Answer to Question No.(a)

Given, $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$

where at rest $\theta = 10°$

so, @ $t = 0$, $\theta = 10 \times \frac{\pi}{180} = \frac{\pi}{18}$

@ $t = 0$, $\omega = \frac{d\theta}{dt} = 0$ [since at rest]

Now rewritting the equation,

$$\theta'' + \frac{g}{l}\theta = 0$$

$$\Rightarrow r^2 + \frac{g}{l} = 0$$

$$\Rightarrow r = \pm\sqrt{-\frac{g}{l}}$$

so the roots are:

$r_1 = 0 \pm i\sqrt{\frac{g}{l}} = i\sqrt{\frac{327}{20}}$

$= 4.0435 i$

$r_2 = -4.0435i$

$\alpha = 0$ & $\beta = \cancel{i}\sqrt{\frac{327}{20}}$

So the solution is $\rightarrow$

$$\theta = e^{\alpha t}\left[c_1 \cos\beta t + c_2 \sin\beta t\right]$$

$$= e^0\left[c_1 \cos\left(\sqrt{\frac{327}{20}}\, t\right) + c_2 \sin\left(\sqrt{\frac{327}{20}}\, t\right)\right]$$

$$\theta' = -\sqrt{\frac{327}{20}}\, c_1 \sin\left(\sqrt{\frac{327}{20}}\, t\right) + \sqrt{\frac{327}{20}}\, c_2 \cos\left(\sqrt{\frac{327}{20}}\, t\right)$$

Now, @ $t=0$; $\theta = \frac{\pi}{18} = c_1 \cos(0) + c_2 \sin(0)$

$$\Rightarrow \frac{\pi}{18} = c_1$$

& @ $t=0$; $\theta' = 0 = -\sqrt{\frac{327}{20}}\, c_1 \sin 0 + \sqrt{\frac{327}{20}}\, c_2 \cos(0)$

$$\Rightarrow 0 = 0 + c_2 \sqrt{\frac{327}{20}}$$

$$\Rightarrow c_2 = 0$$

so the exact/analytical sol^n is $\rightarrow$

$$\theta = \frac{\pi}{18} \cos\left(\sqrt{\frac{327}{20}}\, t\right)$$

Answer to Question No.(b)

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

let, $\theta = y_1$

$$\frac{dy_1}{dt} = y_2$$

$$\frac{d^2\theta}{dt^2} = \frac{dy_2}{dt} = -\frac{g}{l}y_1$$

So, $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ & $Y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} \frac{\pi}{9} \\ 0 \end{bmatrix}$

$$\frac{d}{dt}Y(t) = \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -\frac{g}{l}y_1 \end{bmatrix}$$

the equation can be rewritten as →

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}x_1 \end{bmatrix}$$

$$\Rightarrow \frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\Rightarrow \frac{d}{dt}\{x\} = [A]\{x\}$$

Answer to Question No.(c)

(i) Explicit Euler Method:

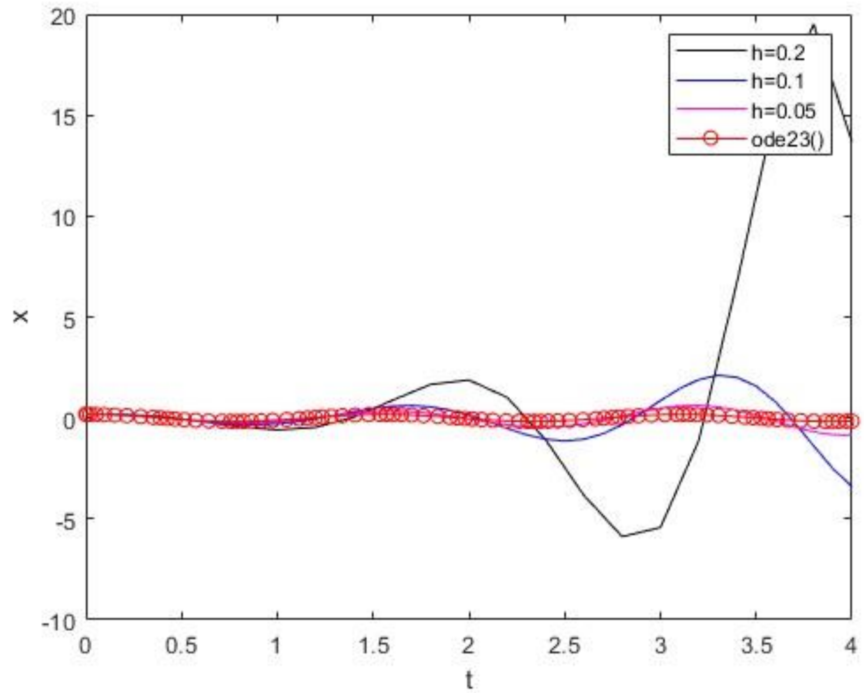


Fig: t vs θ (Numerical solution)

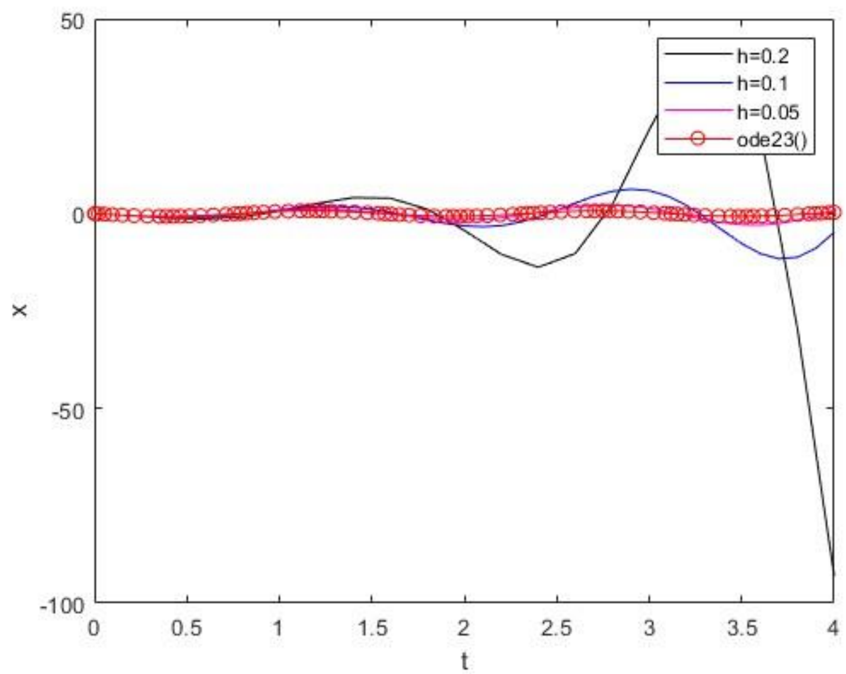


Fig: t vs dθ/dt (Numerical solution)

(ii) Implicit Euler Method:

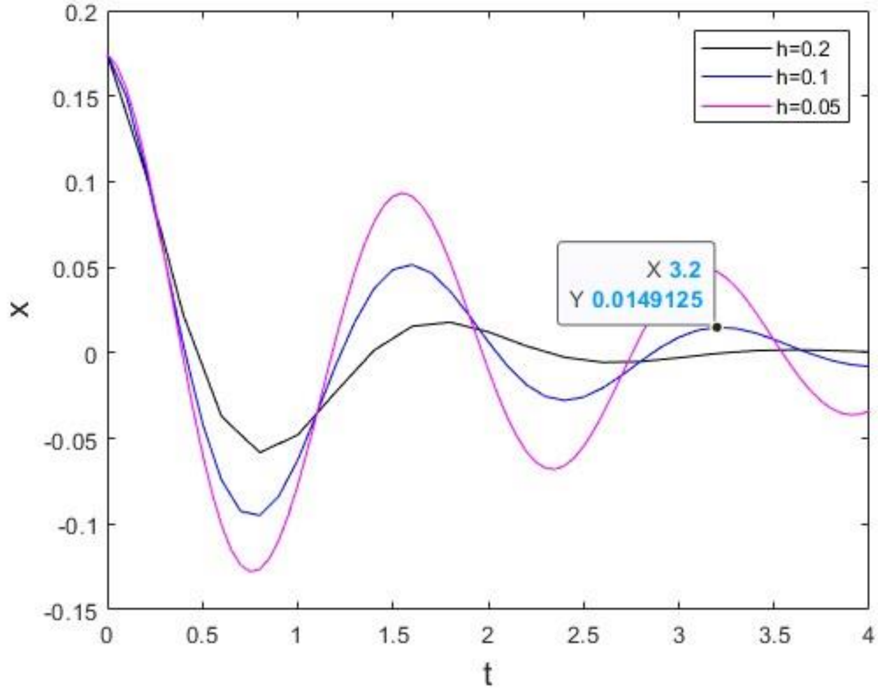


Fig: t vs θ

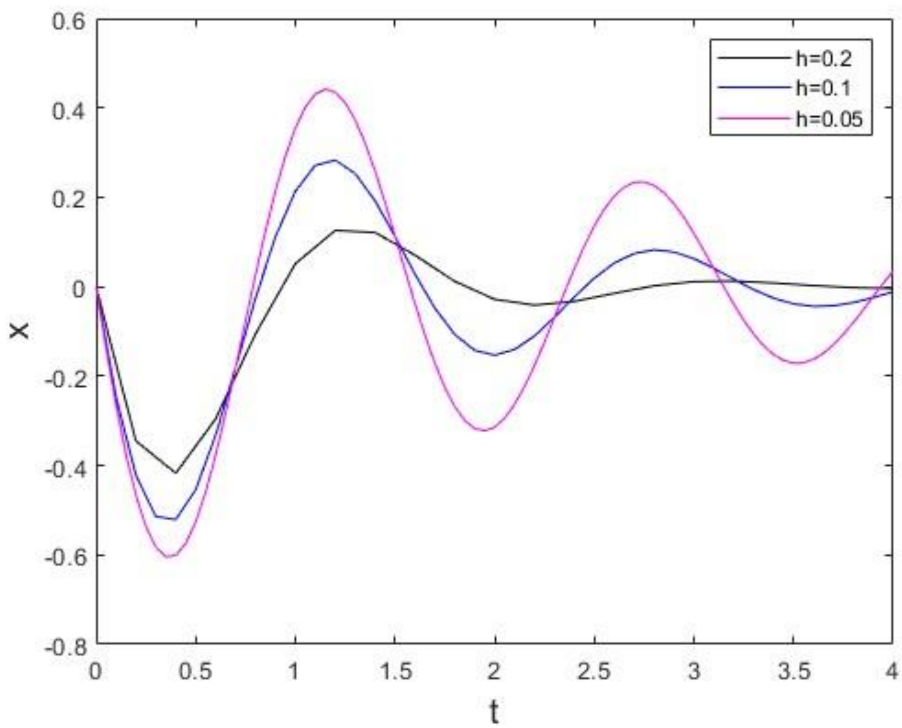


Fig: t vs dθ/dt

(iii) Crank-Nicolson Method:

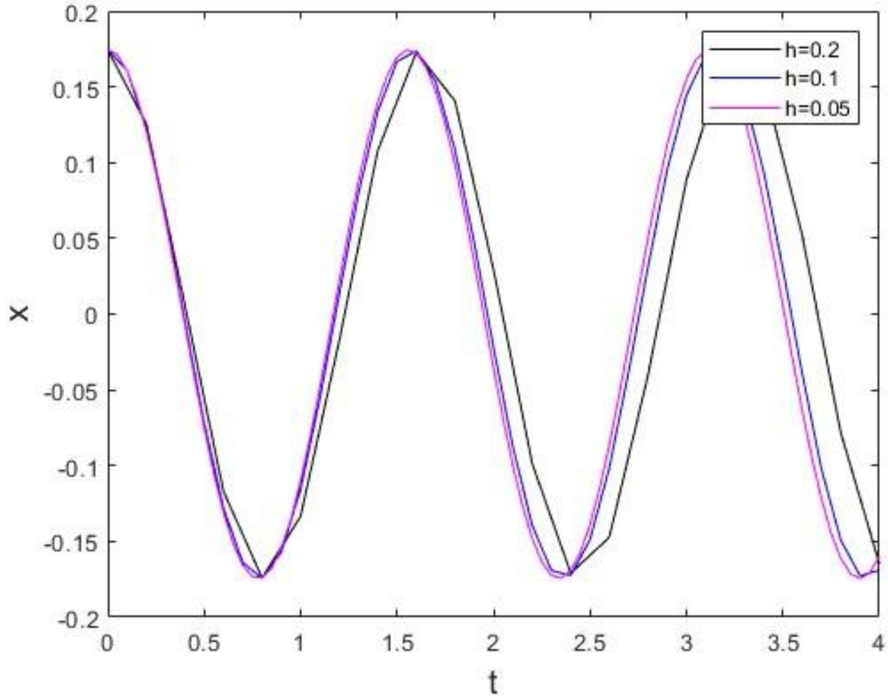


Fig: t vs θ

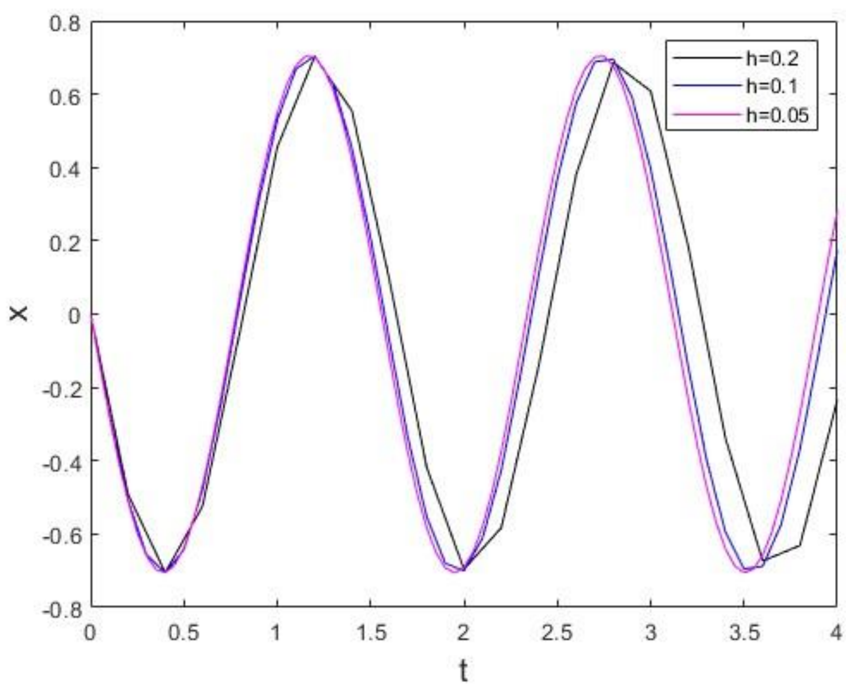


Fig: t vs dθ/dt

(iv) <u>Second Order Runge-Kutta Method:</u>

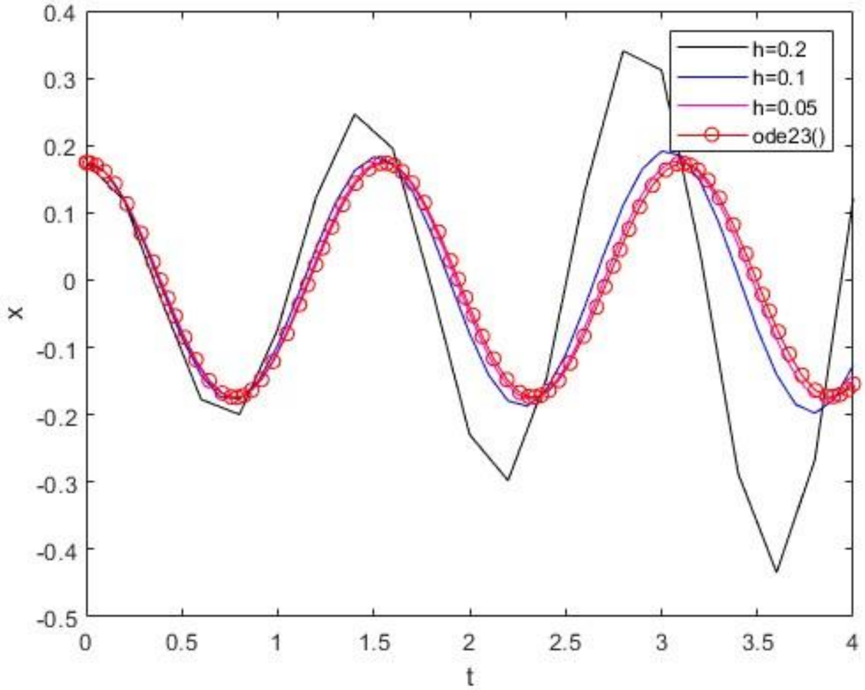


Fig: t vs θ

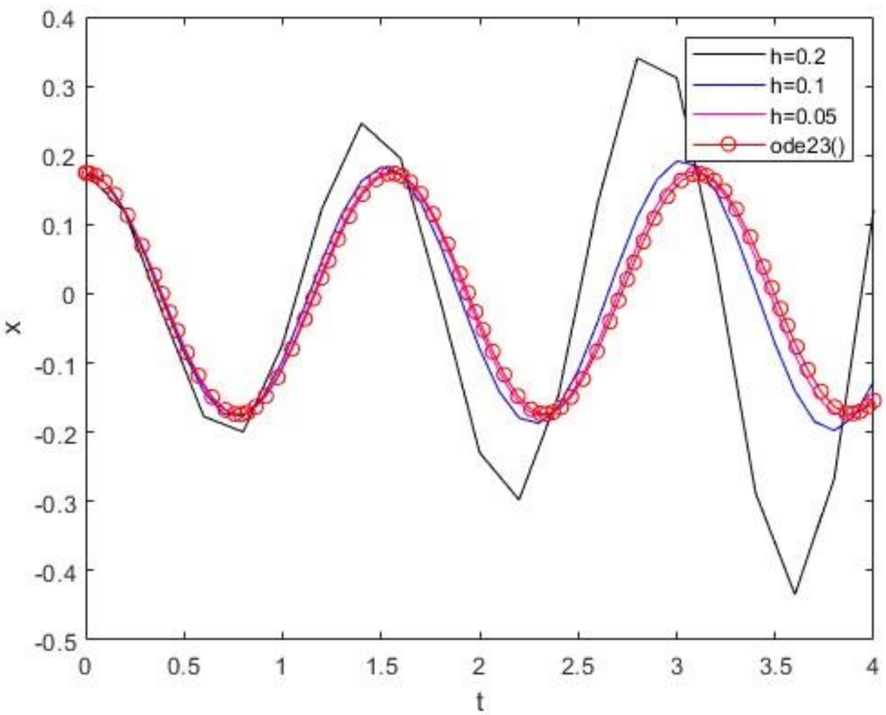


Fig: t vs dθ/dt

(v) Fourth Order Runge-Kutta Method:

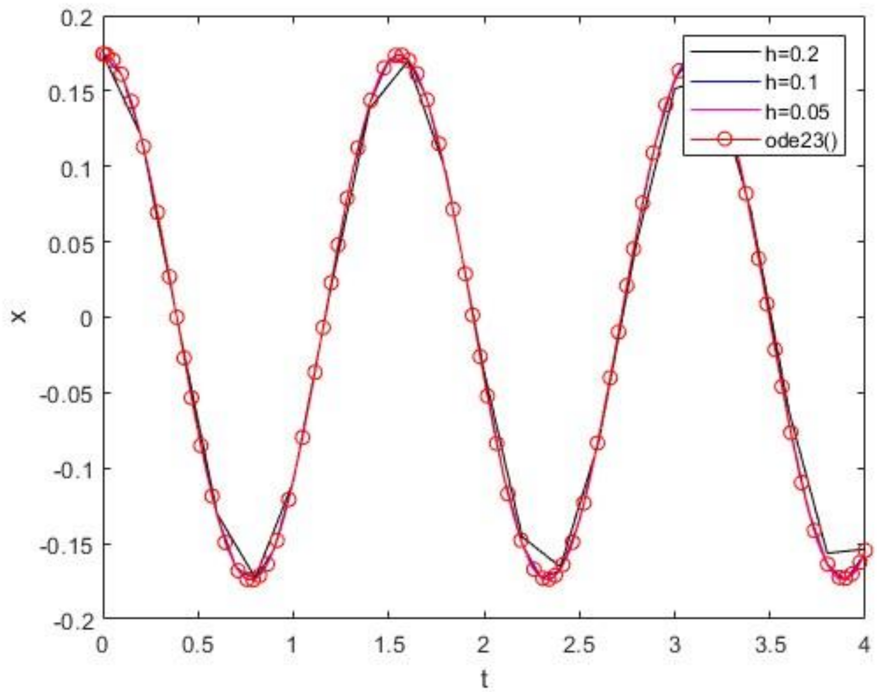


Fig: t vs θ

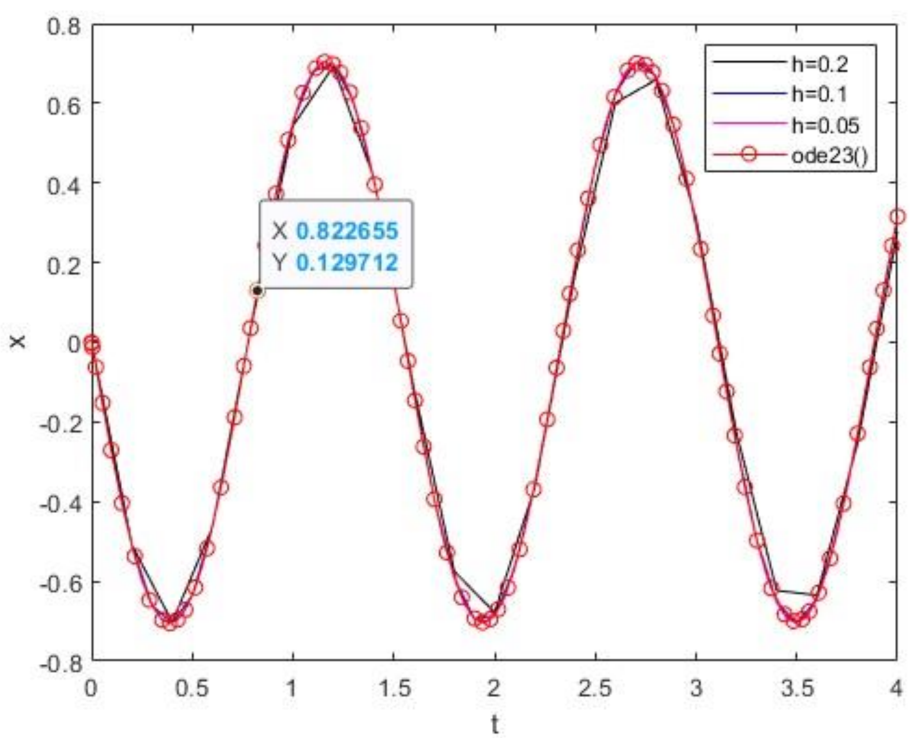


Fig: t vs dθ/dt

Analytical Results:

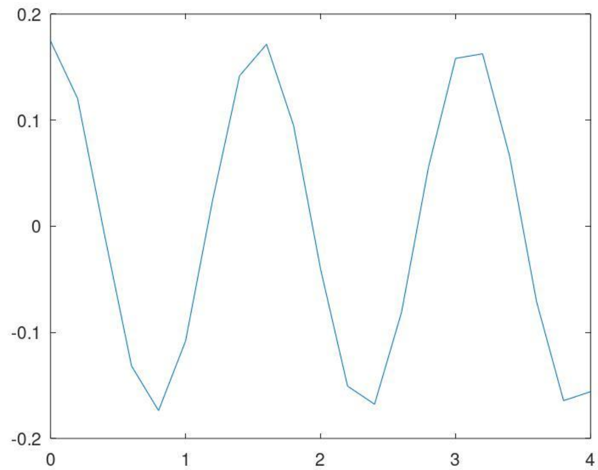


Fig: Analytical Results for h = 0.2

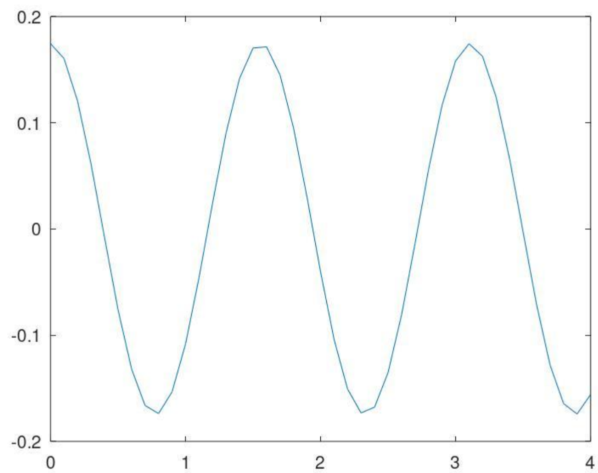


Fig: Analytical Results for h = 0.1

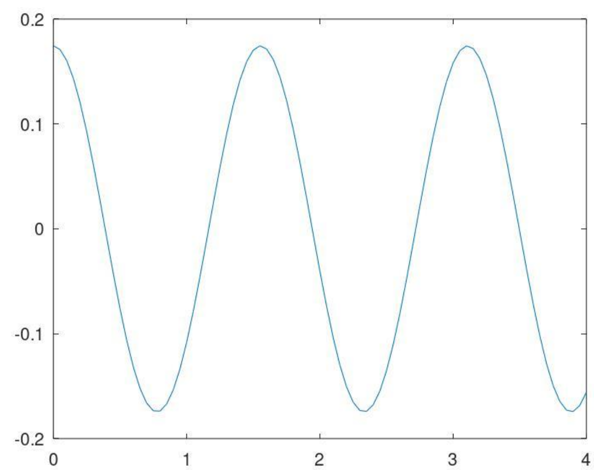
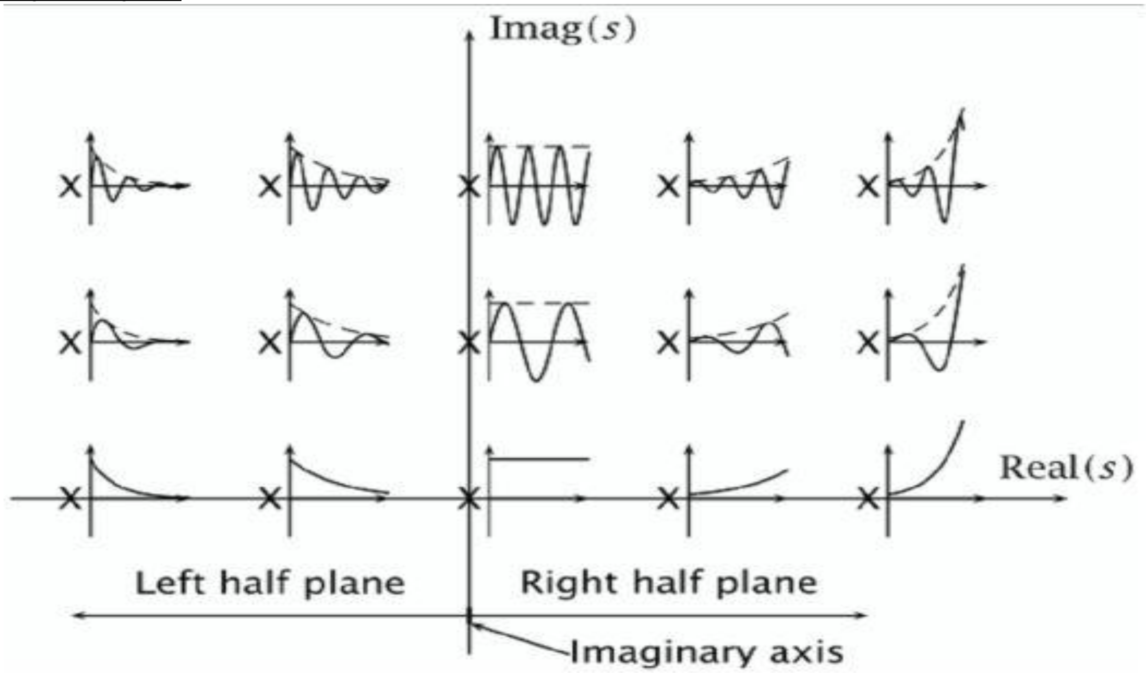


Fig: Analytical Results for h = 0.05

Stability Analysis:



	h=0.2		h=0.1		h=0.05		Ode23()	
	θ	**dθ/dt**	**θ**	**dθ/dt**	**θ**	**dθ/dt**	**θ**	**dθ/dt**
Explicit Euler Method	Unstable	Unstable	Unstable	Unstable	Marginally Stable	Marginally Stable	Marginally Stable	Marginally Stable
Implicit Euler Method	Marginally Stable Stabilizes Rapidly	Marginally Stable Stabilizes Rapidly	Marginally Stable Longer to Stabilize	Marginally Stable Longer to Stabilize	Marginally Stable Slow to Stabilize	Marginally Stable Slow to Stabilize	n/a	n/a
Crank-Nicolson Method	Marginally Stable	Marginally Stable	Marginally Stable	Marginally Stable	Marginally Stable	Marginally Stable	n/a	n/a
2nd Order Runge-Kutta Method	Unstable Slowly blows up	Unstable Slowly blows up	Marginally Stable Less accuracy	Marginally Stable Less accuracy	Marginally Stable Good Accuracy	Marginally Stable Good Accuracy	Marginally Stable	Marginally Stable
4th Order Runge-Kutta Method	Marginally Stable Less Accuracy around extremities	Marginally Stable Less Accuracy around extremities	Marginally Stable Good Accuracy	Marginally Stable Good Accuracy	Marginally Stable Good Accuracy	Marginally Stable Good Accuracy	Marginally Stable	Marginally Stable

We can summarize the question of stability in the form of a table above. By comparing the plots of numerical solutions obtained from MATLAB with figure of stability above we can easily determine which of our plots were stable. The details are discussed below.

Explicit Euler: The stability increases as the value of ‘h’ decreases as we can see from the figures. The built-in MATLAB function ode23() automatically chooses the step size ‘h’ to get the best results. As we can see from the plots for h = 0.2 is completely unstable, and ‘blows up’. The stability for h=0.1 is better than h=0.2 but still unstable. The stability of plots for h = 0.05 looks stable enough and close to the ode23() result. As we can see ode23() takes much smaller value for ‘h’.

Implicit Euler: Implicit methods are by definition always stable. The value of step size ‘h’ only matters for how quickly the solution will become stable. As we see can from the plots for h = 0.2, the curve reaches stability the quickest. As ‘h’ decreases the curves reaches stability comparatively slower, as seen for h=0.05. It doesn’t stabilize within the given time span.

Crank-Nicolson Method: This is also an implicit method so the stability is guaranteed. Also, the plot for crank-nicolson is much closer to analytical plots. From the plot we can see that the curves continuously oscillates. In this case we see the solutions are marginally stable and oscillating.

2^{nd} Order Runge-Kutta Method: The 2^{nd} order Runge-kutta method gives good results as the errors are less than explicit method. We can see that h=0.2 plot is unstable and blows up. But for h = 0.1, 0.05 gives relatively stable result and is close to the plot for ode23(). So, it is already better than explicit Euler but worse than 4^{th} order Runge-kutta.

4^{th} Order Runge-Kutta Method: The 4^{th} order Runge-Kutta method gives accurate results even for h=0.2. As we can see the plots for h=0.2, 0.1, 0.05 are all similar to the ode23() results and the analytical results. The plot for h=0.2 is a little rough when at extremities.

Analytical Results: The smaller the step size ‘h’ the smoother the curves are. So, for h=0.2 the curve is rough around the extremities while h=0.05 curve is really smooth. The analytical solution is free from errors. We can see 4^{th} order runge-kutta and ode23() results are similar to analytical results. So, this gives an idea about the accuracy of the 4^{th} order runge-kutta. In this case we see the solutions are marginally stable and oscillating.

Answer to Question No.(d)

(i) Explicit Euler Method:

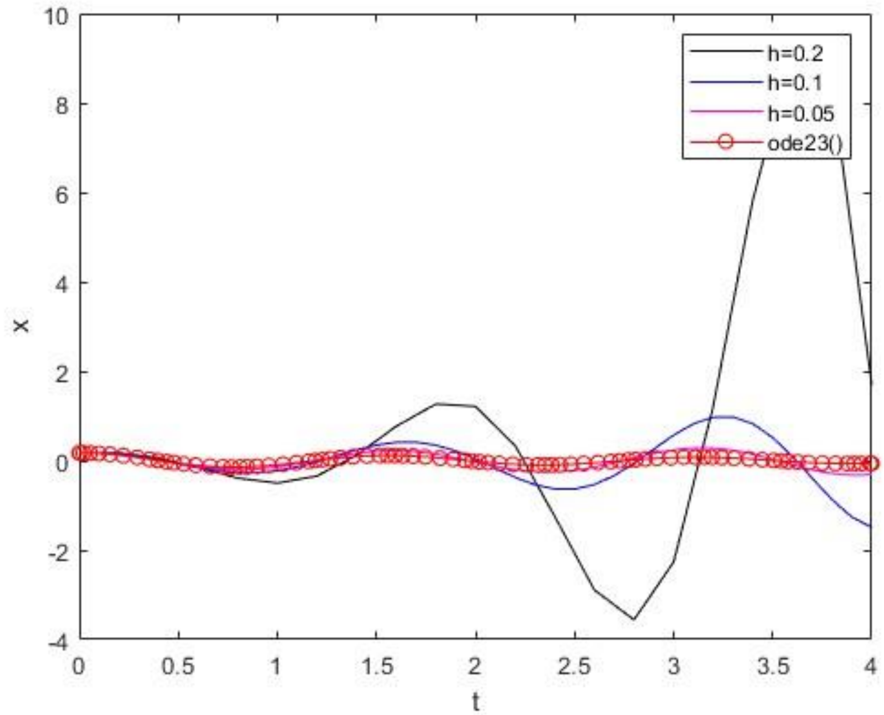


Fig: t vs θ

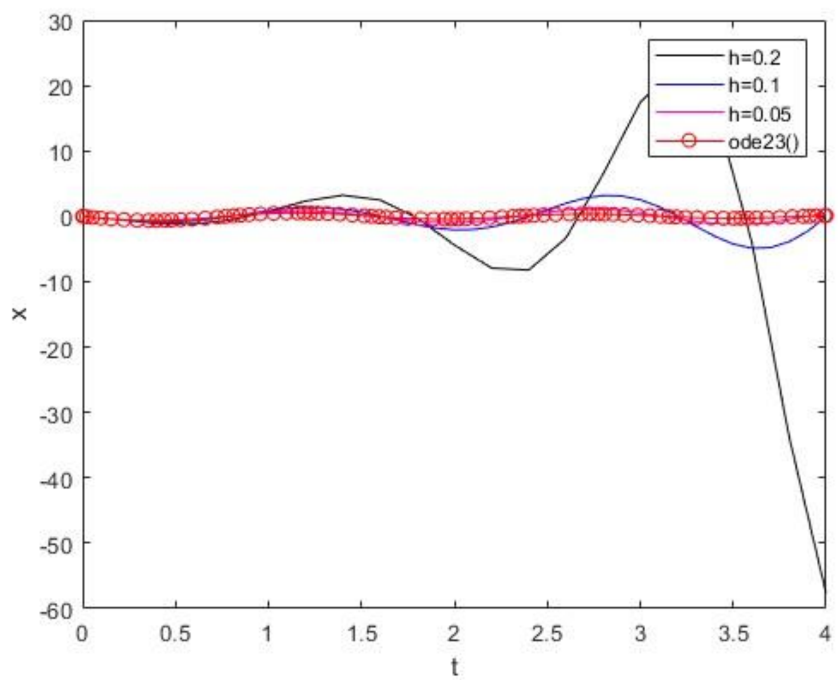


Fig: t vs dθ/dt

(ii) Implicit Euler Method:

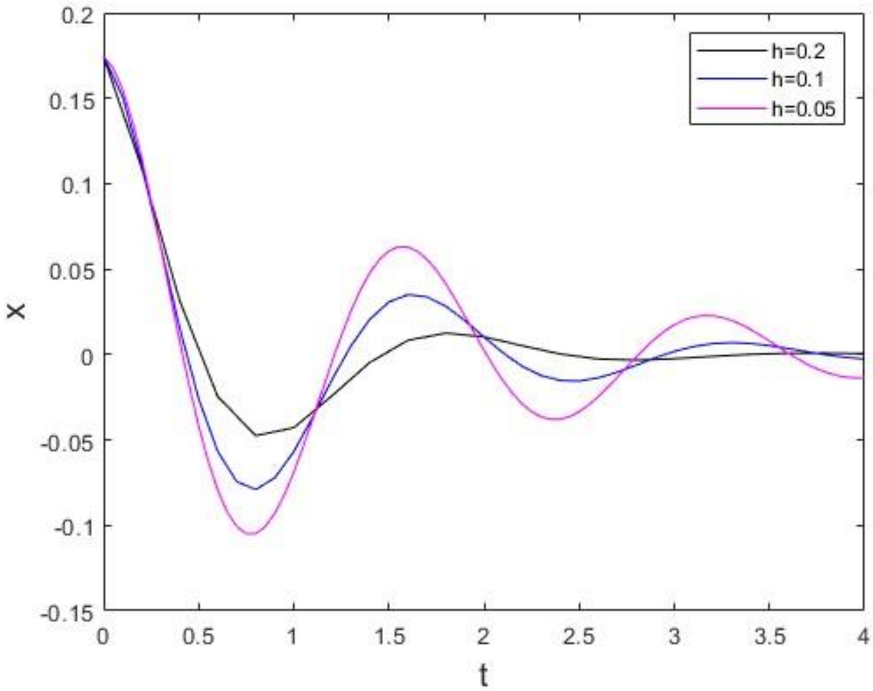


Fig: t vs θ

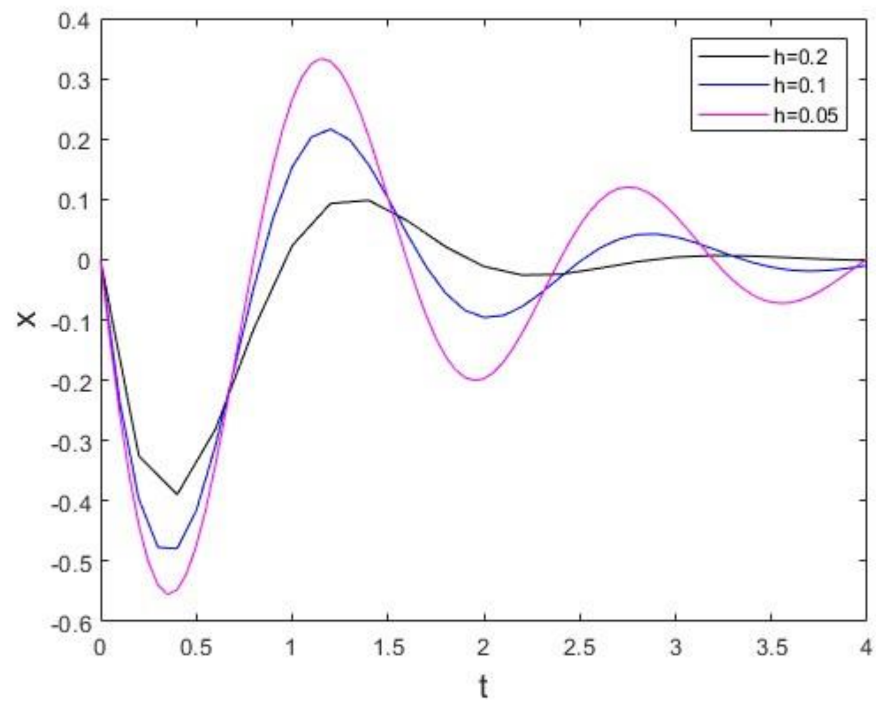


Fig: t vs dθ/dt

(iii) Crank-Nicolson Method:

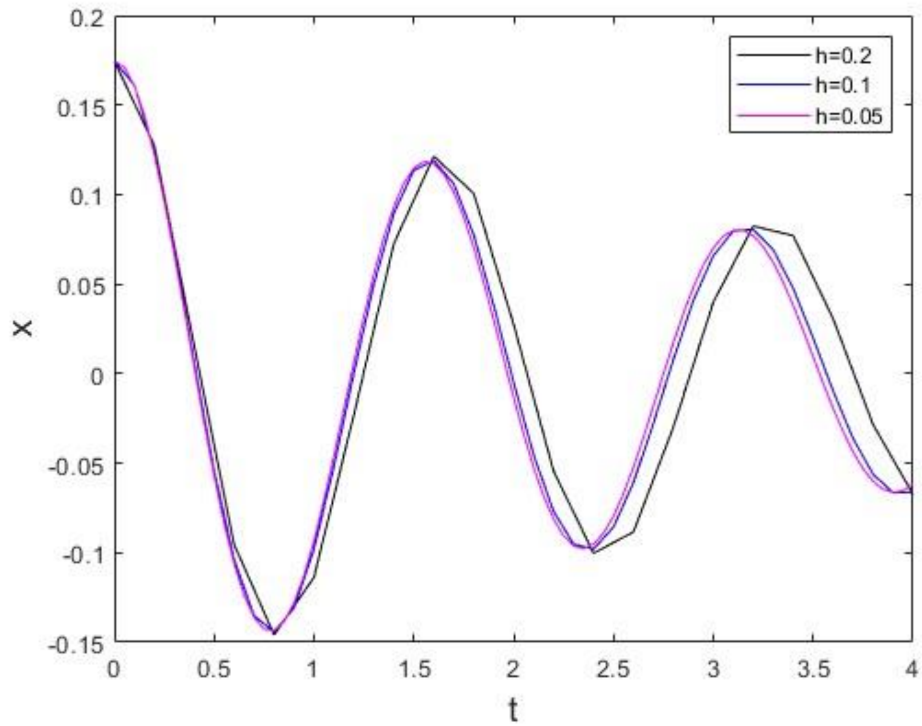


Fig: t vs θ

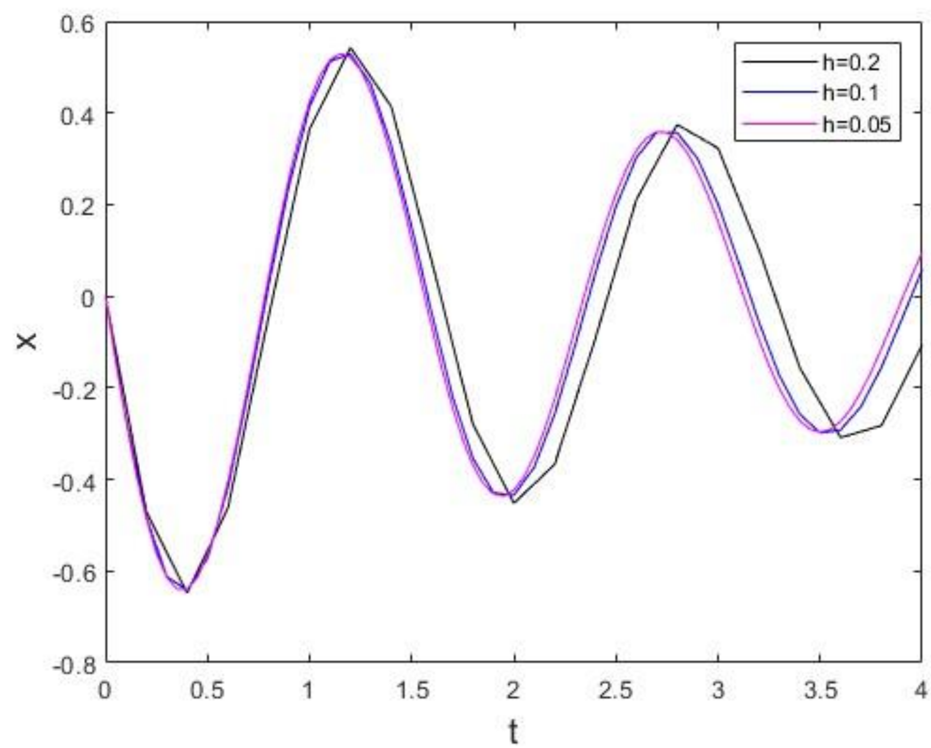


Fig: t vs dθ/dt

(iv) <u>Second Order Runge-Kutta Method:</u>

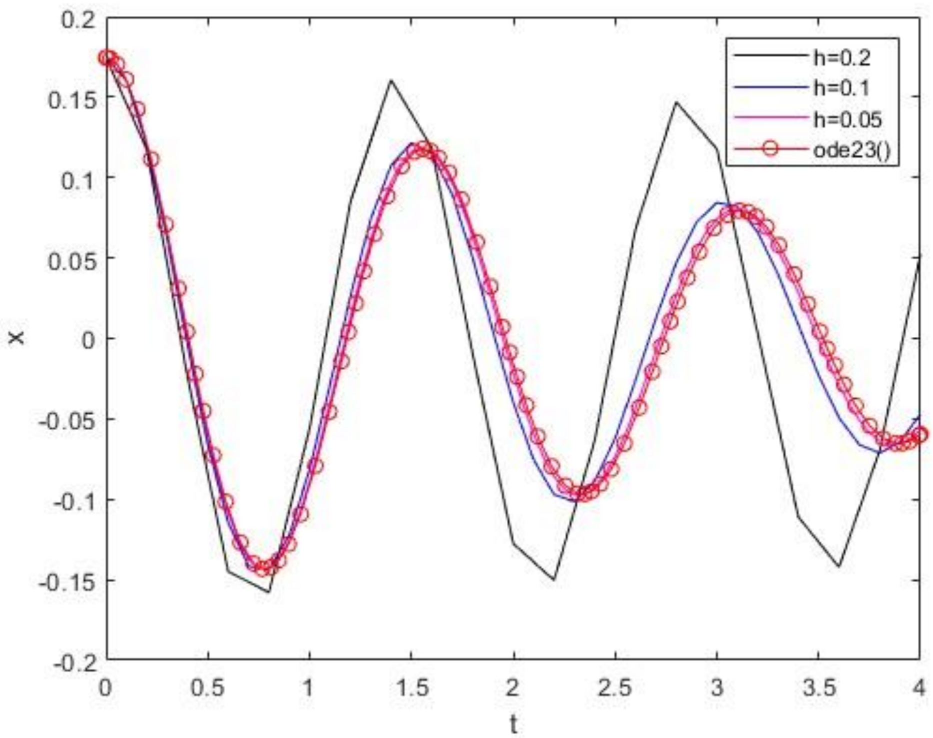


Fig: t vs θ

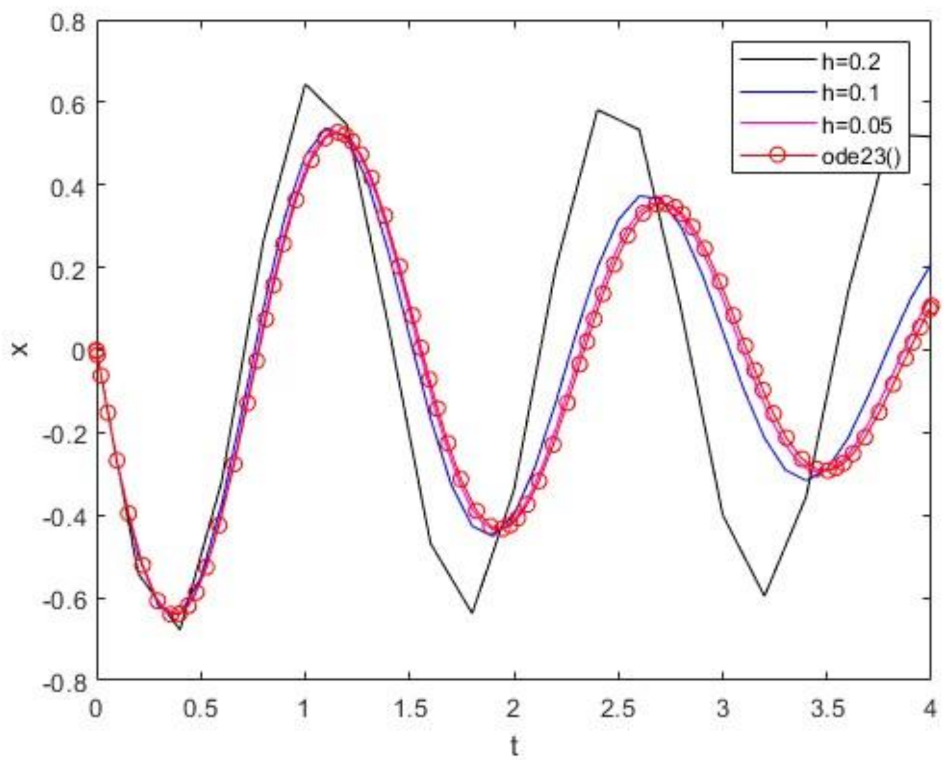


Fig: t vs dθ/dt

(v) <u>Fourth Order Runge-Kutta Method:</u>

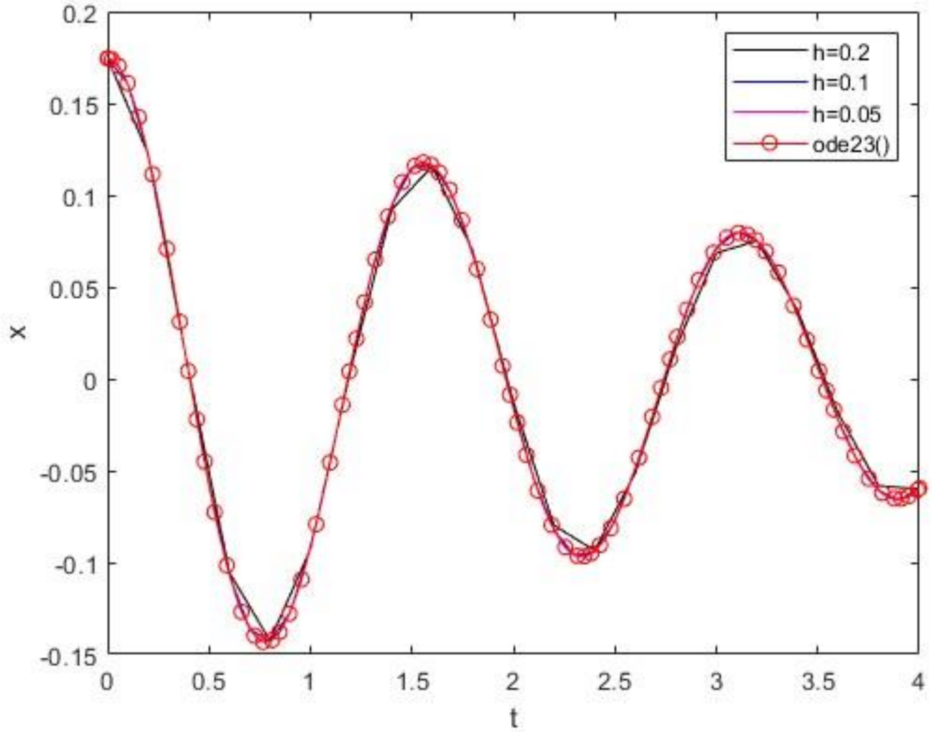


Fig: t vs θ

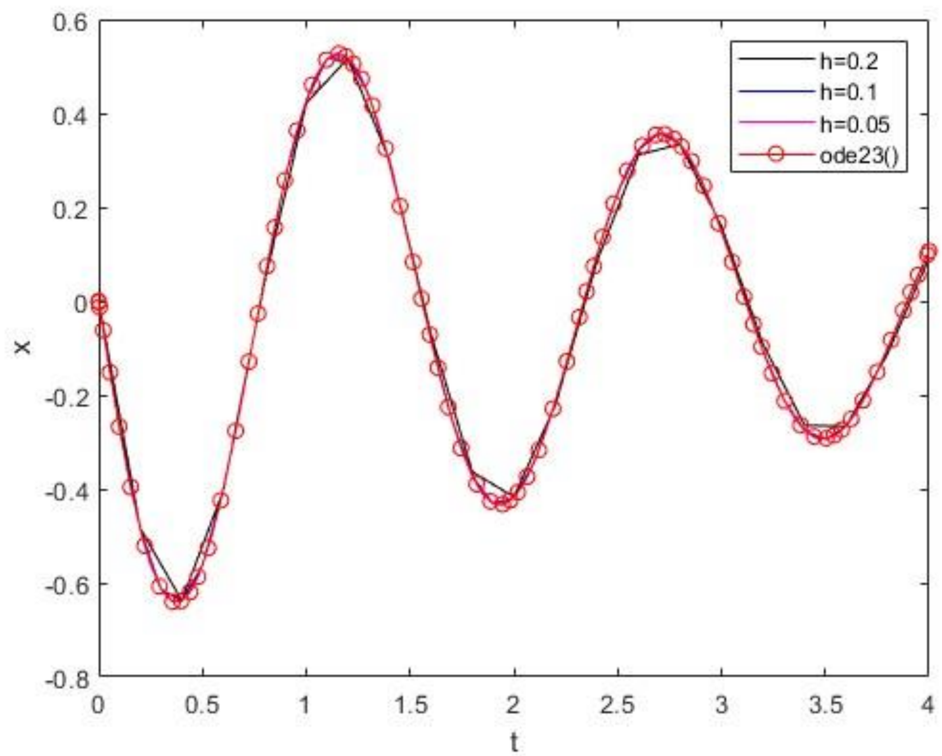


Fig: t vs dθ/dt

Analytical Results:

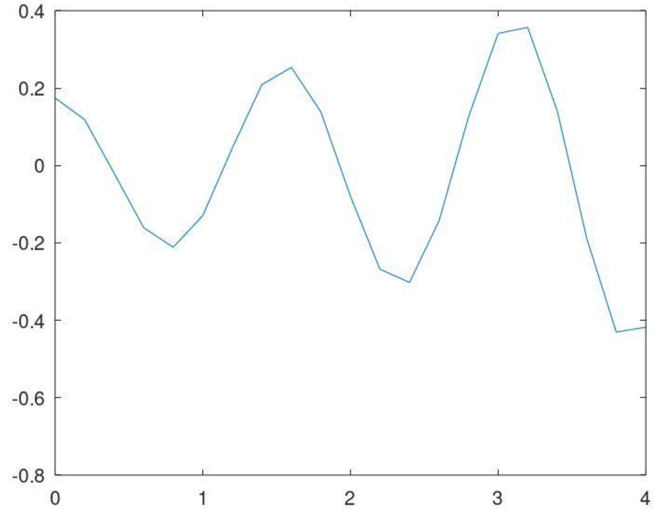


Fig: Analytical Results for h = 0.2

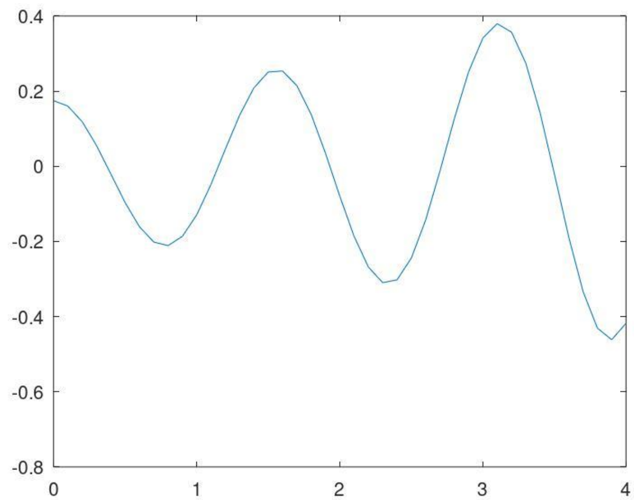


Fig: Analytical Results for h = 0.1

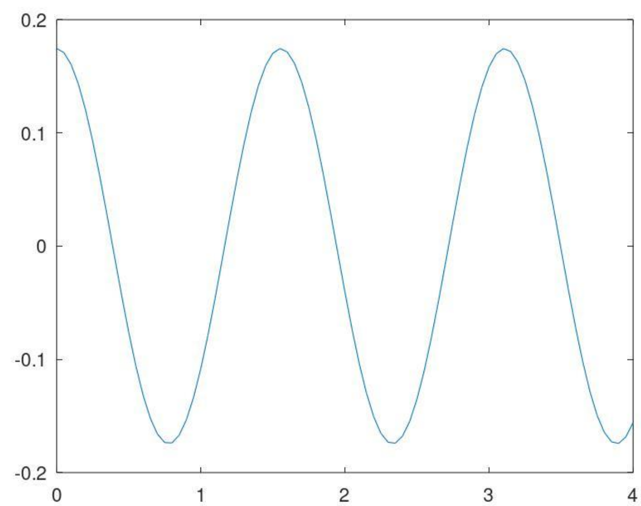
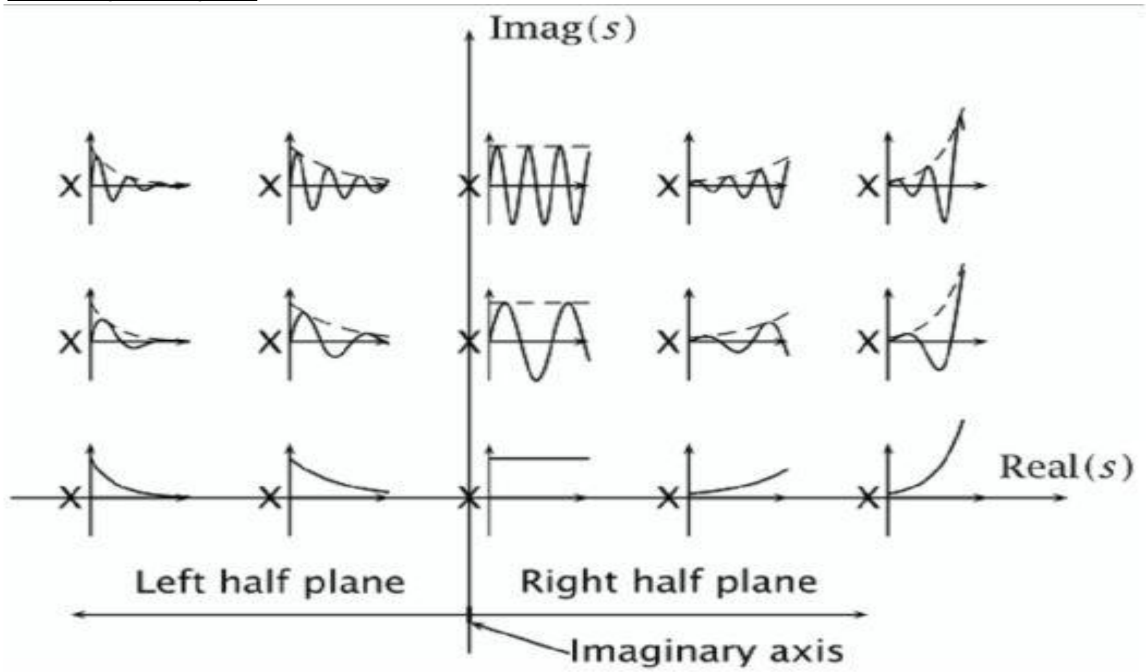


Fig: Analytical Results for h = 0.05

Stability Analysis:



	h=0.2		h=0.1		h=0.05		Ode23()	
	θ	dθ/dt	θ	dθ/dt	θ	dθ/dt	θ	dθ/dt
Explicit Euler Method	Unstable	Unstable	Unstable	Unstable	Marginally Stable	Marginally Stable	Marginally Stable	Marginally Stable
Implicit Euler Method	Marginally Stable Stabilizes Rapidly	Marginally Stable Stabilizes Rapidly	Marginally Stable Longer to Stabilize	Marginally Stable Longer to Stabilize	Marginally Stable Slow to Stabilize	Marginally Stable Slow to Stabilize	n/a	n/a
Crank-Nicolson Method	Stable	Stable	Stable	Stable	Stable	Stable	n/a	n/a
2nd Order Runge-Kutta Method	Marginally Stable Less accuracy	Marginally Stable Less accuracy	Stable Less Accuracy	Stable Less Accuracy	Stable Good Accuracy	Stable Good Accuracy	Stable	Stable
4th Order Runge-Kutta Method	Stable Less accuracy	Stable Less Accuracy	Stable	Stable	Stable	Stable	Stable	Stable

We can summarize the question of stability in the form of a table above. By comparing the plots of numerical solutions obtained from MATLAB with figure of stability above we can easily determine the stability of our plots. The details are discussed below.

Explicit Euler: This method has the worst stability out of all the numerical methods tested here. Lower the value of ‘h’, higher the stability for Euler method. We can see from the plots for h=0.2 the plot blows up. h=0.1 is also unstable. For h=0.05 and ode23() the plots are similar to analytical results. In this case we see the solutions are marginally stable and oscillating.

Implicit Euler: By definition implicit methods are stable. And as we can see from plots all the curves marginally stabilizes. For lower values of ‘h’ the stability reaches slower, which is opposite of the explicit methods. As we can see from the plot, for h=0.2 the solution stabilizes rapidly and for h=0.05 the solution stabilizes slowly.

Crank-Nicolson Method: As an implicit method, the stability is guaranteed. The plots are similar to 4^{th} order Runge-kutta plots which has high accuracy and stability. The values of step size ‘h’ didn’t really matter in this case. By comparing with the stability figure above, we see that the plots oscillates and will eventually stabilize.

2^{nd} order Runge-Kutta: The stability is less than 4^{th} order runge-kutta and Nicolson method. We can see that for h=0.2 the stability is marginal. But for h=0.1, 0.05 the plots are similar to ode23() results, they are oscillating and will eventually stabilize. This method has moderate accuracy.

4^{th} order Runge-Kutta: Most accurate method and gives the best stability out of all explicit methods. We can see that for h=0.2,0.1,0.05 the stability is similar to ode23() plot and analytical plot. The solution oscillates and will eventually stabilize.
We can see from all of the plots that the curves start to stabilizes around t = 2.5

Analytical Solution: The analytical solution is error free and decreasing the values of ‘h’ smoothens out the plot. Like for h=0.2 the accuracy is a little rough while for h=0.05 the plot smooths out. For h=0.2 the results is unstable and accuracy is low. For h=0.1 the plot is still unstable. But for h=0.05 the plot oscillates constantly and is marginally stable.