Question 4

(a) With the help of a Python script, I firstly generated all possible combinations of how the three paintings from Francis Bacon can be distributed amongst the three bidders, Sussane, Lakshmi and Bill. Keeping in mind that one player/bidder can make at most one successful bid, I calculated the total payoff for each of the possible combinations of dividing the paintings.

By sorting the values based on their total cost, we observe that there are two scenarios where the combination of highest bids adds up to 55:

Scenario 1:

Player	Paintings	Bid
Sussane	T1	16
Lakshmi	$\emptyset$	0
Bill	T2, T3	39

Scenario 2:

Player	Paintings	Bid
Sussane	$\emptyset$	0
Lakshmi	T1 T3	37
Bill	T2	18

Now since the VCG algorithm does not care about the difference, we can choose either of the two. I chose to consider Scenario 1 first.

The interesting concept behind the VCG algorithm is that it tries to ensure socially-optimal solution, where the final price paid by the highest bidder is not necessarily what that player had bid. Players are being "charged" for how much damage they have caused to other players by placing their bids. Accounting for that, their actual payment price can be calculated the following way:

$$\text{Player } X = \max\left(\sum_{j \in V/\{x\}} v_j(C')\right) - \sum_{j \in V/\{x\}} v_j(C^*)$$

Where the first part accounts for the maximum sum of the combination of distribution of paintings excluding player X. The second part accounts for the sum of the bids put forward by other players for the remaining paintings in that given combination.

Let us now calculate the price paid by the players for Scenario 1.

$$\text{Sussane} = \max\left(\sum_{j \in V/S/j} v_j(C')\right) - \sum_{j \in V/f3} v_j(C') = 55 - 39 = 16$$

Sussane = ∅	0
Lakshmi = T1 T3	37
Bill = T2	16

Bill = T2 T3	39

$$\text{Lakshmi} = \max\left(\sum_{j \in V/S/j} v_j(C')\right) - \sum_{j \in V/f3} v_j(C') = 55 - 16 - 39 = 0$$

Sussane = T1	16
Lakshmi = ∅	∅
Bill = T2 T3	39

Sussanne = T1	16
Bill = T2 T3	39

$$\text{Bill} = \max\left(\sum_{j \in V/S/j} v_j(C')\right) - \sum_{j \in V/f3} v_j(C') = 54 - 16 = 38$$

Sussane = T1 T2 T3	54
Lakshmi = ∅	0
Bill = ∅	0

Susane = T1	16

So, for Scenario 1 the players will pay the following amounts:

Sussanne = 16

Lakshmi = 0

Bill = 38

(b)

Scenario 1 from upstair is not a unique outcome. We have shown that Scenario 2 has the same total valuation of 55 units, and the players would allocate the paintings in a different arrangement.

In particular, the outcome for Scenario 2 would be:

Sussane = $\{\emptyset\}$ # no paintings

Lakshmi = $\{T_1, T_2\}$

Bill = $\{T_2\}$

In order to check if the prices paid by each player are uniquely determined we need to calculate the price paid by each player in Scenario 2

$$\text{Sussane} = \max\left(\sum_{j \in V/\{i\}} v_j(O')\right) - \sum_{j \in V/\{i\}} v_j(O') = \left(V_L(\{T_1, T_3\}) + V_B(\{T_2\}) - V_L(\{T_1, T_3\}) + V_B(\{T_2\})\right) =$$
$$= 37 + 18 - (37 + 18) = \underline{0}$$

$$\text{Lakshmi} = \max\left(\sum_{j \in V/\{i\}} v_j(O')\right) - \sum_{j \in V/\{i\}} v_j(O') = \left(V_S(\{T_1\}) + V_B(\{T_2, T_3\})\right) - \left(V_S(\{\emptyset\}) + V_B(\{T_2\})\right) =$$
$$= 55 - 18 = \underline{37}$$

$$\text{Bill} = \max\left(\sum_{j \in V/\{i\}} v_j(O')\right) - \sum_{j \in V/\{i\}} v_j(O') = \left(V_S(\{T_1, T_2, T_3\}) + V_L(\{\emptyset\})\right) - \left(V_S(\{\emptyset\}) + V_L(\{T_1, T_3\})\right) =$$
$$= 54 - 37 = \underline{17}$$

So, for Scenario 2 the players will pay the following amounts:

Sussanne = 0

Lakshmi = 37

Bill = 17

So to answer the question of whether the VCG prices paid by the players are uniquely determined; no they are not. As we have shown above, we have found two VCG outcomes with same total outcome, but the players pay a different price, henceforth, the VCG prices paid by the players are not uniquely determined.

(c)

In this particular example the VCG mechanism seems like a reasonable mechanism option. It is incentive-compatible, meaning players should be providing honest valuations. However, it's vurneable to bidder collision if players cooperate or even just when one player acts as multiple bidders.
Another limitation of VCG is large auctioning with multiple pieces, such as the example with 20 Andy Warhols. Reason being the fact that VCG is NP-hard problem and we don't think there are tractable solutions to all NP problems. With 20 paintings there are simply too many combinations we need to compute and would not do it in a reasonable time.
As an alternative auction mechanism we could use the Simultaneous Ascending Auction (SAA), which is not a combinatorial auction and hence allows bidders to withdraw their bids and provides them with the information that reduces their uncertainty. It also makes it easier since each piece can be introduced to the market/bidder seperately.