Assignment-1

ME6185 – Advanced Numerical Analysis

Submitted by: Abrar Salekin Raiyan

Submitted to: Dr. Sheikh Mohammad Shavik

Gauss – Elimination Method with Partial Pivoting:

Gauss - Elimination (Partial Pivoting)

$$9x_1 + 4x_2 + x_3 = -17$$
$$x_1 - 2x_2 - 6x_3 = 14$$
$$x_1 + 6x_2 + 0x_3 = 4$$

Augmented matrix →
$$\left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & -2 & -6 & 14 \\ 1 & 6 & 0 & 4 \end{array}\right]$$

$$R_2' = R_2 - R_1 \times \frac{R_{21}}{R_{11}}$$
$$R_2' = R_2 - R_1 \times \frac{1}{9}$$

$$= \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 0 & -\frac{22}{9} & -\frac{55}{9} & \frac{143}{9} \\ 1 & 6 & 0 & 4 \end{array}\right]$$

$$R_3' = R_3 - R_1 \times \frac{R_{31}}{R_{11}}$$
$$\frac{R_{31}}{R_{11}} = \frac{1}{9}$$

$$= \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 0 & -\frac{22}{9} & -\frac{55}{9} & \frac{143}{9} \\ 0 & \frac{50}{9} & -\frac{1}{9} & \frac{59}{9} \end{array}\right]$$

$$R_3' = R_3 - R_2 \times \frac{R_{32}}{R_{22}}$$
$$= -\frac{22}{50}$$
$$\frac{R_{32}}{R_{22}} = \frac{\frac{50}{9}}{-\frac{22}{9}} = \frac{-50}{22}$$

$$= \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 0 & -22/9 & -55/9 & \frac{143}{9} \\ 0 & 0 & -14 & 42 \end{array}\right]$$

Back Ward Substitution

So, $-14\,x_3 = 42$

$$x_3 = -3$$

& $-\frac{22}{9} x_2 - \frac{55}{9} x_3 = \frac{143}{9}$

$\Rightarrow \quad -22 x_2 - 55 \times (-3) = 143$

$\Rightarrow \quad x_2 = 1$

& $9x_1 + 4x_2 + x_3 = -17$

$\Rightarrow \quad x_1 = \frac{-17 - 4 \times 1 - (-3)}{9} = -2$

So the exact Solution by Gauss-Elimination

$$\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

Answer from MATLAB:

```
>> Assignment_4methods
The required solution is:

ans_gauss_elimination =

   -2.0000
    1.0000
   -3.0000

   .
```

The exact solution by solving the equations manually is : [X1, X2, X3] = [-2, 1, -3]

It is the same as the result obtained from MATLAB implementation of Gauss-Elimination.

Gauss – Jordan Elimination Method with Partial Pivoting:

Gauss - Jordan Elimination (Partial Pivoting):

$$9x_1 + 4x_2 + x_3 = -17$$
$$x_1 - 2x_2 - 6x_3 = 14$$
$$x_1 + 6x_2 + 0x_3 = 4$$

Augmented Matrix $\rightarrow$ $\left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & -2 & -6 & 14 \\ 1 & 6 & 0 & 4 \end{array}\right]$

Reduced Low Echelon form $\rightarrow$ $\left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 0 & -\frac{22}{9} & -\frac{55}{9} & \frac{143}{9} \\ 0 & \frac{50}{9} & -\frac{1}{9} & \frac{53}{9} \end{array}\right]$

① $r_2' = r_2 - r_1 \times \left(\frac{r_{21}}{r_{11}}\right)_{1/9}$

$r_3' = r_3 - r_1 \times \left(\frac{r_{31}}{r_{11}}\right)_{1/9}$

② $r_3' = r_3 - r_2 \times \left(\frac{r_{32}}{r_{22}}\right)$ $\left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 0 & -\frac{22}{9} & -\frac{55}{9} & \frac{143}{9} \\ 0 & 0 & -14 & 42 \end{array}\right]$

$\frac{r_{32}}{r_{22}} = -\frac{50}{9} \Big/ \frac{22}{9} = \frac{-50}{22}$

③ $r_1' = r_1 - r_3 \times \frac{r_{13}}{r_{33}}$ $\left[\begin{array}{ccc|c} 9 & 4 & 0 & -14 \\ 0 & -\frac{22}{9} & 0 & -\frac{22}{9} \\ 0 & 0 & -14 & 42 \end{array}\right]$

$\frac{r_{13}}{r_{33}} = \frac{-1}{14}$ $\quad \frac{r_{23}}{r_{33}} = \frac{55}{14 \times 9}$

$r_2' = r_2 - r_3 \times \frac{r_{23}}{r_{33}}$

(4) $r_1' = r_1 - r_2 \times \frac{r_{12}}{r_{22}}$

$$\frac{r_{12}}{r_{22}} = \frac{4}{-\frac{22}{9}} = -\frac{4\times 9}{22}$$

$$\left[\begin{array}{ccc|c} 9 & 0 & 0 & -18 \\ 0 & -\frac{22}{9} & 0 & -\frac{22}{9} \\ 0 & 0 & -14 & 42 \end{array}\right]$$

(5) $r_1' = r_1/9$

$r_2' = r_2/(-22/9)$

$r_3' = r_3/-14$

[Normalizing left side of the Augmented Matrix]

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

So, exact solution by Gauss - Jordan method is $\rightarrow$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

Answer from MATLAB:

```
>> Assignment_4methods
The required solution is:

ans_jordan_elimination =

   -2.0000
    1.0000
   -3.0000
```

The exact solution by solving the equations manually is : [X1, X2, X3] = [-2, 1, -3]

It is the same as the result obtained from MATLAB implementation of Gauss-Jordan-Elimination.

Jacobi Method:

Jacobi Method

$$9x_1 + 4x_2 + x_3 = -17$$
$$x_1 - 2x_2 - 6x_3 = 14$$
$$x_1 + 6x_2 + 0x_3 = 4$$

$$\cancel{x_1 = \frac{-17 - 4x_2 - x_3}{9}}$$

$$\cancel{x_2 = 14}$$

Reordering the equations →
(So that diagonally dominant)

$$9x_1 + 4x_2 + x_3 = -17$$
$$x_1 + 6x_2 + 0x_3 = 4$$
$$x_1 - 2x_2 - 6x_3 = 14$$

Now,

$$x_1 = \frac{-17 - 4x_2 - x_3}{9}$$

$$x_2 = \frac{4 - x_1 - 0x_3}{6}$$

$$x_3 = -\frac{14 + 2x_2 - x_1}{6}$$

Taking initial Guess of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

	Initial	1st	2nd	3rd
x_1	0	$-\frac{17}{9}$	$\frac{-17-4\left(\frac{2}{3}\right)-\left(-\frac{7}{3}\right)}{9} = -\frac{52}{27}$	$-\frac{325}{162} = -2.006$
x_2	0	$\frac{2}{3}$	$\frac{4-\left(-\frac{17}{9}\right)}{6} = \frac{53}{54}$	$\frac{80}{81} = 0.98765$
x_3	0	$-\frac{7}{3}$	$-\frac{14+2\left(\frac{2}{3}\right)-\left(-\frac{17}{9}\right)}{6} = -\frac{155}{54}$	$-\frac{161}{54} = -2.98148$

so after 3 iteration we get $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2.006 \\ 0.98765 \\ -2.98148 \end{bmatrix}$

Tolerance, $|\varepsilon_{a,i}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} \right| \times 100\% < \varepsilon_s$

$$\varepsilon_{a,1} = \frac{x_1^{(3)} - x_1^{(2)}}{x_1^{(3)}} = \frac{-2.006-(-1.9259)}{-2.006}$$

$$= 0.039930 \times 100\%$$

$$= 3.99\% \approx 4\%$$

$$\varepsilon_{a,2} = 0.625\%$$

$$\varepsilon_{a,3} = 3.7267\%$$

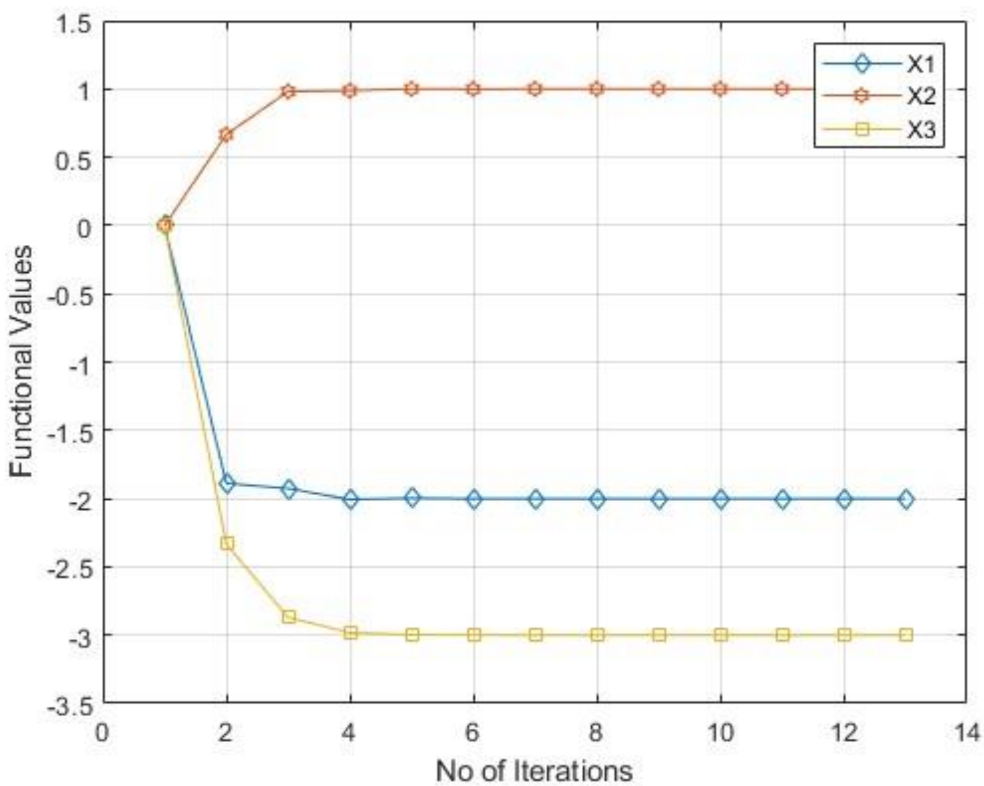
Answer from MATLAB:

```
>> Assignment_4methods
The solution by Jacobi Method after 13 itertion is
ans_jacobi =

   -2.0000
    1.0000
   -3.0000
```

From the MATLAB implementation of Jacobi Method, we get the exact result after just 13 iterations. The exact result from Jacobi matches the exact result obtained from both Gauss-Elimination and Gauss-Jordan Elimination.

Convergence Graph:



Values from each iteration:

```
values =

  Columns 1 through 9

         0   -1.8889   -1.9259   -2.0062   -1.9966   -2.0008   -1.9998   -2.0001   -2.0000
         0    0.6667    0.9815    0.9877    1.0010    0.9994    1.0001    1.0000    1.0000
         0   -2.3333   -2.8704   -2.9815   -2.9969   -2.9998   -2.9999   -3.0000   -3.0000

  Columns 10 through 13

   -2.0000   -2.0000   -2.0000   -2.0000
    1.0000    1.0000    1.0000    1.0000
   -3.0000   -3.0000   -3.0000   -3.0000
```

The results from manual iteration (3-iteration):

	Initial	1st	2nd	3nd
x_1	0	$-\frac{17}{9}$	$\frac{-17-4\left(\frac{2}{3}\right)-\left(-\frac{7}{3}\right)}{9} = -\frac{52}{27}$	$-\frac{325}{162} = -2.006$
x_2	0	$\frac{2}{3}$	$\frac{4-\left(-\frac{17}{9}\right)}{6} = \frac{53}{54}$	$\frac{80}{81} = 0.98765$
x_3	0	$-\frac{7}{3}$	$-\frac{14+2\left(\frac{2}{3}\right)-\left(-\frac{17}{9}\right)}{6} = -\frac{155}{54}$	$-\frac{161}{54} = -2.98148$

We see from the step-by-step iteration results from both MATLAB implementation and the manual iteration that the step by step results are similar for both. So the MATLAB implementation is correct and perfectly captures the Jacobi Iterative method.

Gauss – Seidel Method:

Gauss - Seidel :

$$9x_1 + 4x_2 + x_3 = -17$$
$$x_1 - 2x_2 - 6x_3 = 14$$
$$x_1 + 6x_2 + 0x_3 = 4$$

$$\rightarrow \quad 9x_1 + 4x_2 + x_3 = -17$$
$$x_1 + 6x_2 + 0x_3 = 4$$
$$x_1 - 2x_2 - 6x_3 = 14$$

$$x_1 = \frac{-17 - 4x_2 - x_3}{9}$$

$$x_2 = \frac{4 - x_1 - 0x_3}{6}$$

$$x_3 = -\frac{14 + 2x_2 - x_1}{6}$$

Initial Guess,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

	Initial	1st	2nd	3rd
x_1	0	$\frac{-17-0-0}{9} = -\frac{17}{9}$	$\frac{-17 - 4\frac{53}{54} + \frac{241}{81}}{9} = -\frac{1454}{729}$	-1.999738
x_2	0	$\frac{4 - \left(\frac{-17}{9}\right) - 0}{6} = \frac{53}{54}$	$\frac{4 - \left(\frac{-1454}{729}\right)}{6} = \frac{2185}{2187}$	0.999956
x_3	0	$\frac{14 + 2\frac{53}{54} + \frac{17}{9}}{-6} = -\frac{241}{81}$	$\frac{14 + 2\frac{2185}{2187} + \frac{1454}{729}}{-6} = -2.9987$	-2.9999417

Error, $|\varepsilon_{a,i}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} \right| \times 100\%$

$$\varepsilon_{a,1} = \frac{-1.999738 - \left(-\frac{1454}{729}\right)}{-1.999738} = 0.261\%$$

$$\varepsilon_{a,2} = 0.087\%$$

$$\varepsilon_{a,3} = 0.0414\%$$

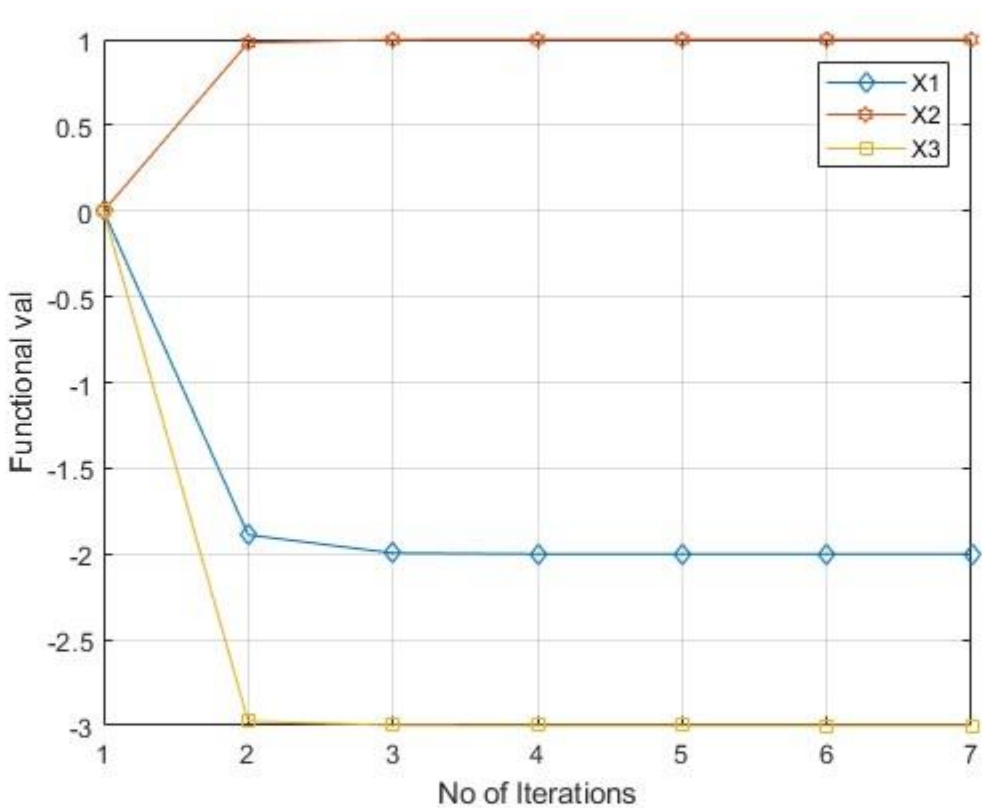
<u>Answer From MATLAB:</u>

```
>> Assignment_4methods
The solution by Gauss-Seidel Method after 6 itertion is
ans_seidel =

   -2.0000
    1.0000
   -3.0000
```

From the MATLAB implementation of Gauss-Seidel Method, we get the exact result after just 6 iterations. The exact result from Gauss-Seidel matches the exact result obtained from both Gauss-Elimination and Gauss-Jordan Elimination.

<u>Convergence Graph:</u>



Values from each iteration:

```
val =

         0   -1.8889   -1.9945   -1.9997   -2.0000   -2.0000   -2.0000
         0    0.9815    0.9991    1.0000    1.0000    1.0000    1.0000
         0   -2.9753   -2.9988   -2.9999   -3.0000   -3.0000   -3.0000
```

The results from manual iteration (3-steps):

	Initial	1st	2nd	3rd
x_1	0	$\frac{-17-0-0}{9} = -\frac{17}{9}$	$\frac{-17-4\frac{53}{54}+\frac{241}{81}}{9} = -\frac{1454}{729}$	-1.999738
x_2	0	$\frac{4-\left(\frac{-17}{9}\right)-0}{6} = \frac{53}{54}$	$\frac{4-\left(\frac{-1454}{729}\right)}{6} = \frac{2195}{2187}$	0.999956
x_3	0	$\frac{14+2\frac{53}{54}+\frac{17}{9}}{-6} = -\frac{241}{81}$	$\frac{14+2\frac{2185}{2187}+\frac{1454}{729}}{-6} = -2.99987$	-2.9999417

We see from the step-by-step iteration results from both MATLAB implementation and the manual iteration that the step-by-step results are similar for both. So, the MATLAB implementation is correct and perfectly captures the Gauss-Seidel Iterative method.

Final Remark:

The Gauss Elimination and Gauss-Jordan Elimination both are exact methods. The implementation of Gauss-Jordan is more efficient in MATLAB since there is built in library for solving matrix's into reduced echelon form as "rref". This method is showcased in the MATLAB file as gauss_jordan_easy.

The results from every method match-up. So the MATLAB implementation is correct.

For the iterative methods Jacobi gives results in 13 iterations and Gauss-Seidel gives results in 6 iterations. So, implementation of Gauss-Seidel is more efficient for iterative methods.