Algorithmic Game Theory and Applications

Lecture 2: Mixed Strategies, Expected Payoffs, and Nash Equilibrium

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Finite Strategic Form Games

Recall the "strategic game" definition, now "finite":

Definition A finite *strategic form game* Γ, with n-players, consists of:

all pure strategies player
i has in the game

1. A set $N = \{1, \ldots, n\}$ of Players.
2. For each $i \in N$, a finite set $S_i = \{1, \ldots, m_i\}$ of *(pure) strategies*.
   Let $S = S_1 \times S_2 \times \ldots \times S_n$ be the set of possible combinations of (pure) strategies.

   maps each combination of strategies into some real value => for evaluation

3. For each $i \in N$, a *payoff (utility) function*:
   $u_i : S \mapsto \mathbb{R}$, describes the payoff $u_i(s_1, \ldots, s_n)$ to player i under each combination of strategies.

player: 1 plays s_1
2 plays s_2
⋮
n plays s_n

(Each player wants to maximize its own payoff.)

Mixed (Randomized) Strategies

We define "mixed" strategies for general finite games.

Definition A **mixed** (i.e., **randomized**) **strategy** x_i for Player i, with $S_i = \{1, \ldots, m_i\}$, is a probability distribution over S_i. In other words, it is a vector $x_i = (x_i(1), \ldots, x_i(m_i))$, such that $x_i(j) \geq 0$ for $1 \leq j \leq m_i$, and

it doesn't need to be uniformly distributed:

$$x_i(1) + x_i(2) + \ldots + x_i(m_i) = 1$$

Intuition: Player i uses randomness to decide which strategy to play, based on the probabilities in x_i.

$X_i = \{x_i(1), x_i(2) \ldots x_i(m_i)\}$

Let X_i be the set of mixed strategies for Player i.
For an n-player game, let

$$X = X_1 \times \ldots \times X_n$$

denote the set of all possible combinations, or "**profiles**", of mixed strategies.

Expected Payoffs

PS – pure strategy

Let $x = (x_1, \ldots, x_n) \in X$ be a profile of mixed strategies.
For $s = (s_1, \ldots, s_n) \in S$ a combination of pure strategies, let

probability of this combination of pure strategies:

$$x(s) := \prod_{j=1}^{n} x_j(s_j)$$

each player makes his/her decision independently of the rest!

be the probability of combination s under mixed profile x. (We're assuming players make their random choices independently.)

Definition: The **expected payoff** of Player i under a mixed strategy profile $x = (x_1, \ldots, x_n) \in X$, is:

payoff for player i under PSs combination

$$U_i(x) \; := \sum_{s \in S} x(s) * u_i(s)$$

probability of that combination of PSs

I.e., the "weighted average" Player i's payoff under each pure combination s, weighted by the probability of that combination.
Key Assumption: Every player's goal is to maximize its own expected payoff. (This can somtimes be a dubious assumption.)

some notation

! if one strategy has a probability of 1 !

We call a mixed strategy $x_i \in X_i$ *pure* if $x_i(j) = 1$ for some $j \in S_i$, and $x_i(j') = 0$ for $j' \neq j$. We denote such a pure strategy by $\pi_{i,j}$.
I.e., the "mixed" strategy $\pi_{i,j}$ does not randomize at all: it picks (with probability 1) exactly one strategy, j, from the set of pure strategies for player i.

player i's strategy spot

Given a profile of mixed strategies $x = (x_1, \ldots, x_n) \in X$, let

$$x_{-i} = (x_1, x_2, \ldots, x_{i-1}, \texttt{empty}, x_{i+1}, \ldots, x_n)$$

I.e., x_{-i} denotes everybody's strategy except that of player i.
For a mixed strategy $y_i \in X_i$, let $(x_{-i}; y_i)$ denote the new profile:

$$(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$$

every player's strategy stays the same except for x_i ; (x_{-i} = empty) gets replaced by y_i

In other words, $(x_{-i}; y_i)$ is the new profile where everybody's stategy remains the same as in x, except for player i, who switches from mixed strategy x_i, to mixed strategy y_i.

Best Responses

Definition: A (mixed) strategy $z_i \in X_i$ is a **best response** for Player i to x_{-i} if for all $y_i \in X_i$,

$$U_i(x_{-i}; z_i) \geq U_i(x_{-i}; y_i)$$

Clearly, if any player were given the opportunity to "cheat" and look at what other players have done, it would want to switch its strategy to a best response.
Of course, players in a strategic form game can't do that: players pick their strategies simultaneously/independently.

But suppose, somehow, the players "arrive" at a profile where everybody's strategy is a best response to everybody else's.
Then no one has any incentive to change the situation.
We will be in a "stable" situation: an "*Equilibrium*".
That's what a "Nash Equilibrium" is.

Nash Equilibrium

Definition: For a strategic game Γ, a strategy profile $x = (x_1, \ldots, x_n) \in X$ is a mixed **Nash Equilibrium** if for every player, i , x_i is a best response to x_{-i}.
In other words, for every Player $i = 1, \ldots, n$, and for every mixed strategy $y_i \in X_i$,

$$U_i(x_{-i}; x_i) \geq U_i(x_{-i}; y_i)$$

In other words, *no player can improve its own payoff by unilaterally deviating from the mixed strategy profile* $x = (x_1, \ldots, x_n)$.

x is called a **pure Nash Equilibrium** if in addition every x_i is a pure strategy $\pi_{i,j}$, for some $j \in S_i$.

Nash's Theorem

Finite Game – each player has a finite amount of options, the number of players is finite, and the game cannot go on indefinitely

This can, agruably, be called
"The Fundamental Theorem of Game Theory"

Theorem(Nash 1950) Every finite *n*-person strategic game has a mixed Nash Equilibrium.

We will prove this theorem next time.
To prove it, we will "cheat" and use a fundamental result from topology: the Brouwer Fixed Point Theorem.

The crumpled sheet experiment

Let's all please conduct the following experiment:

1. Take two identical rectangular sheets of paper.
2. Make sure neither sheet has any holes in it, and that the sides are straight (not dimpled).
3. "Name" each point on both sheets by its "(x, y)-coordinates".
4. Crumple one of the two sheets any way you like, *but make sure you don't rip it in the process*.
5. Place the crumpled sheet completely on top of the other flat sheet.

Fact! There must be a point named (a, b) on the crumpled sheet that is directly above the same point (a, b) on the flat sheet. (Yes, really!)

As crazy as it sounds, this fact, in its more formal and general form, will be the key to why every game has a mixed Nash Equilibrium.