Methods for Causal Inference
Lecture 2

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Causal theory and data

Requires 4 steps:

1. Definition of Causation
2. Clearly formulating causal **assumptions** and creating the **causal model**
3. Link the structure of casual model to features of data
4. **Estimation** given the causal model and data

Defining causation:
A variable X is a cause of a variable Y if Y in any way relies on X for its value.
(Intuitively: X is a cause of Y if Y listens to X and decides its value in response to what it hears)

Pre-requisites: Elementary concepts from probability theory, statistics, graph theory

Causal Inference in Statistics, Pearl (2016)

Basics of Probability

Most causal statements are uncertain: “drinking causes liver disease”, does not mean every person who consumes alcohol is certain to have liver disease

→ Need language and laws of probability.

Variables: Any property or descriptor that can take multiple values, e.g., age (x=40), sex (x’=F), family history of disease (x”=0), … .

Events: An event is any assignment of a **value or set of values** to a variable or set of variables.

Discrete (binary/categorical): Are being treated or not, have a disease or not, …

Continuous (can take infinite set of values): age, weight, …

Drug (yes/no) vs dose of drug (categorical). Sun intake (time is continuous),

Basics of Probability

For probabilistic modelling (of a random experiment) we need to:

- Describe possible outcomes: **sample space**
- **Event:** A subset of sample space
- Describe beliefs about likelihood of these events: **probability law**

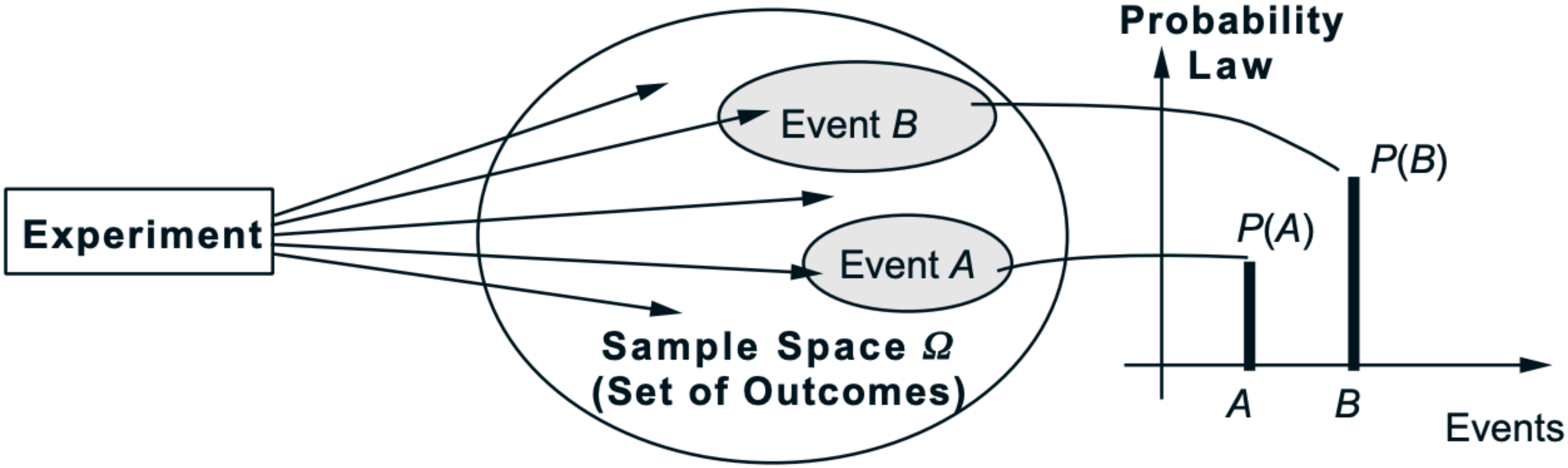
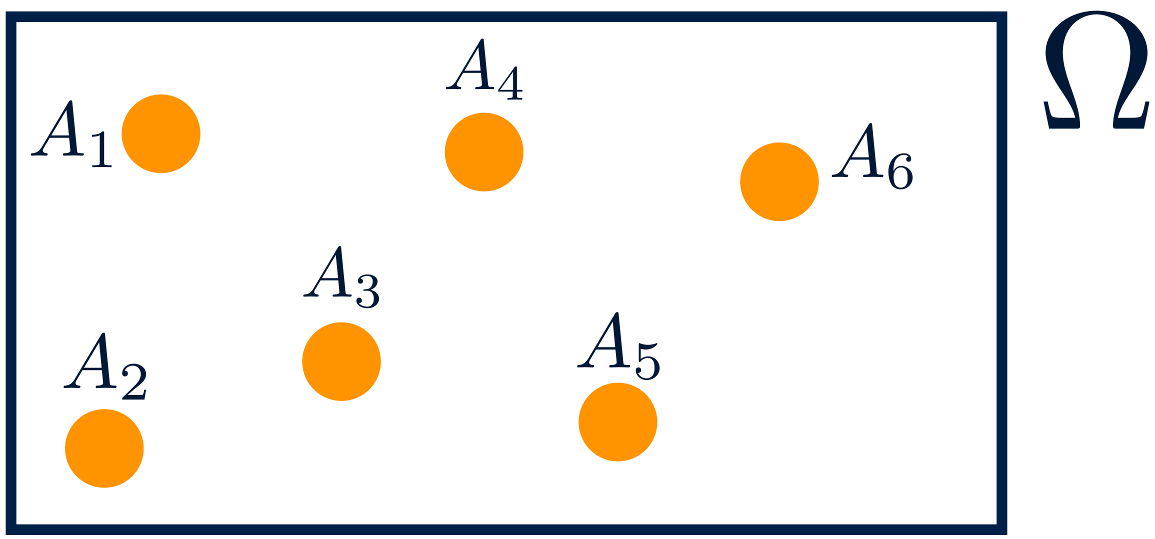


Figure from Tsitsiklis MIT Lecture Notes: Introduction to probability

Sample Space

The sample space is the set of all possible outcomes of the experiment:

e.g. Rolling a dice



Outcomes must be:

- **Mutually Exclusive**: If I tell you, after the experiment, that A_1 happened, then it should not be possible for that A_6 also happened.
- **Collectively Exhaustive**: Collectively, all the outcomes in Ω exhaust all possibilities

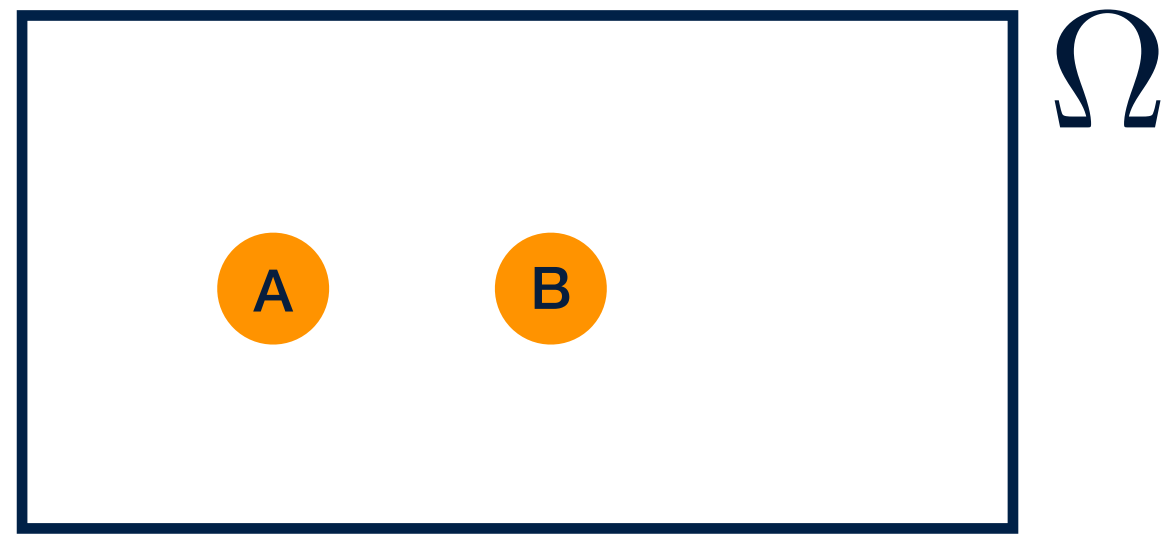
Tsitsiklis: Introduction to probability

Probability Axioms

Non-negativity: $P(A) > 0$

Normalisation: $P(\Omega) = 1$

For any two mutually exclusive events
(e.g. A and B cannot co-occur) we have:



$P(A \text{ or } B) = P(A) + P(B)$,

which implies, $P(A) = P(A, B) + P(A, \text{'not B'})$

A and B are mutually exclusive. If A is true, then either “A and B” or “A and not B” must be true. Generalise for exhaustive, mutually exclusive partitions of B:

Generalise: $P(A) = P(A, B1) + P(A, B2) + \ldots + P(A,Bn)$

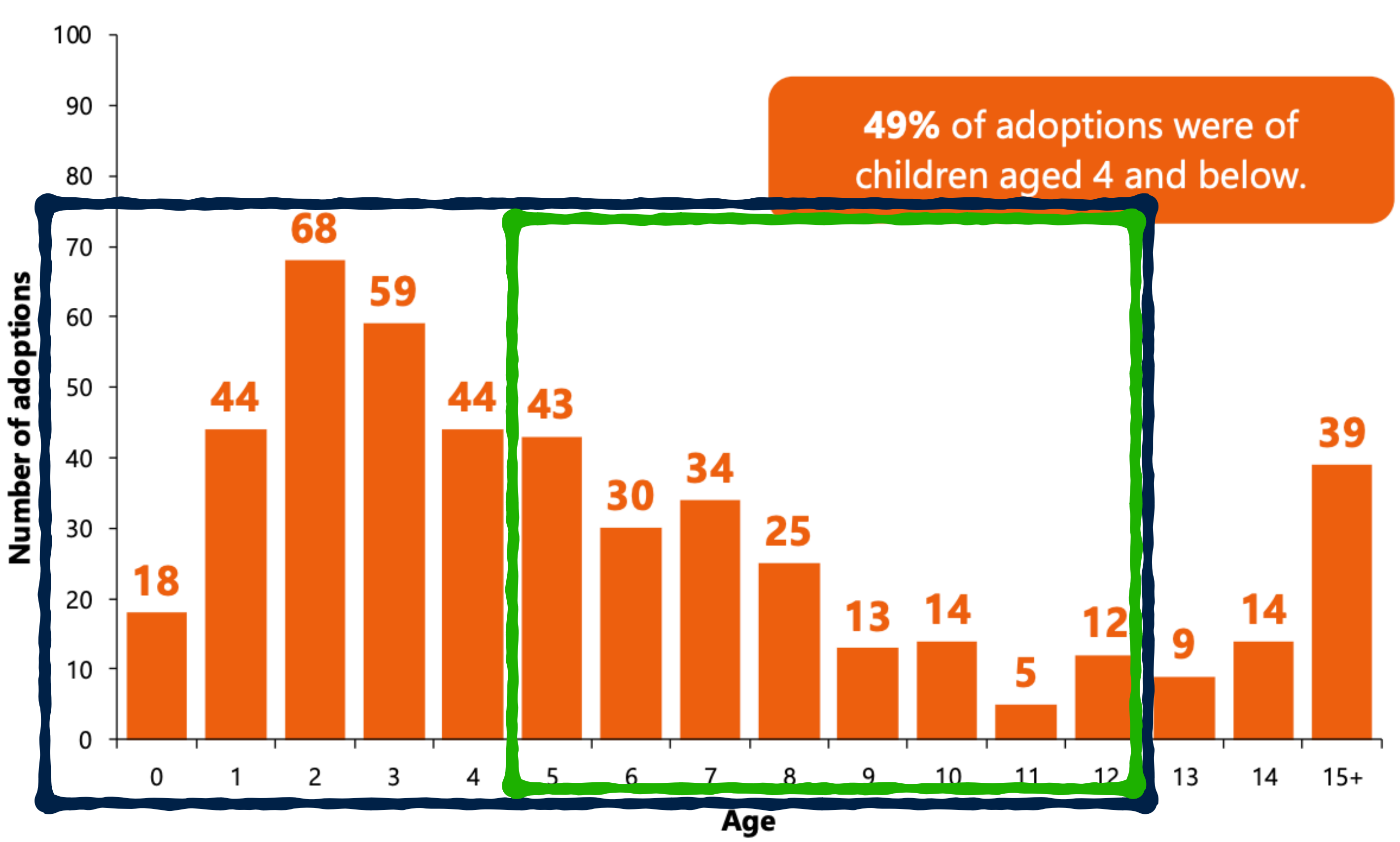
Causal Inference in Statistics, Pearl (2016)

Intervals

P(age > 4) = 1 - P(age <= 4) = 1- 0.49 = 0.51

P(4 < age < 12) =
(43+30+34+25+13+14+5+12) / 471 = 0.37

Figure 7.2: Age at adoption, Scotland, 2018



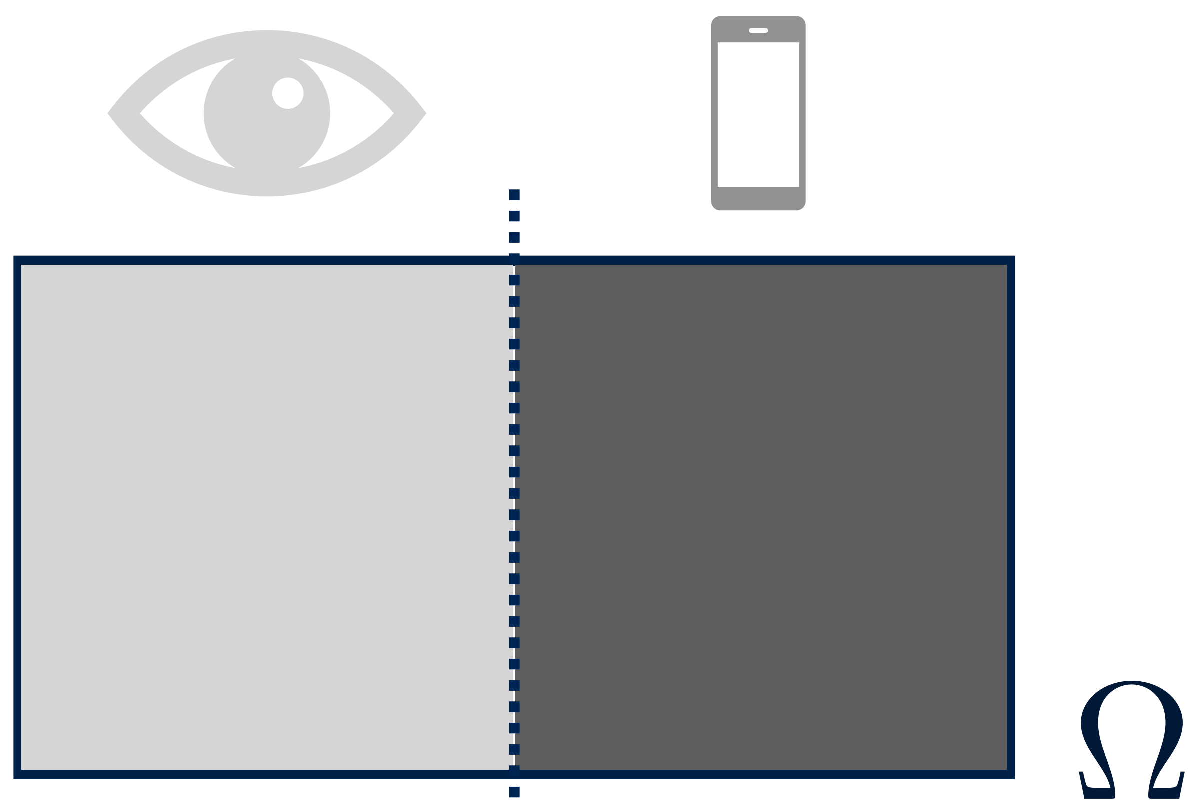
Total = 471

National Records of Scotland 2018

Law of Total probability: Example

Assuming ‘no multi-tasking’, the event:
“Passing the causality exam AND not being on your phone during the lectures”
is **mutually exclusive** from
“Passing the causality exam and being entirely on your phone during the lectures”

P(passing the causality exam) =
P(passing the exam, being entirely on your phone during the lecture) +
P(passing the exam, fully paying attention during the lecture)



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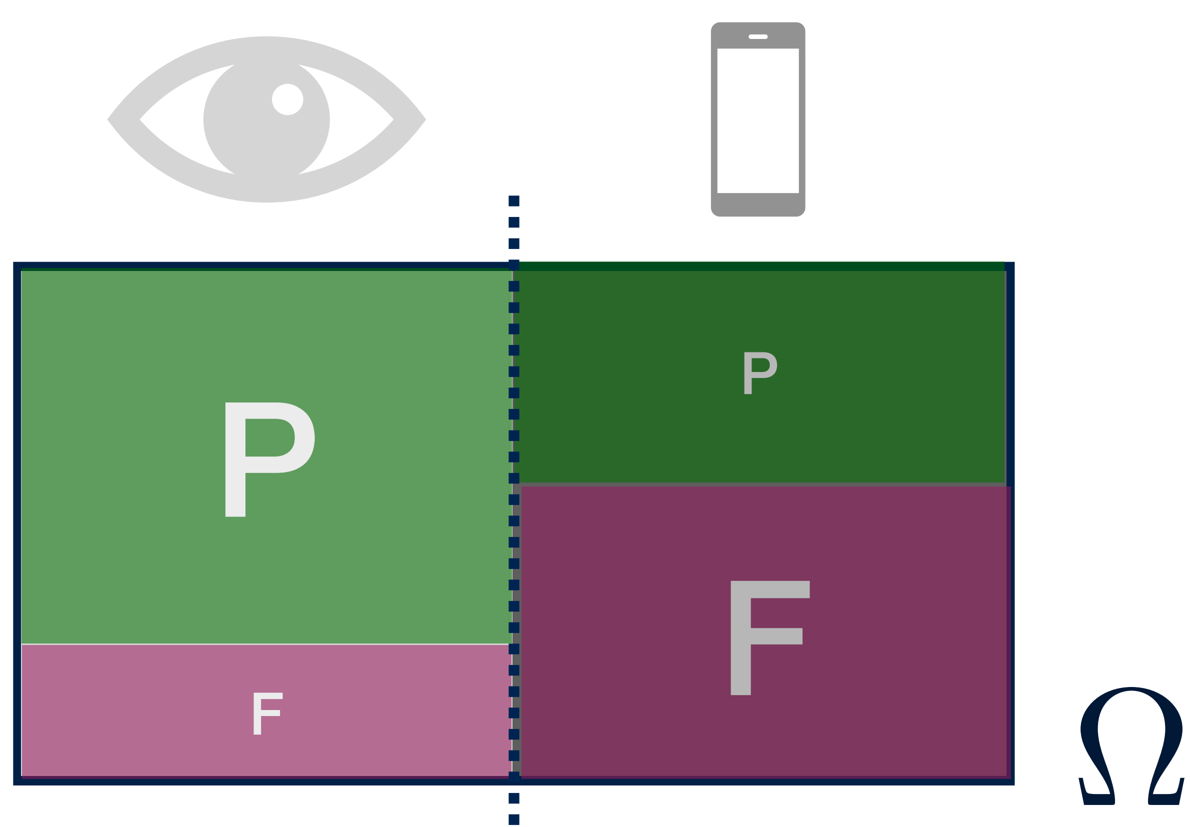


Conditional Probability

The probability that event A occurs, given that we know some other event B has occurred. (Think of filtering the data based on the value of some variable)

P(X=x) vs P(X=x|Y=y): The probability of X=x can drastically change depending on the knowledge Y=y

Example: P(lung cancer | smoker) vs

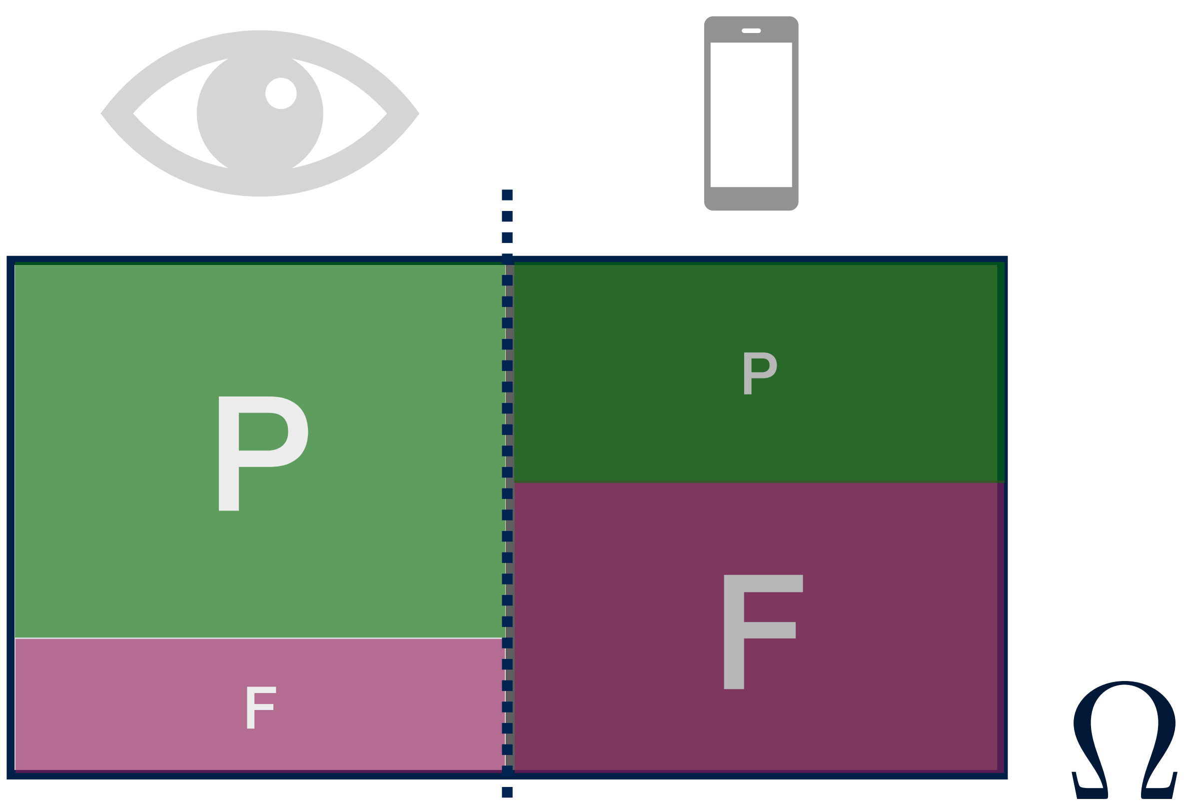
$$P(\underbrace{\text{lung cancer | smoker}}_{x}\ ,\ \underbrace{\text{socio-economic status}}_{Y})$$

Given that the patient is a smoker, does knowing their socio-economic status add further information to the probability of lung cancer?

$$P(X,Y) = P(X|Y)P(Y)$$

Conditional Probabilities

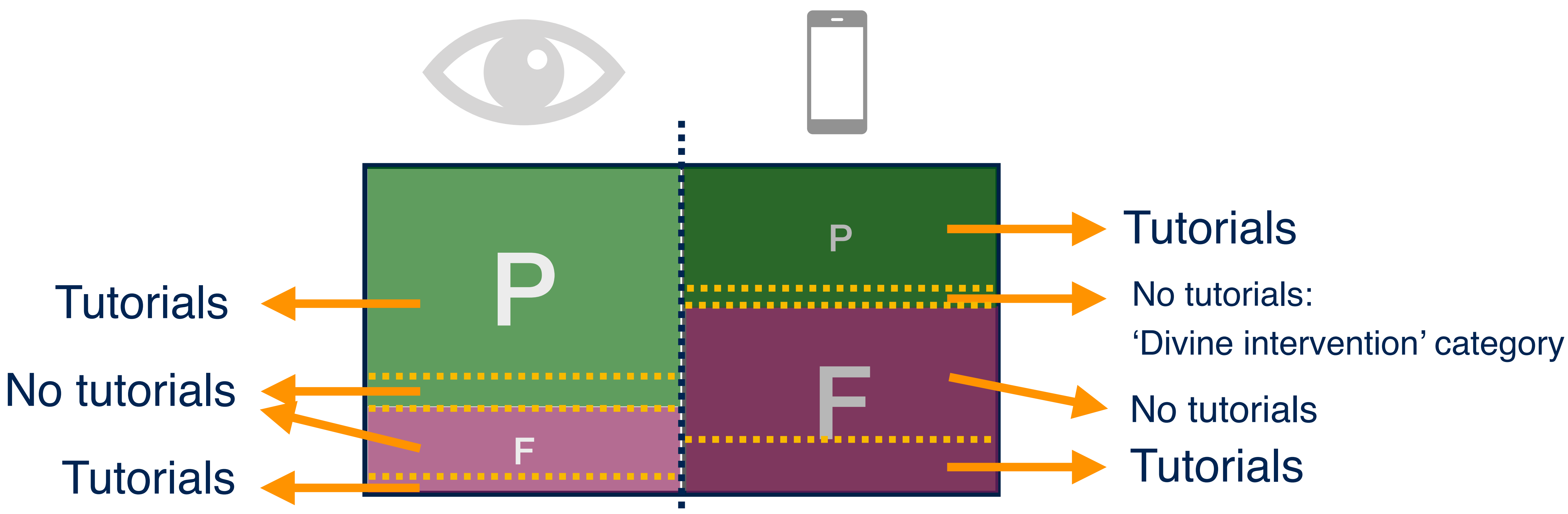
P(passing the causality exam | paying attention) >
P(passing the causality exam | being on your phone)



Conditional Law of Total probability: Example

P(passing the causality exam | fully paying attention during the lecture) =
P(passing the exam , attending tutorials | attention in lecture) +
P($\underbrace{\text{passing the exam}}_{X}$, $\underbrace{\text{not attending tutorials | attention in lecture}}_{Y}$)

P(passing the causality exam | being on one's phone during the lectures) =
P(passing the exam , attending tutorials | being on phone during lecture) +
P(passing the exam, not attending tutorials | being on phone lecture)



Bayes' Rule

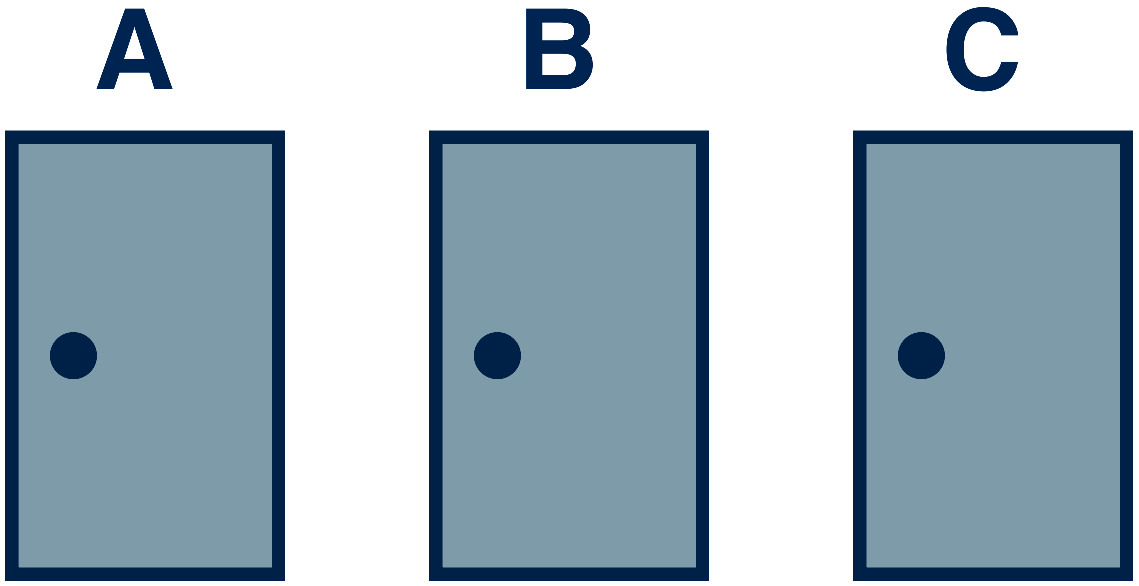
$X_1, X_2, \ldots, X_n$ are disjoint events forming a partition of the sample space

and $P(X_i) > 0$, $\forall X_i$. Then for any event Y , $P(Y) > 0$, Bayes' rule states:

$$P(X_i|Y) = \frac{P(X_i)P(Y|X_i)}{P(Y)}$$

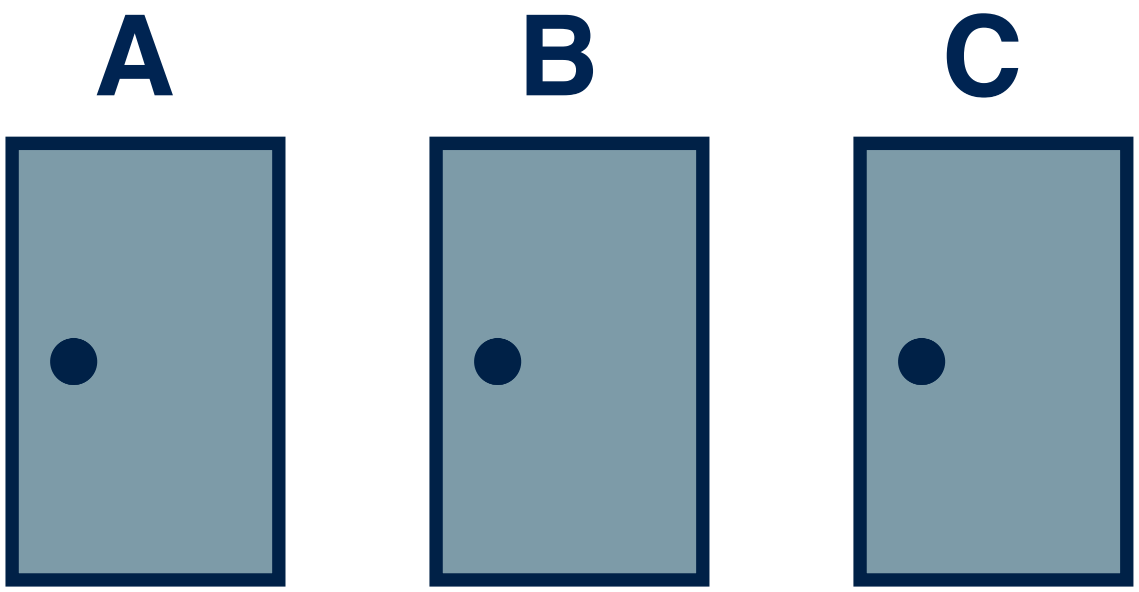
$$= \frac{P(X_i)P(Y|X_i)}{\underbrace{P(X_1)P(Y|X_1) + \cdots + P(X_n)P(Y|X_n)}_{\text{this is just normalised notation}}} = P(Y)$$

Monty Hall Problem & Application of Bayes' Rule



Car or Goat?

Monty Hall Problem & Application of Bayes' Rule

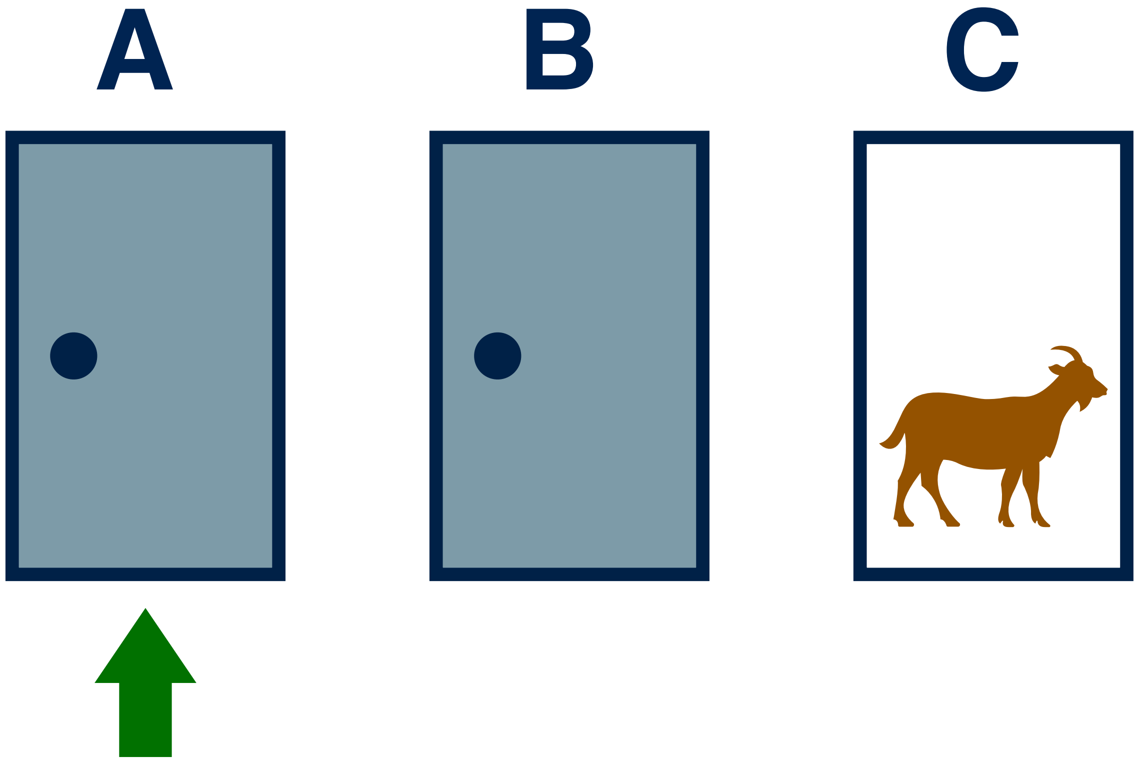


X = Door chosen by player

Y = Door hiding the car

Z = Door opened by host

Monty Hall Problem & Application of Bayes' Rule



X = Door chosen by player

Y = Door hiding the car

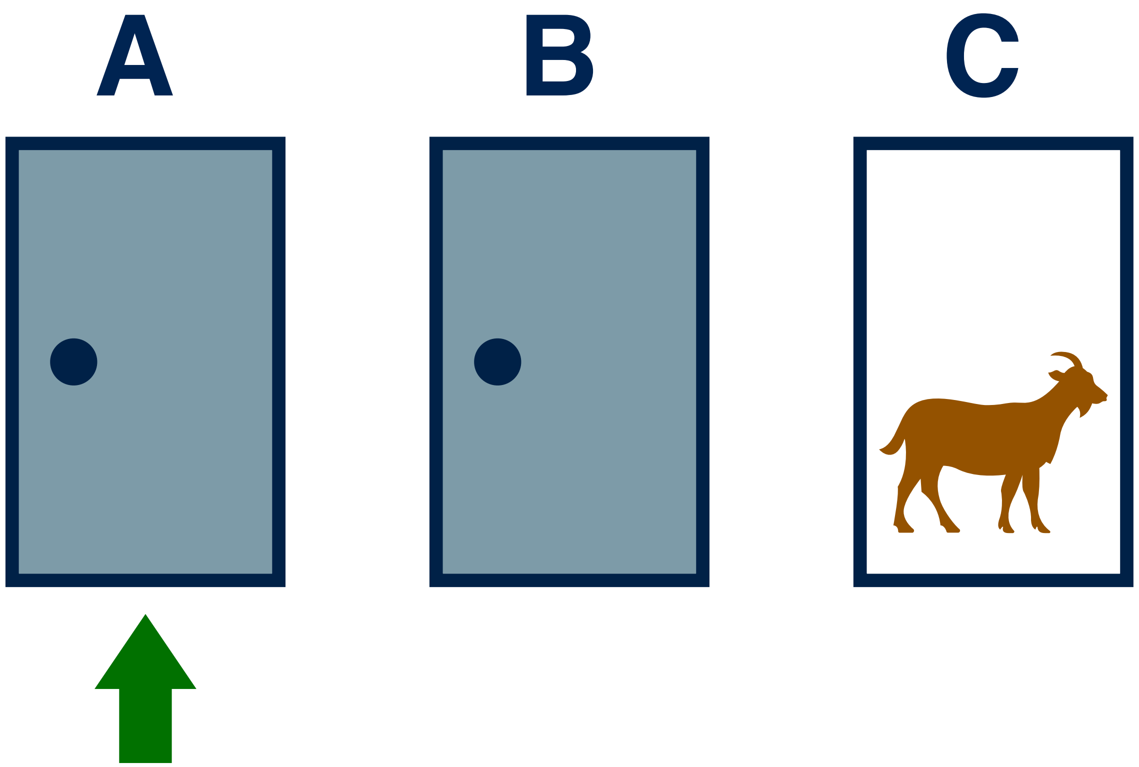
Z = Door opened by host

Prove that switching doors improves our chance of winning the car.

Note the assumptions:

1. The host will not open the door we have chosen
2. **The host will never open a door with a car behind**
3. Given a choice of doors, the host will choose at **random** (whilst 2)
4. Given no info, the car is equally likely to be behind any door

Monty Hall Problem & Application of Bayes' Rule



X = Door chosen by player

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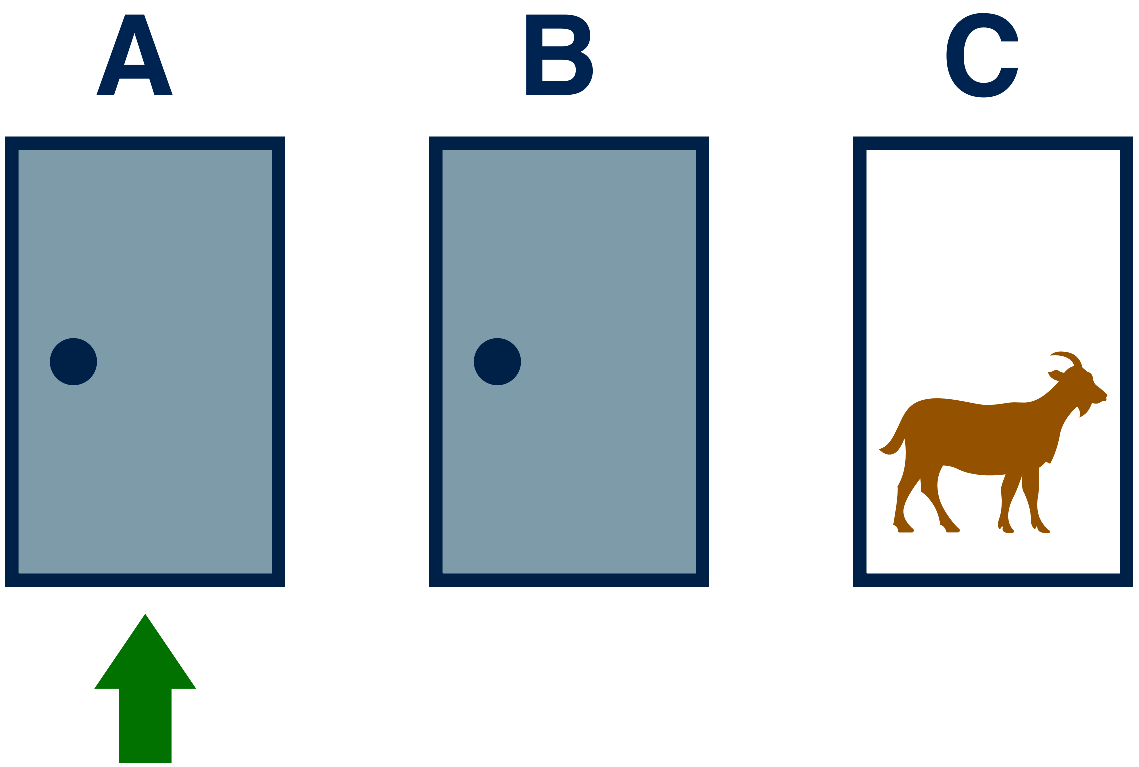
Prove that switching doors improves our chance of winning the car.

Need to show (given the we have selected A and host has shown us C):

$$P(Y = A | X = A, Z = C) < P(Y = B | X = A, Z = C)$$

Is the car more likely to be behind B than A, i.e. switching improves our chance.

Monty Hall Problem & Application of Bayes' Rule



X = Door chosen by player

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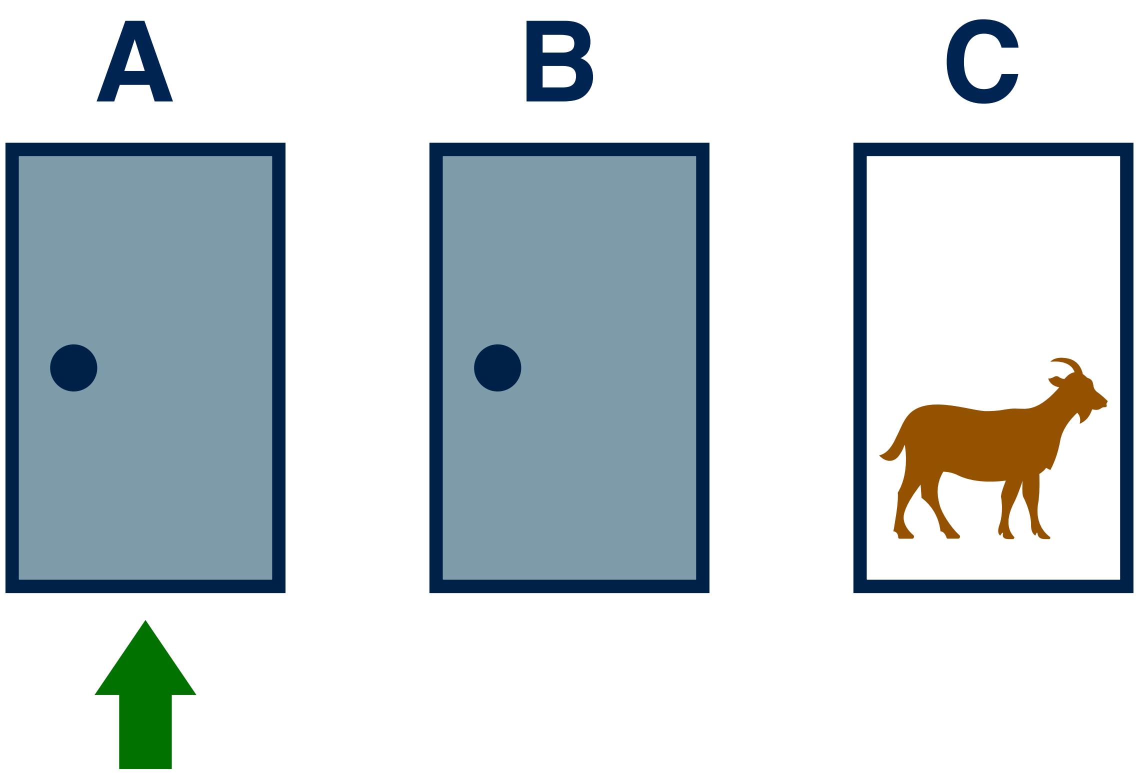
$$P(Y=A|X=A,Z=C)=\frac{P(Z=C|X=A,Y=A)P(Y=A|X=A)}{P(Z=C|X=A)}$$

Monty Hall Problem & Application of Bayes' Rule



X = Door chosen by player

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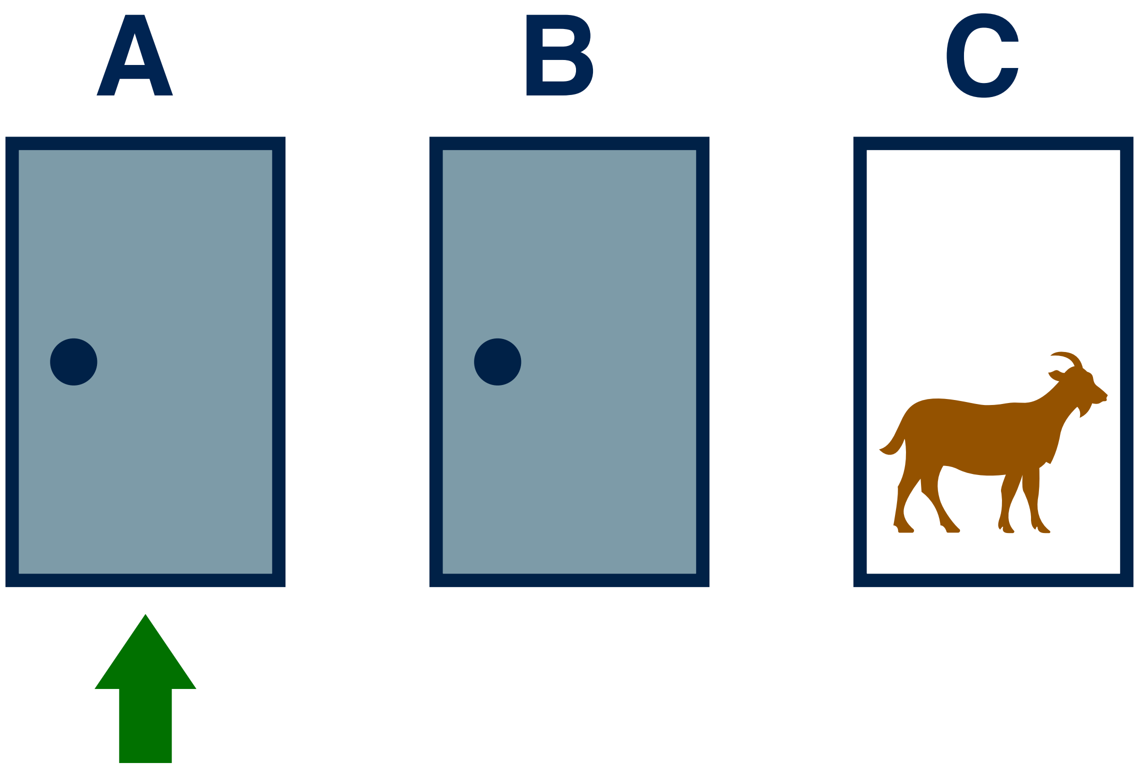
Z = Door opened by host

$$P(Y = A|X = A, Z = C) = \frac{\overset{1/2}{\boxed{P(Z = C|X = A, Y = A)}}P(Y = A|X = A)}{P(Z = C|X = A)}$$

Given we choose A (X=A), and the car is in A (Y=A), then the host is allowed to choose either B or C, as neither has the car behind it. Since the host choses randomly (assumption 3), we get 1/2.

Monty Hall Problem & Application of Bayes' Rule



X = Door chosen by player

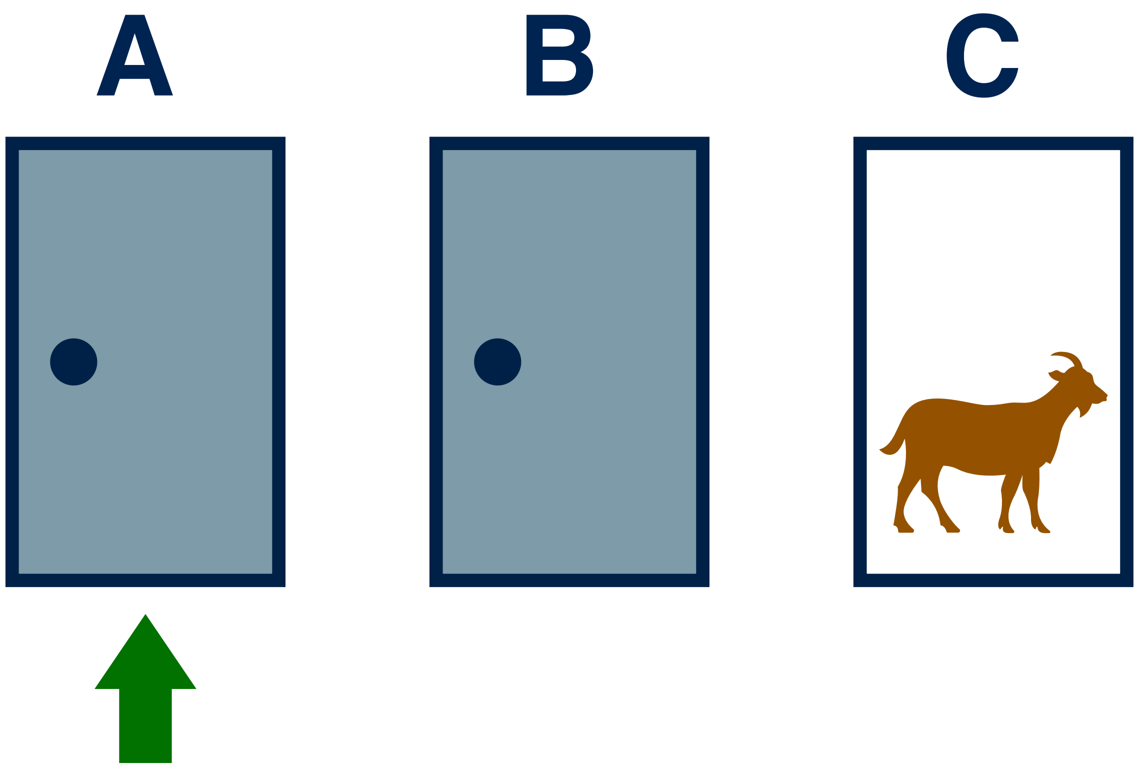
Y = Door hiding the car

Z = Door opened by host

$$P(Y=A|X=A,Z=C) = \frac{P(Z=C|X=A,Y=A)\overset{1/3}{\boxed{P(Y=A|X=A)}}}{P(Z=C|X=A)}$$

Given we choose A (X=A), what is the probability that the car is behind A? With no further information, this is equal to 1/3.

Monty Hall Problem & Application of Bayes' Rule



X = Door chosen by player

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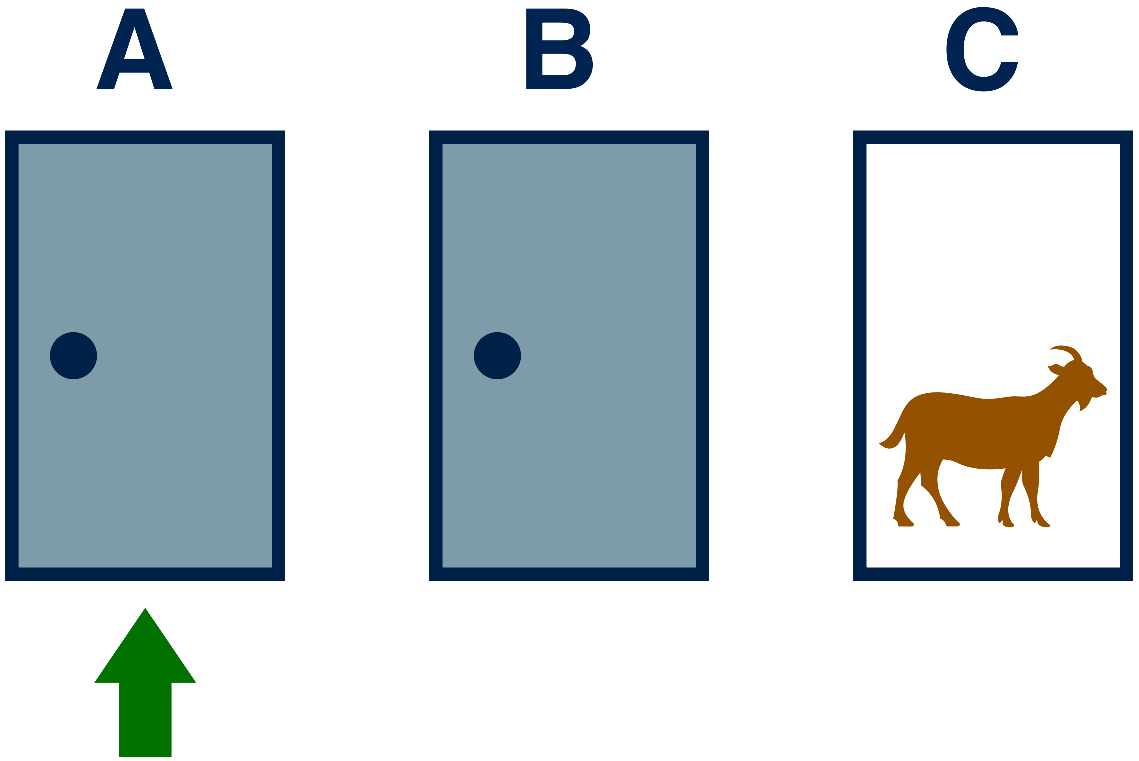
$$P(Y=A|X=A,Z=C)=\frac{P(Z=C|X=A,Y=A)P(Y=A|X=A)}{\boxed{P(Z=C|X=A)}}$$ 1/2

Total law of prob

Product rule

$$P(Z=C|X=A)=\sum_{d=A,B,C}P(Z=C,Y=d|X=A)=\sum_{d=A,B,C}P(Z=C|X=A,Y=d)P(Y=d)$$

Monty Hall Problem & Application of Bayes' Rule



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$$P(Y=A|X=A,Z=C) = \frac{P(Z=C|X=A,Y=A)P(Y=A|X=A)}{\boxed{P(Z=C|X=A)}}$$ 1/2

Total law of prob — Product rule

$$P(Z=C|X=A) = \sum_{d=A,B,C} P(Z=C,Y=d|X=A) = \sum_{d=A,B,C} P(Z=C|X=A,Y=d)P(Y=d)$$

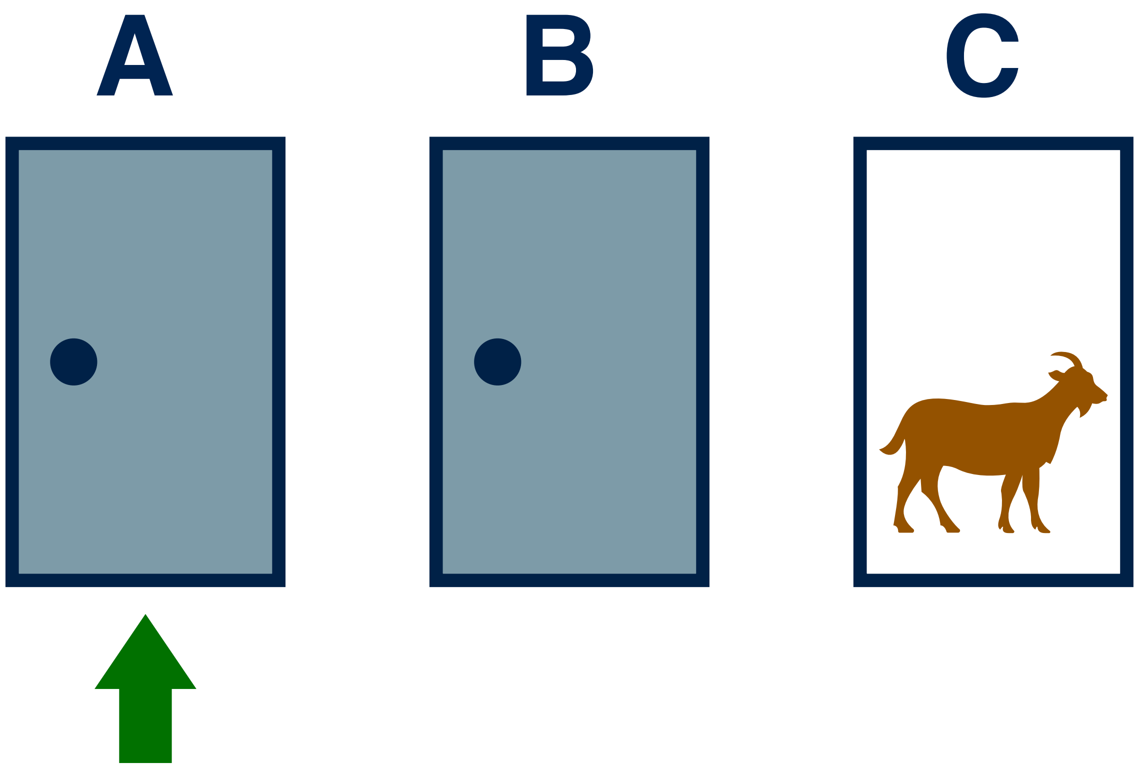
$$= \frac{1}{3}\Big(P(Z=C|X=A,Y=A) + P(Z=C|X=A,Y=B) + P(Z=C|X=A,Y=C)\Big) = 1/2$$

1/2 as above

1: Given we chose A and car is behind B, host is **forced** to choose C (Assumption 2)

0: Given we chose A and car is behind C, the host cannot choose C (Assumption 2)

Monty Hall Problem & Application of Bayes' Rule



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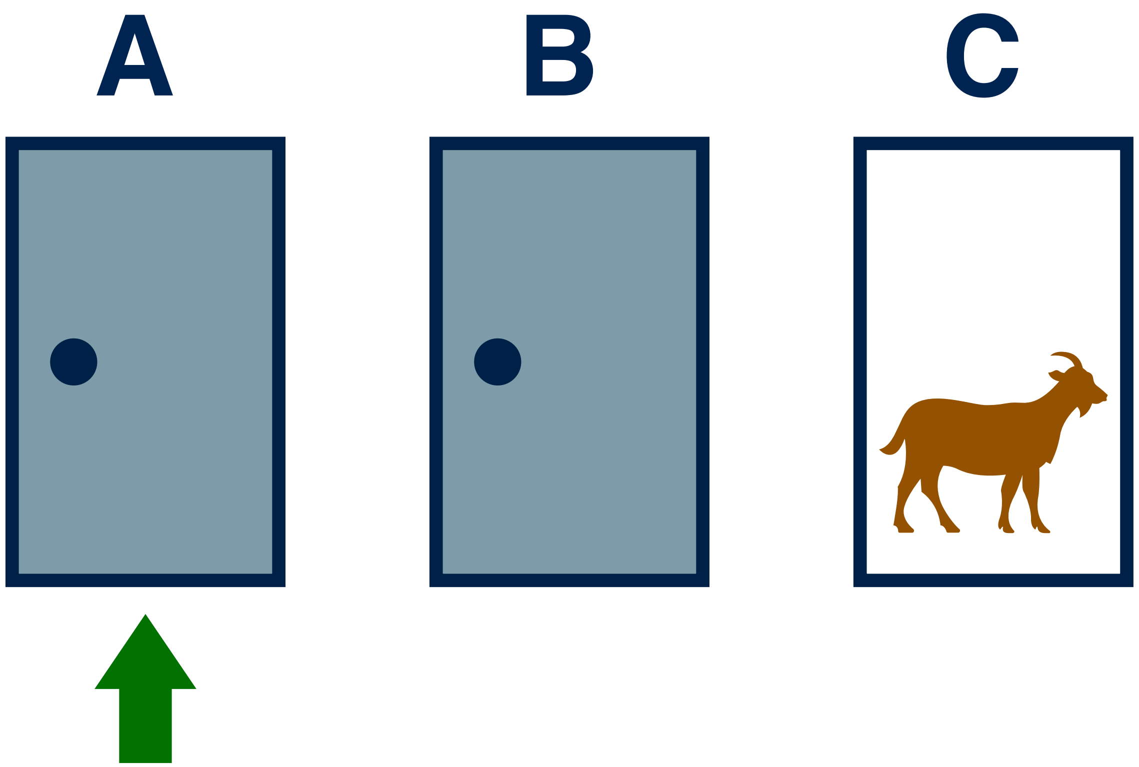
Monty Hall Problem & Application of Bayes' Rule



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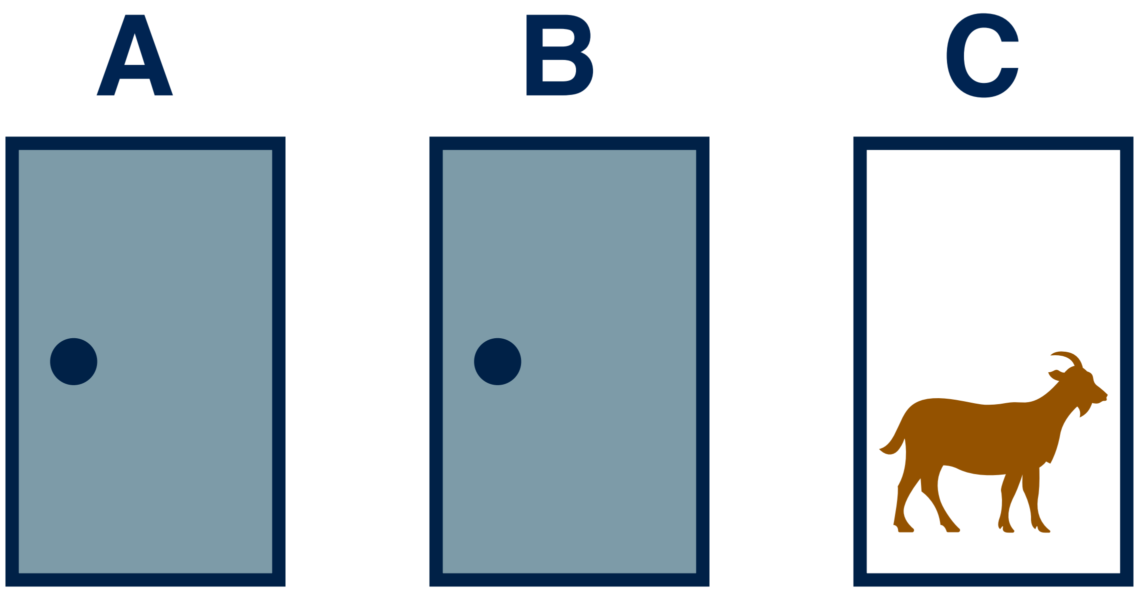
$$\begin{aligned} P(Y=B|X=A,Z=C) =& 1 - P(Y=A|X=A,Z=C) - P(Y=C|X=A,Z=C) \\ =& 1 - \frac{1}{3} - 0 = 2/3 \end{aligned}$$

Mamma Mia!

Monty Hall Problem & Application of Bayes' Rule



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Importance: Incorporating knowledge about the process that generated the data. The first step towards **causal inference**.

'Host could have opened', 'he was forced to open', 'randomly opened', 'about to open', …

Independence

X and Y are independent events: $P(X,Y) = P(X)P(Y)$

Equivalently: $P(X|Y) = P(X)$ (where $P(Y)$ is non-zero, otherwise $P(X|Y)$ not defined)

Conditional independence: $P(X,Y|Z) = P(X|Z)P(Y|Z)$

Equivalently: $P(X|Y,Z) = P(X|Z)$ (again, for $P(Y,Z)$ non-zero)

Independence of several events: $$P(X_1, X_2, \cdots, X_N) = \prod_{i=1}^{N} P(X_i)$$

Remark: Pairwise independence does not imply independence

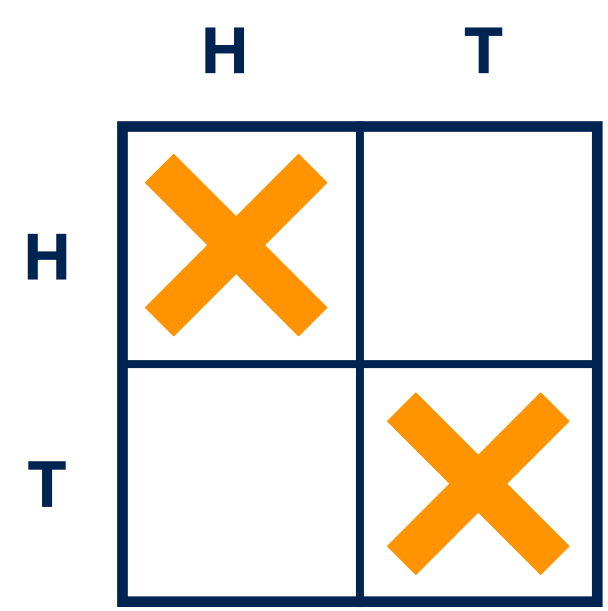
Example: 2 independent fair coin tosses (p1, p2 = 0.5)

Consider 3 events:

H1 = first coin is a head

H2 = second coin is a head

J = the two tosses have the same results



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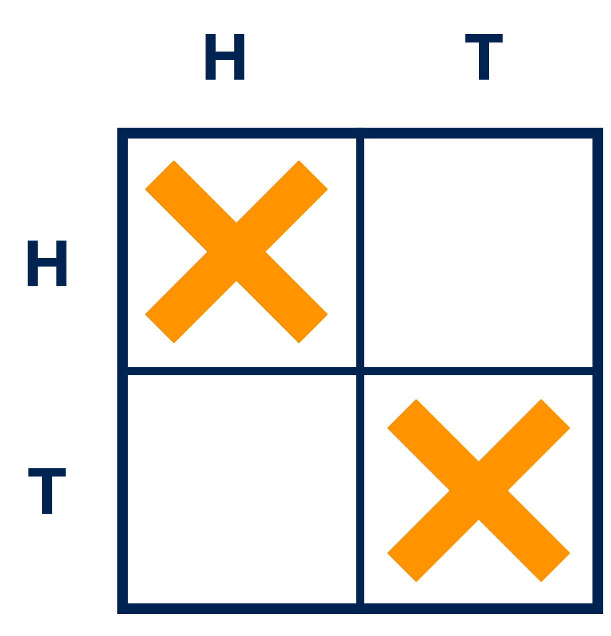
Example: 2 independent fair coin tosses (p1, p2 = 0.5)

H1 & H2: independent coin tosses

P(H1,H2) = P(H1|H2)P(H2) = 0.5x0.5 = P(H1)P(H2)



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Example: 2 independent fair coin tosses (p1, p2 = 0.5)

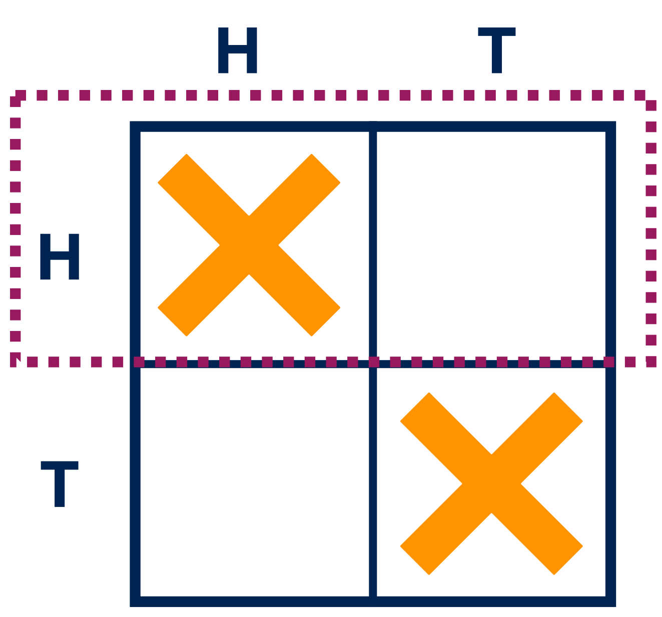
H1 & H2: independent coin tosses

P(H1,J) = P(J l H1)P(H1) =

Given H1, what is the probability of J

(i.e second toss also being a head)

So: P(J l H1) = 0.5



Independence

X and Y are independent events: P(X,Y) = P(X)P(Y)

Equivalently: P(XIY) = P(X) (where P(Y) is non-zero, otherwise P(XIY) not defined)

Conditional independence: P(X,YIZ) = P(XIZ)P(YIZ)

Equivalently: P(XIY,Z) = P(XIZ) (again, for P(Y,Z) non-zero)

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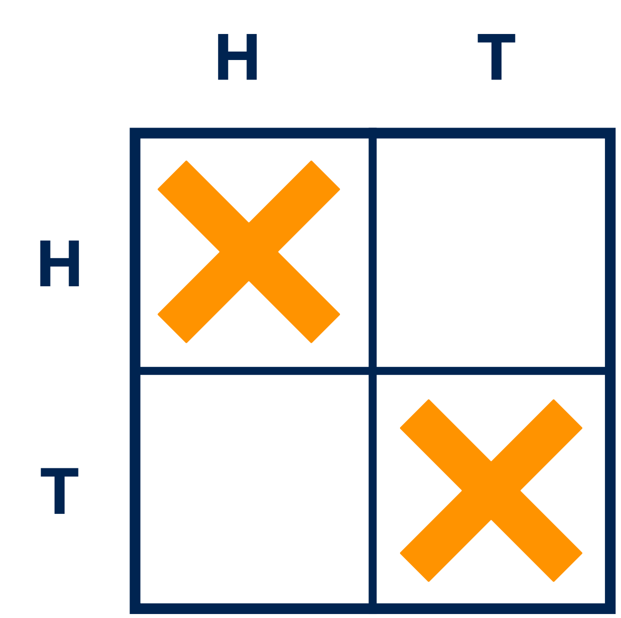

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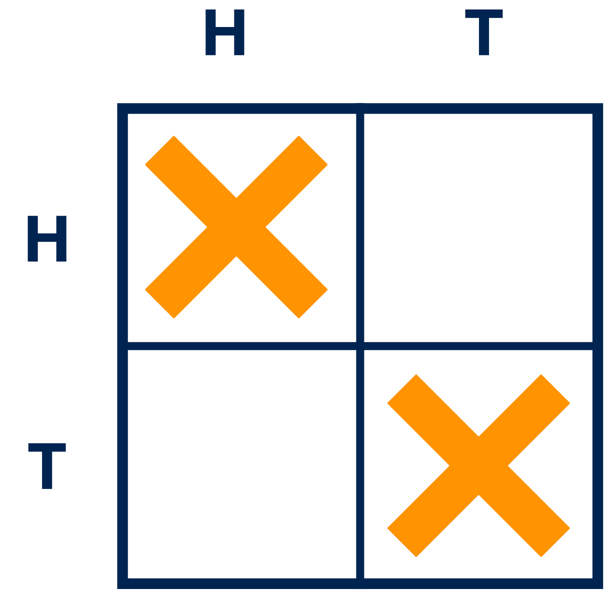
Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses (p1, p2 = 0.5)

H1 & H2: independent coin tosses

P(H2,J) = P(J | H2)P(H2) = 0.5 x 0.5 = P(J)P(H2)

So pair-wise independent. BUT …



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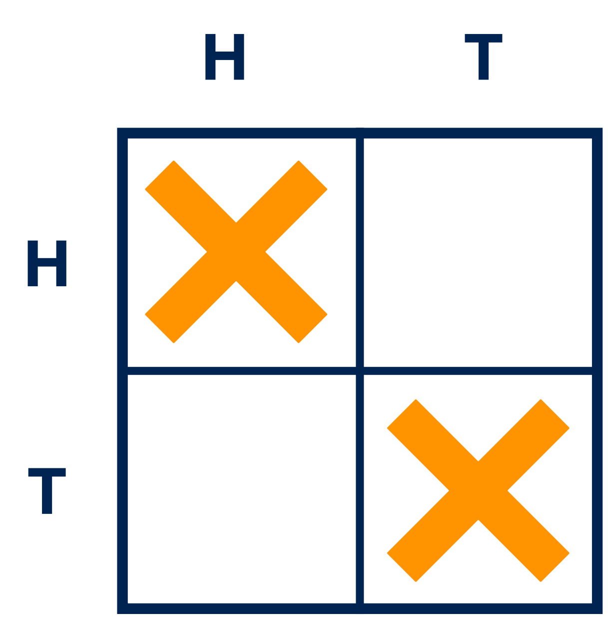
Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses (p1, p2 = 0.5)

H1 & H2: independent coin tosses

P(H1,H2,J) = P(H1 | H2,J) P(H2,J) = 1 x 0.25 = 0.25



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Remark: Pairwise independence does not imply independence

Example: 2 independent fair coin tosses (p1, p2 = 0.5)

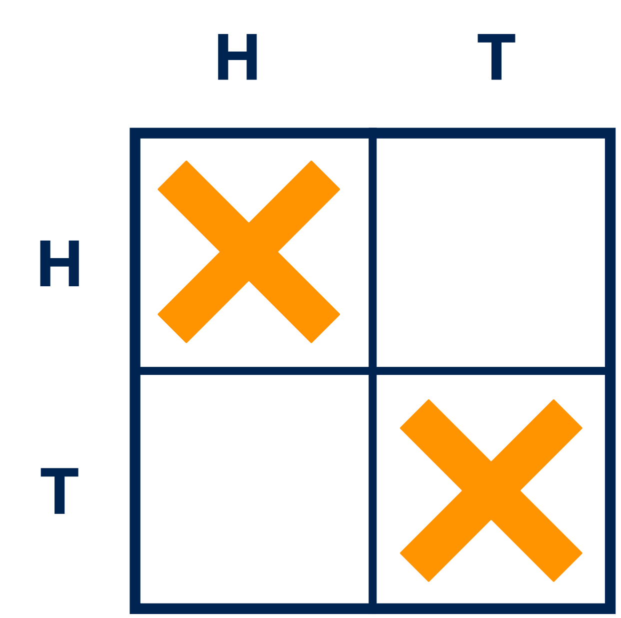
H1 & H2: independent coin tosses

P(H1,H2,J) = P(H1 | H2,J) P(H2,J) = 1 x 0.25 = 0.25

$\neq$

However, P(H1)P(H2)P(J)=0.5x0.5x0.5=0.125

i.e. not jointly independent



Expected Values

The probability distribution of a random variable X provides us with probabilities of all possible values of X.

Summarise information, with some loss of information, represented by:
The **expected value** or **mean**:

$$\mathbb{E}[X] = \sum_{x} x \; P(X = x)$$

For a dice: (1x1/6) + (2x1/6) + (3x1/6) + (4x1/6) + (5x1/6) + (6x1/6) = 3.5

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The expected value of any function of X, e.g. g(x):

$$\mathbb{E}[g(X)] = \sum_{x} g(x) \; P(X = x)$$

Dice: (1x1/6) + (4x1/6) + (9x1/6) + (16x1/6) + (25x1/6) + (36x1/6) = 15.17

Expected Values

The probability distribution of a random variable X provides us with probabilities of all possible values of X.

Summarise information, with some loss of information, represented by:
The **expected value** or **mean**:

$$\mathbb{E}[X] = \int x \ P(x) dx$$

for a continuous variable X.

Variance

The variance of a random variable X, denoted Var(X) or σ_X^2:

$$var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

and can be calculated as

$$var(X) = \sum_x (X - \mathbb{E}[X])^2 p_X(x)$$

(Integral of continuous variables), and measure how "spread out" the values of X in a data set are relative to their mean.

The standard deviation σ_X , (has the same units as X).

For a normal distribution, ~2/3 of the population values of X fall within one σ_X, 95% fall between 2 σ_X , etc.

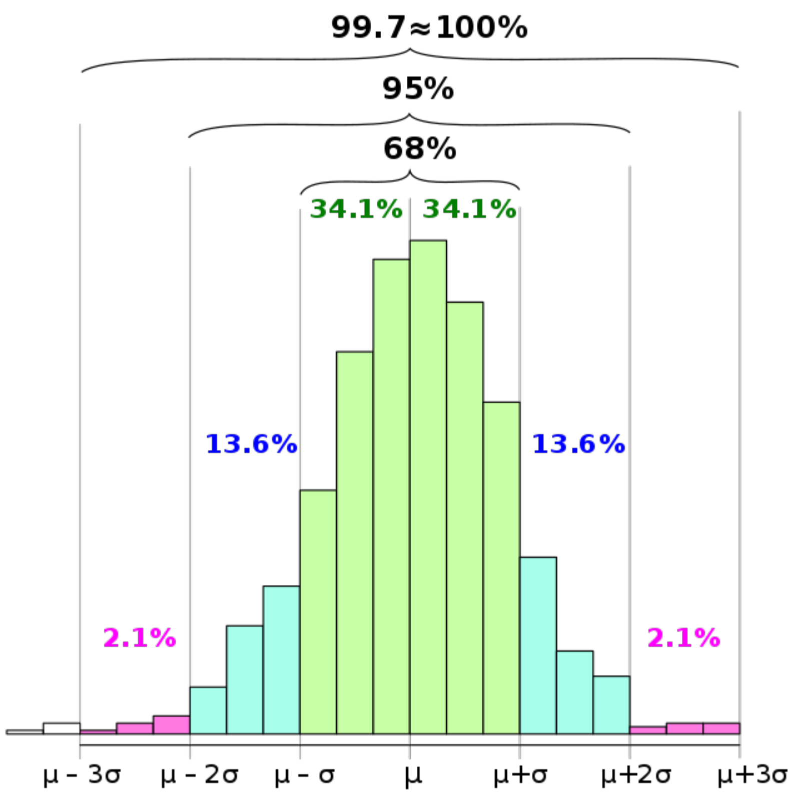

Figure from WikiPedia

Covariance

The degree to which two random variables X and Y covary (degree associated):

$$\sigma_{XY} = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

and measures a specific way X and Y covary, i.e., **linearly**. When normalised, it yields the correlation coefficient (Pearson correlation):

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

a dimensionless quantity between -1 and 1.

When X and Y are independent, then $\rho_{XY} = 0$.
The reverse is not true!
(e.g. ρ_{XY} may be zero, but not linear-correlation, hence dependence exists. This requires more complex methods of demonstrating if $P(Y|X) = P(Y)$)

Anscombe's Quartet

Group of 4 datasets with nearly identical simple descriptive statistical properties:

- Mean and sample variance of X
- Mean and sample variance of Y
- Correlation between X and Y
- Linear regression line (coefficient the same up to 2 or 3 decimal places)
- R^2 coefficient

A note on R^2: A measure for goodness-of-fit

$$R^2 = 1 - \frac{\sum_i (y_i - f_i)}{\sum_i (y_i - \bar{y})} \ , \ y_i = f(x_i) \ , \ \bar{y} = \frac{1}{n} \sum_i y_i$$

If the fit y=f(x) is a perfect fit, the numerator is zero, $R^2 = 1$, and
$R^2 = 0$ implies the fit f(x) is no better than baseline average $\bar{y}$.
Negative values corresponds to models worse than the baseline average.

Anscombe’s Quartet

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Yet, very different distributions, which can be observed by plotting the graphs

Same Pearson correlation, but,
different dependence structure
(X causes Y, but it different ways)

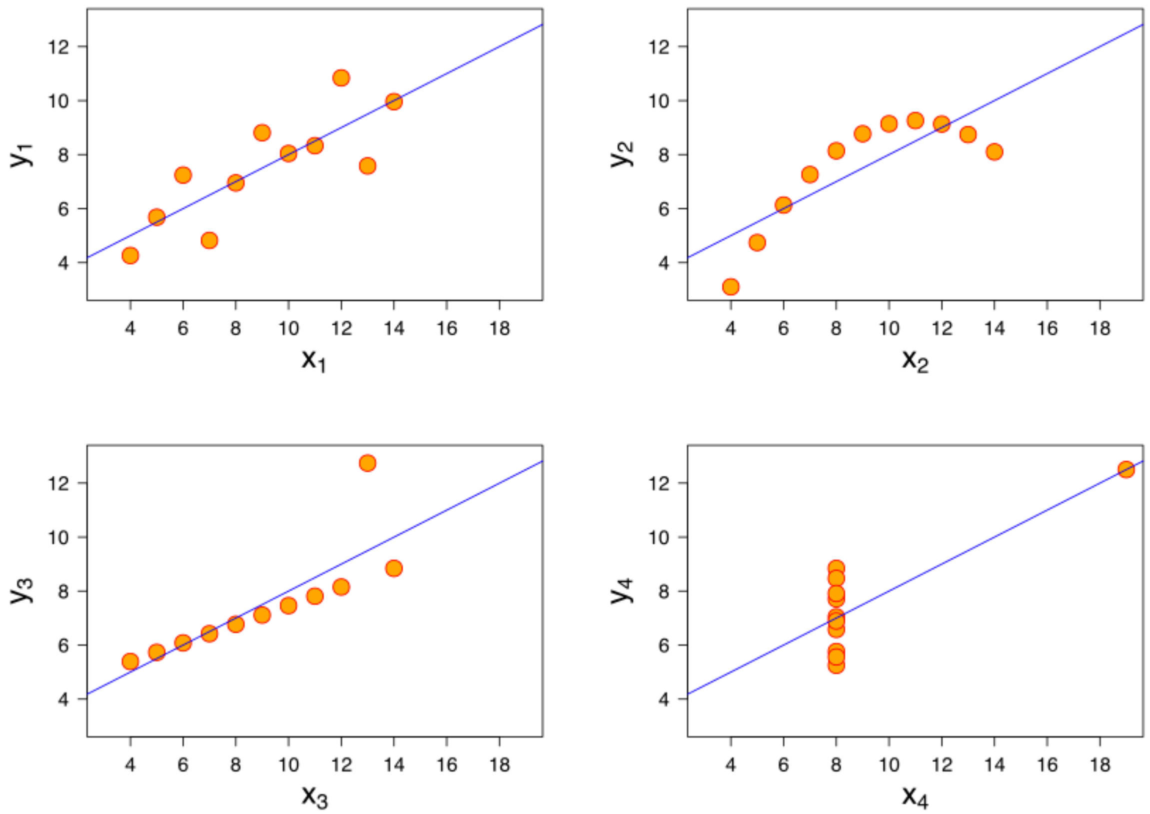

Figure from WikiPedia

Next time: Lecture 3

Regression, graphs, Structural Causal Models

Methods for Causal Inference
Lecture 2

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