + \dots + x_1(m_1) = 1$$

$$x_1(j) \geq 0 \text{ for } j = 1, \dots, m$$

which follows from the minimax theorem stating that the optimal solution (x_1^*, v^*) would give precisely the minimax value v^*, and a minmaximizer x_1^* for Player 1. And by providing slight changes to our linear constraints, and changing the aim to be minimizing for V, we will also obtain optimal solution (x_2^*, V') where, V' is precisely the minimax value, hence $V = V'$, and x_2^* is maxminimizer for Player 2.

Luckily there are powerful algorithms available for solving linear problems which is also why I decided to use Python and the Gekko library to find solutions.

The computed minimax value $V = 4.667$

Player 1 solution:

$$x_1^* = [x_1, x_2, x_3, x_4, x_5] = [0.0, 0.5, 0.0, 0.278, 0.222]$$

Player 2 solution:

$$x_2^* = [y_1, y_2, y_3, y_4, y_5] = [0.333, 0.0, 0.333, 0.0, 0.333]$$

Question 2 b

Suppose we have a symmetric 2-player zero sum game, G.
Let B be a skew-symmetric payoff matrix for player 1.

$$B = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix}$$

Now suppose there exist vectors $x' \in \mathbb{R}^n$ and $y' \in \mathbb{R}^m$, such that:

① $Ax' < b$ ③ $A^T y' > 0$

② $x' \geq 0$ ④ $y' \geq 0$

Now let's prove that for the game G, every minmaximizer strategy $w = (y^*, x^*, z)$ for Player 1, and hence every maxminimizer strategy for Player 2 since the game is symmetric, has the property that $z > 0 \Rightarrow$ the last pure strategy is played with positive probability.

$$(y^*)^T (Ax' - b) < 0 \qquad (x^*)^T (A^T y' - c) > 0$$

$$(y^*)^T Ax' - (y^*)^T b < 0 \quad \text{A} \qquad (x^*)^T A^T y' - (x^*)^T c > 0 \quad \text{B}$$

$$(y^*)^T Ax' - y^* b < (x^*)^T A^T y' - (x^*)^T c \qquad \text{according to A \& B}$$

$$(y^*)^T Ax' - (x^*)^T A^T y' < (y^*)^T b - (x^*)^T c \qquad \text{C}$$

Since we know that:

$$Bw \leq 0 \; ; \; \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix} \begin{bmatrix} y^* \\ x^* \\ z \end{bmatrix} \leq 0$$

$\Rightarrow$ using the last row we must satisfy:

$$b^T y^* - c^T x^* \leq 0$$

$$b^T y^* \leq c^T x^* \qquad \text{D}$$

From C and D we can further obtain:

$$(y^*)^T Ax' - (x^*)^T A^T y' < 0$$

So hence we can write as:

$$(y^*)^T A x' < (x^*)^T A^T y' \leq 0$$

Which is a contradiction! Since $(y^*)^T A x' \geq 0$ and $(x^*)^T A^T y' \leq 0$ since $x', y' \geq 0$

$$(y^*)^T A x' < (x^*)^T A^T y' \qquad \text{D}$$

Since the initial conditions are: $Ax' < b$, $y^* > 0$

we can deduce: $(y^*)^T A x' < (y^*)^T b$ E

and other initial conditions: $A^T y' > c$, $x^* > 0$

we can deduce: $(x^*)^T A^T y' > (x^*)^T c$ F

Now looking at D, E and F we see that we get a contradiction. Therefore it is infeasable to find a solution!

$\Rightarrow$ we also know that for: $\underbrace{b^T x^* - c^T y^* = 0}_{}$

$b^T x^* = c^T y^*$ must hold!

~~So~~ hence we can write (1) as:

$$(y^*)^T A x' < (x^*)^T A^T y' \leq 0$$

Which is a contradiction! Since $(y^*)^T A x' < 0$ and $(x^*)^T A^T y' > 0$

Question 4 a

Farkas Lemma states that a linear system of inequalities $Ax \leq b$ has a solution x if and only if there is no vector y satisfying: ① $y \geq 0$, ② $y^TA = 0$, and ③ $y^Tb < 0$.

Which can be written as:

$$Ax \leq b \text{ where } x \text{ is a solution} \iff \forall y, \neg y ; y \geq 0 \land y^TA = 0 \land y^Tb < 0$$

To prove this claim we shall use Fourier Motzkin Elimination algorithm (FME), which allows for easier elimination of variables.

As stated, $Ax \leq b$, where A is a $m \times n$ matrix, and x is a $n \times 1$ vector, and b is a $m \times 1$ vector.

Let us use the method of contradiction in this proof, and for that let's suppose that there is such y ; $y \geq 0$.

Case I:

$$Ax \leq b \land x \text{ is a solution} \Longleftarrow \forall y, y ; y \geq 0 \land y^TA = 0 \land y^Tb < 0.$$

$$Ax \leq b \quad / \cdot y^T$$

$$y^TAx \leq y^Tb$$

Which is a contradiction!

Since $y^TA = 0$ and $y^Tb < 0$, this inequality cannot hold.
Hence we have proven that x can be a solution to $Ax \leq b$ only if there is no such y satisfying the conditions.

Now let us consider Case II, where we prove the other implication.

Case II:

$$\forall x, \neg x ; Ax \leq b \Rightarrow \exists y, y ; y \geq 0 \land y^TA = 0 \land y^Tb < 0$$

Which states that if there is no such x to satisfy $Ax \leq b$, then there must be some y satisfying ① $y \geq 0$ & ② $y^TA = 0$ & ③ $y^Tb < 0$.

$\Rightarrow$ meaning $Ax \leq b$ is infeasable

In order to prove that, let's first define $D = \begin{pmatrix} A \\ -A \\ I \end{pmatrix}$ and $h = \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$.

We shall now apply FME on $Dx \leq d$ in order to simplify the equations and eventually end up with equations with one variable. Hence, we want to remove all variables $1, 2, \dots n$ in A, where A is a $m \times n$ matrix.

$\Longrightarrow$ for that, let U^i be a matrix for removing i variable

$U^i \geq 0$

According to FME we can deduce:

$$U^n \cdot U^{n-1} \cdot \ldots \cdot U^1 \cdot D = 0$$
$$U = U^n \cdot U^{n-1} \cdot \ldots \cdot U^1$$
$$U \cdot D = 0$$

But since $Ax \leq b$ is infeasable, it follows that $Dx \leq d$ is also infeasable !

$\hookrightarrow$ since it is constructed from A,

This suggests that there must be a case in FME algorithm, where:

$$u \in U \quad ; \quad u^T D = 0 \;\&\; u^T d < 0$$

Now let's divide vector u into three parts p, q, r where p are the first m elements of u, q the next m elements, and r the last n elements, such that:

$$(p \;\; q \;\; r) \begin{pmatrix} A \\ -A \\ I \end{pmatrix} \Longrightarrow (p - q) A = r$$

since we know that $p, q, r \geq 0$, we can write $(p - q) A = 0$

Now let $p - q$ be y^T, which then suggests $y^T A = 0$. From this follows:

$$(p \;\; q \;\; r) \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix} < 0 \Longrightarrow \begin{array}{l} pb - qb < 0 \\ (p-q) b < 0 \end{array}$$

Let $(p-q)$ be y^T, which then suggests $y^T b < 0$

This completes the proof, where we have shown that the existence of y in case $Ax \leq b$ is infeasible.

Question 4 b

Theorem:
Strong Duality
von Neumann '47

According to the Duality Theorem for LPs, where P and D are a primal-dual pair of LPs, then one of these four cases must occur:

1. Both are infeasable
2. P is unbounded and D is infeasable
3. D is unbounded and P is infeasable
4. Both are feasable and there exists optimal solution x, y to P and D such that $c^T x = b^T y$.

focusing on the ① case, let's find such an example.
Let's first define our primal as well as dual from definition:

primal LP:

$$P = \max\left(c^T X \mid AX \leq b,\ X \geq 0,\ X \in \mathbb{R}^N\right)$$

dual LP:

$$D = \min\left(b^T y \mid A^T y \geq c,\ y \geq 0,\ y \in \mathbb{R}^N\right)$$

Now let's suppose we have an example where $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $b < 0$; $c > 0$.

Then in case of our primal:

$$P = \max\left(c^T X \mid Ax \leq b,\ x \geq 0,\ x \in \mathbb{R}^N\right)$$

$$0 = Ax \leq b < 0$$

$\Rightarrow$ Which is the first contradiction and hence finding a solution to P is infeasable

And then the same goes for our dual:

$$D = \min\left(b^T y \mid A^T y \geq c;\ y \geq 0,\ y \in \mathbb{R}^N\right)$$

$$0 = A^T y \geq c > 0$$

$\Rightarrow$ Which is the first contradiction and hence finding a solution to D is infeasable

This completes the proof that the chosen example indeed satisfies the given case of the Duality Theorem for LPs.