Morphological Multi-scale Decomposition and efficient representations with Auto-Encoders

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Internship Defense - 18 septembre 2018

Agenda

01. Introduction

02. Part-based Representation using Non-Negative Matrix Factorization

03. Part-based Representation using Auto-Encoders

04. Using a Deeper Architecture

05. Conclusion

01 - Introduction

Representation Learning and Part-Based representation

Representation Learning:

- Learning an underlying structure/process explaining the M input images $\mathbf{x}^{(i)} \in E^N, i \in [1, M]$ (of N pixels), that can somehow be represented as a set of latent features $\mathbf{h}^{(i)} \in E^k, i \in [1, M]$ in a space of dimension k
- If the data points live on a manifold of lesser dimension than the original space: $k < N$

Sparse coding and dictionary learning:

- The input images is assumed to be well represented as a weighted linear combination of a few elements from a dictionary, called the **atom images,** $\mathbf{w}_j \in E^N, j \in [1, k]$:

$$\forall i \in [1, M], \mathbf{x}^{(i)} \approx \sum_{j=1}^{k} h_{i,j}\mathbf{w}_j = \mathbf{h}^{(i)}\mathbf{W} = \hat{\mathbf{x}}^{(i)}$$

Part-based representation:

- Introduced by Lee and Seung in their 1999 work about NMF: atom images representing localized features corresponding with intuitive notions of the parts of the input image family.

Max-Approximation to Morphological Operators

“*Sparse mathematical morphology using non-negative matrix factorization*” , Angulo, Velasco-Forero 2017: Exploring how image sparse representations can be useful to efficiently calculate approximations to morphological operators, applying the operators only to the reduced set of atoms of the representation rather than to whole set of images.

Sparse Max-Approximation to gray-level dilation and erosion:

$$\forall i \in [1, M], D_{SE}(\mathbf{x}^{(i)}) = \sum_{j=1}^{k} h_{i,j}\delta_{SE}(\mathbf{w}_j)$$

$$E_{SE}(\mathbf{x}^{(i)}) = \sum_{j=1}^{k} h_{i,j}\varepsilon_{SE}(\mathbf{w}_j) = \sum_{j=1}^{k} h_{i,j}n\left(\delta_{SE}(n(\mathbf{w}_j))\right)$$

Motivation for Non-Negative and Sparse representation:

$$\forall i \in [1, M], \forall (j, l) \in [1, k]^2, h_{i,j}\mathbf{w}_j \bigwedge h_{i,l}\mathbf{w}_l \approx 0$$

$$\Longrightarrow \forall i \in [1, M], \bigvee_{j \in [1,k]} h_{i,j}\mathbf{w}_j \approx \sum_{j=1}^{k} h_{i,j}\mathbf{w}_j$$

$$\Longrightarrow \forall i \in [1, M], D_{SE}(\mathbf{x}^{(i)}) \approx \delta_{SE}(\mathbf{x}^{(i)})$$

Objectives and Motivations of the Internship

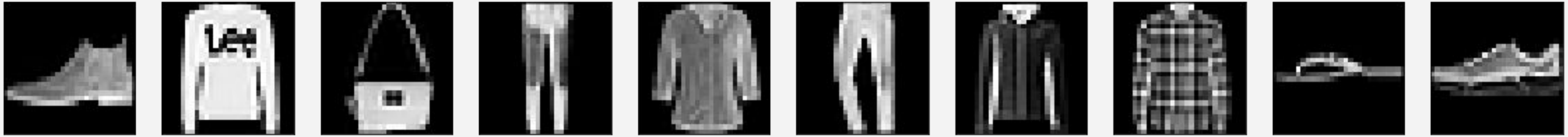
Using Neural Networks to learn a non-negative and sparse part-based representation:

- No need to re-train the model to encode new, previously unseen, images, unlike NMF.
- Ability to approximate the application of various morphological operators (dilations, erosions, openings, closings, morphological gradient, black top-hat, etc.) to an unlimited number of images by applying these operators only to the k atom images.
- Ability to capture complex and hierarchical relationships, similar to the ones present in the human's brain visual cortex.
- **Universal approximator theorem:** a feed-forward neural network with at least one hidden layer can represent an approximation of any function (within a broad class) to an arbitrary degree of accuracy, provided that it has enough hidden units.

The most intuitive and common way to perform representation learning in the Deep Learning paradigm is to use Auto-Encoders.

Evaluation and Data of the Proposed Models

The Fashion-MNIST database of images:



Evaluation criteria of the learned representation of a test set of images not used to train the model (except for the NMF):

- **Approximation error of the representation**: mean-squared error between the original input images and their approximation by the learned representation
- **Max-approximation error to the dilation** by a disk of radius 1: mean-squared error between the max-approximation to the dilation and the dilation of the original input images
- **Sparsity of the encoding**, measured using the metric introduced by Hoyer (2004)
- **Classification Accuracy** of a linear Support-Vector Machine, taking as input the encoding of the images.

02 - Non-Negative Matrix Factorization

General Presentation

"***Learning the parts of objects by non-negative matrix factorization***", **Lee and Seung, 1999:**

- Matrix factorization algorithm:

  $$\hat{\mathbf{X}} = \mathbf{HW} \approx \mathbf{X}$$

  with $\mathbf{X} \in E^{M \times N}$ the **data matrix** containing the M images of N pixels, as row vectors

  $\mathbf{W} \in E^{k \times N}$ the **dictionary matrix**, containing the k atom images as row vectors

  $\mathbf{H} \in E^{M \times k}$ the **encoding matrix**, containing the representation of each of the images as row vectors

- Proven to actually recover the parts of the images if the data set of images is a **separable factorial articulation family:**
  - Each image actually generated by a linear combination of positive atom images associated with non-negative weight
  - All atom images have separated supports.
  - All different combinations of parts are exhaustively sampled in the data set of images.

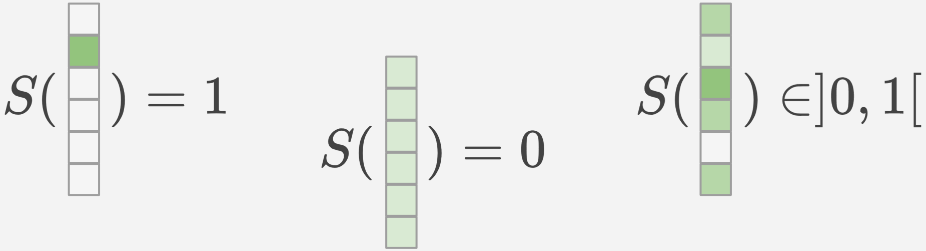
Addition of sparsity constraints (Hoyer 2004)

“*Non-negative matrix factorization with sparseness constraints*”, **Hoyer, 2004:**

- Enforcing sparsity of the encoding and/or of the atoms of the NMF representation: most coefficients taking values close to zero, while only a few take significantly non-zero values.

Sparsity measure of vector $\mathbf{v} \in E^d$:

$$S(\mathbf{v}) = \frac{\sqrt{d} - \frac{||\mathbf{v}||_1}{||\mathbf{v}||_2}}{\sqrt{d}-1} \in [0, 1]$$

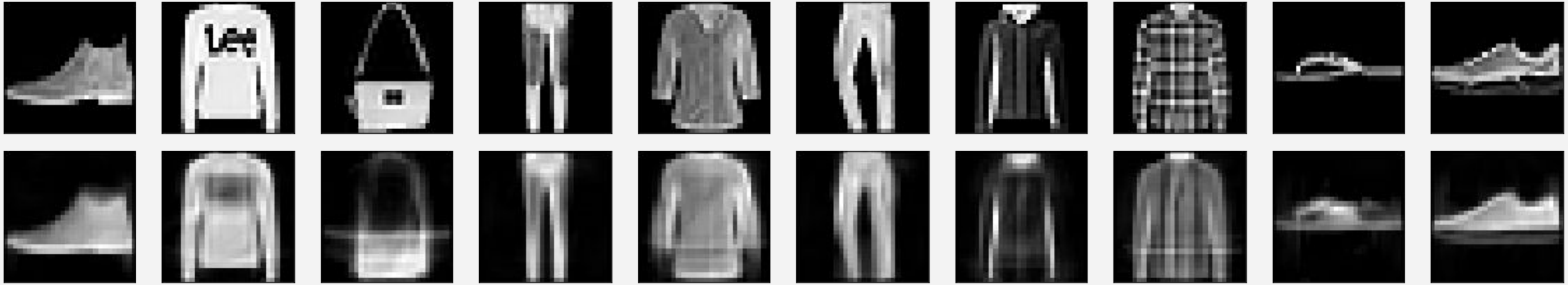


After each update of $\mathbf{H}$ and $\mathbf{W}$ in the NMF algorithm, the encodings and atoms are projected on the space verifying:

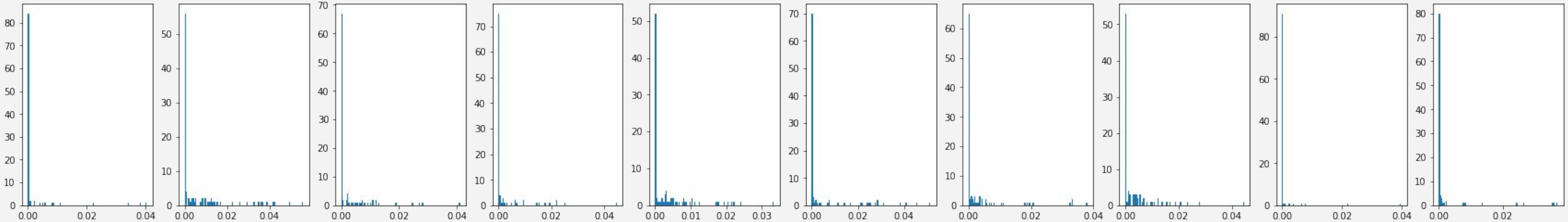
$$S(\mathbf{h}^{(i)}) = S_h, \forall i \in [1, M]$$
$$S(\mathbf{w}_j) = S_w, \forall j \in [1, k]$$

Results - $S_h = 0.6$

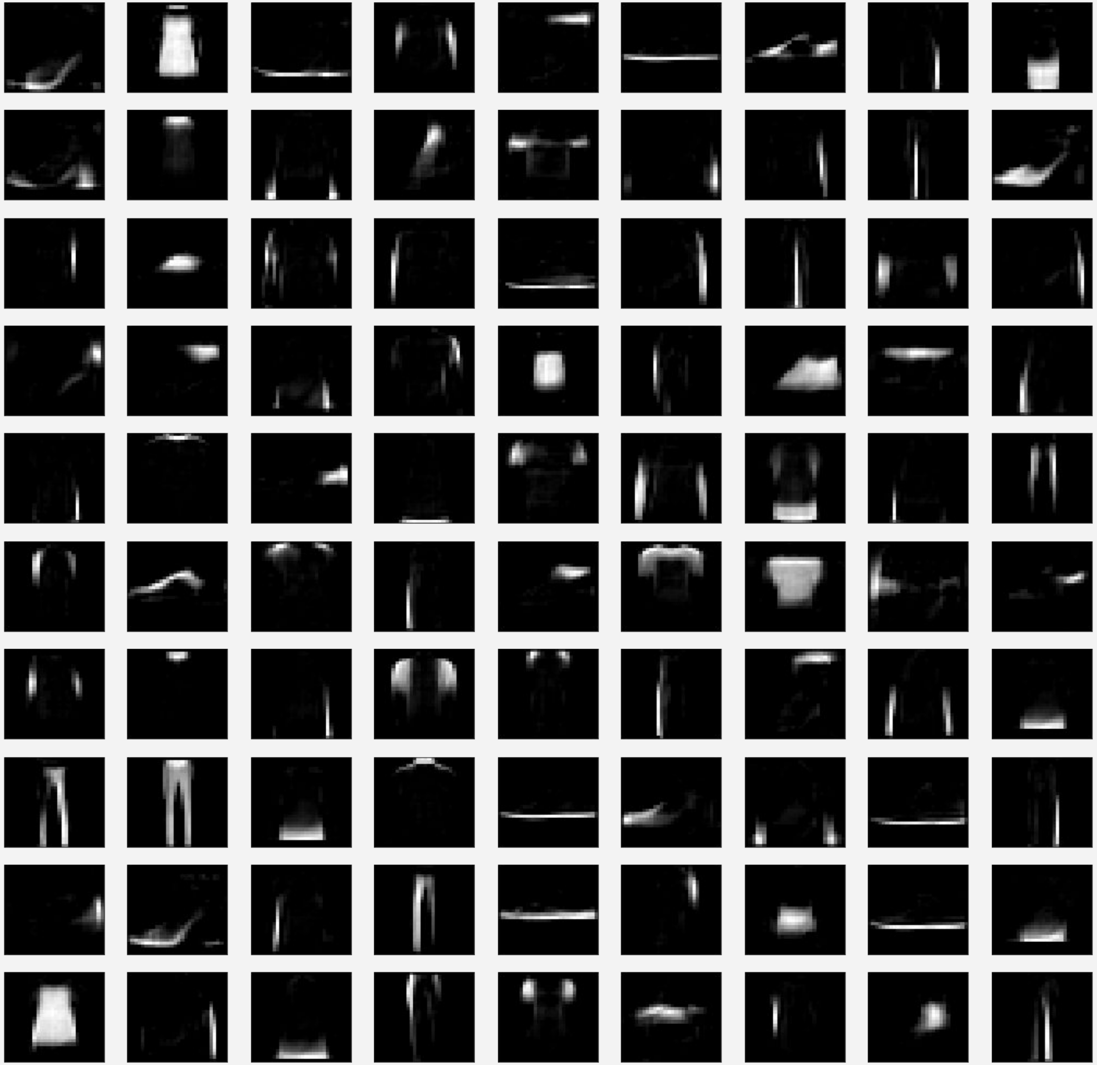
Original images and reconstruction - *Reconstruction error: 0.0109*



Histogram of the encodings - *Sparsity metric: 0.650*

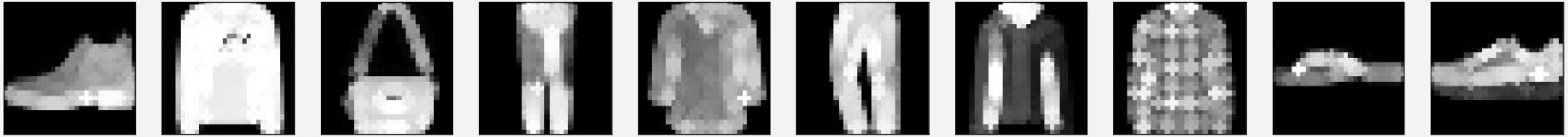


Atom images of the representation

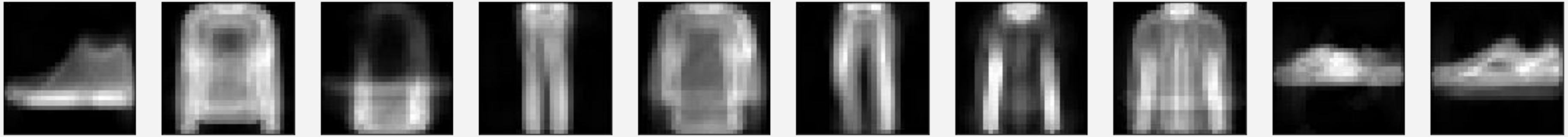


Results - Max-Approximation to dilation

Dilation of the original images by a disk of radius 1

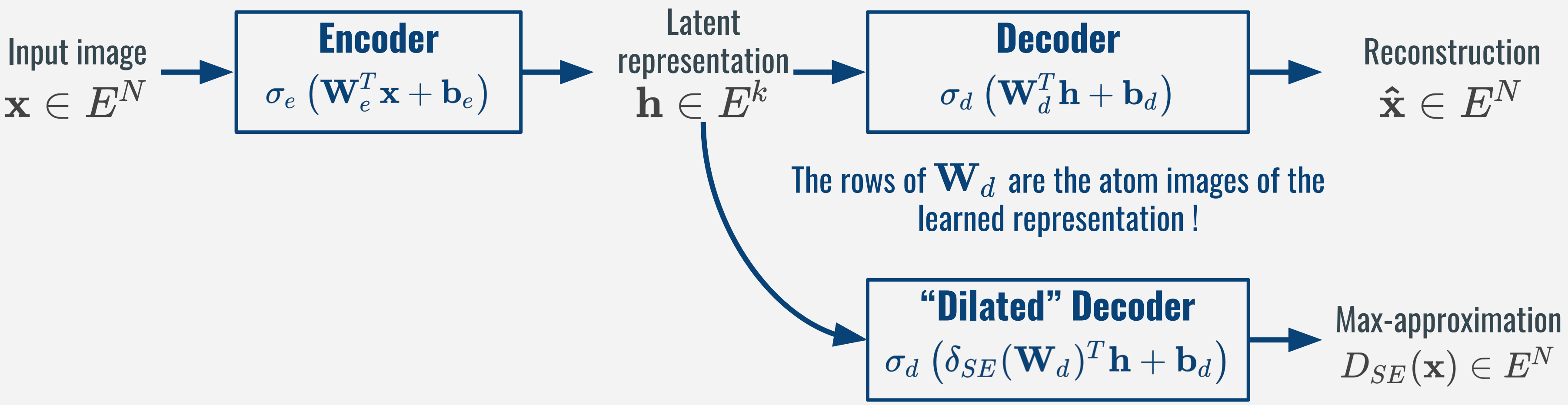


Max-approximation to the dilation by a disk of radius 1 - ***Max-approximation error: 0.107***



03 - Part-Based Representation using Auto-Encoders

Shallow Auto-Encoders



Input image $\mathbf{x} \in E^N$

Encoder $\sigma_e\left(\mathbf{W}_e^T\mathbf{x} + \mathbf{b}_e\right)$

Latent representation $\mathbf{h} \in E^k$

Decoder $\sigma_d\left(\mathbf{W}_d^T\mathbf{h} + \mathbf{b}_d\right)$

Reconstruction $\hat{\mathbf{x}} \in E^N$

The rows of $\mathbf{W}_d$ are the atom images of the learned representation !

"Dilated" Decoder $\sigma_d\left(\delta_{SE}(\mathbf{W}_d)^T\mathbf{h} + \mathbf{b}_d\right)$

Max-approximation $D_{SE}(\mathbf{x}) \in E^N$

Auto-encoder loss function, minimized during training:

$L_{AE} = \frac{1}{M}\sum_{i=1}^{M} L(\mathbf{x}^{(i)}, \hat{\mathbf{x}}^{(i)})$ where $L(\mathbf{x}^{(i)}, \hat{\mathbf{x}}^{(i)})$ is the reconstruction error (MSE)

Enforcing the Sparsity of the Encoding

Regularization of the auto-encoder:

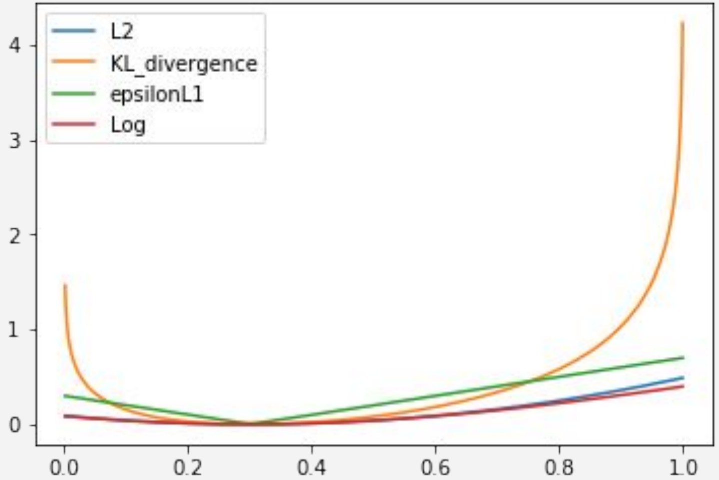
Sparsity constraint

$$L_{AE} = \frac{1}{M}\sum_{i=1}^{M} L(\mathbf{x}^{(i)}, \hat{\mathbf{x}}^{(i)}) + \boxed{\beta \sum_{j=1}^{k} S(\frac{1}{M}\sum_{i=1}^{M} h_j^{(i)}, p)}$$

Penalizes a deviation of the **expected activation of each hidden unit** from a (low) **fixed level**

Various choices for the sparsity-regularization function:

$$S_{KL}(t_j, p) = p \log \frac{p}{t_j} + (1-p) \log \frac{1-p}{1-t_j}$$



Enforcing Non-Negativity of the Atoms of the Dictionary

Two common approaches:

- ○ Asymmetric weight decay:

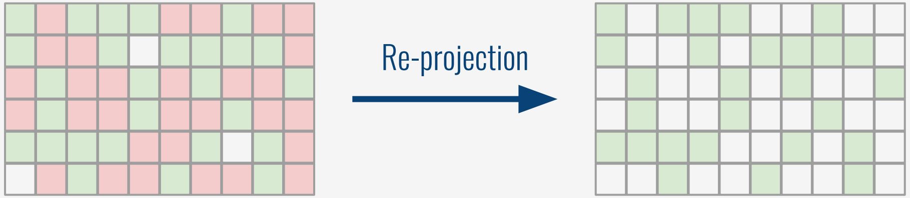
$$L_{AE} = \frac{1}{M}\sum_{i=1}^{M} L(\mathbf{x}^{(i)}, \hat{\mathbf{x}}^{(i)}) + \overset{\text{Non-Negativity constraint}}{\boxed{\lambda \sum_{i,j} \Phi(W_{d,(i,j)})}} + \overset{\text{Sparsity constraint}}{\boxed{\beta \sum_{j=1}^{k} S\left(\frac{1}{M}\sum_{i=1}^{M} h_j^{(i)}, p\right)}}$$

Stronger decay of the negative weights

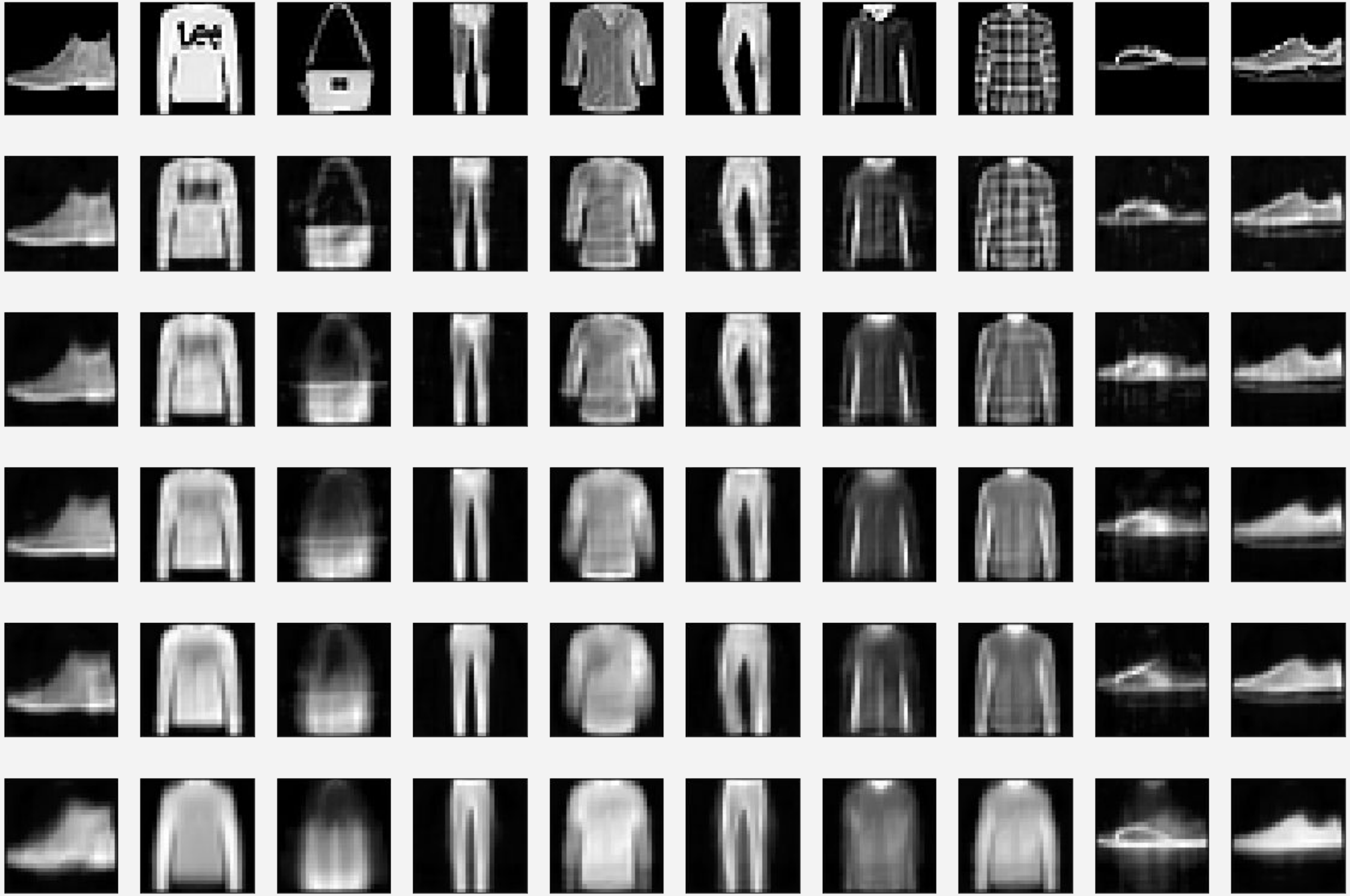
$$\Phi(W_{d,(i,j)}) = \begin{cases} \alpha |W_{d,(i,j)}|^2 & \text{if } W_{d,(i,j)} \geq 0 \\ |W_{d,(i,j)}|^2 & \text{if } W_{d,(i,j)} < 0 \end{cases} \qquad 0 \leq \alpha < 1$$

- ○ Re-Projection on the positive orthant:
  - ● Non-Parametric constraint
  - ● Ensured non-negativity

After each iteration of the optimization algorithm (e.g.: Stochastic Gradient Descent):



Results - Reconstructions



Original images

No Constraint -
Reconstruction error: 0.00697

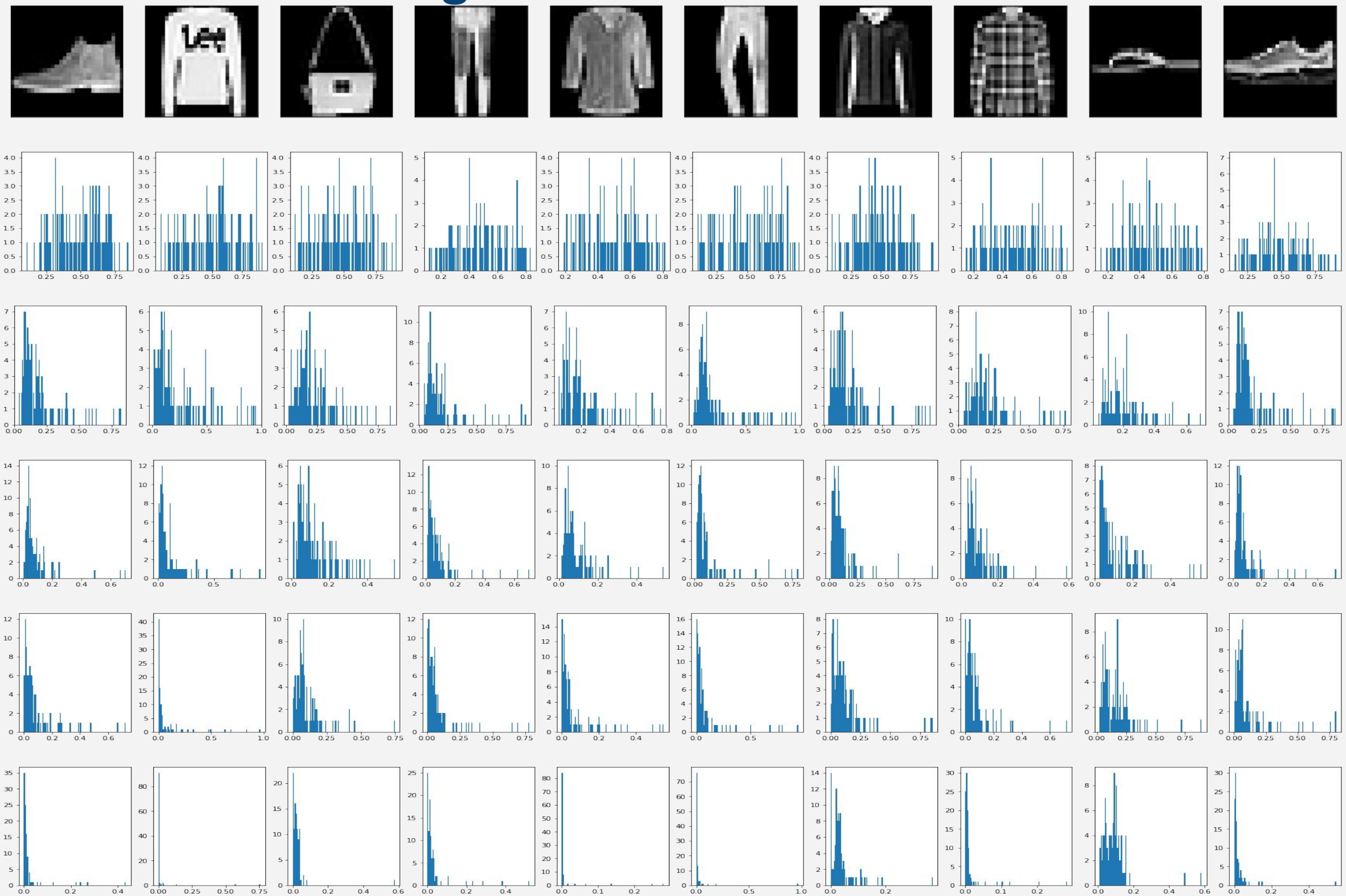
p=0.2, beta=0.001 -
Reconstruction error: 0.0103

p=0.1, beta=0.01 -
Reconstruction error: 0.0139

p=0.05, beta=0.001 -
Reconstruction error: .0164

p=0.01, beta=0.005 -
Reconstruction error: 0.0288

Results - Encodings



Original images

No Constraint -
Sparsity: 0.0678

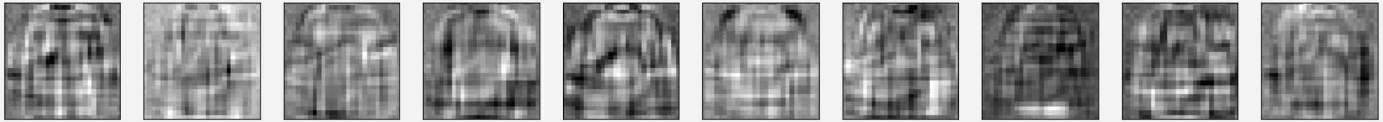
p=0.2, beta=0.001 -
Sparsity: 0.230

p=0.1, beta=0.01 -
Sparsity: 0.363

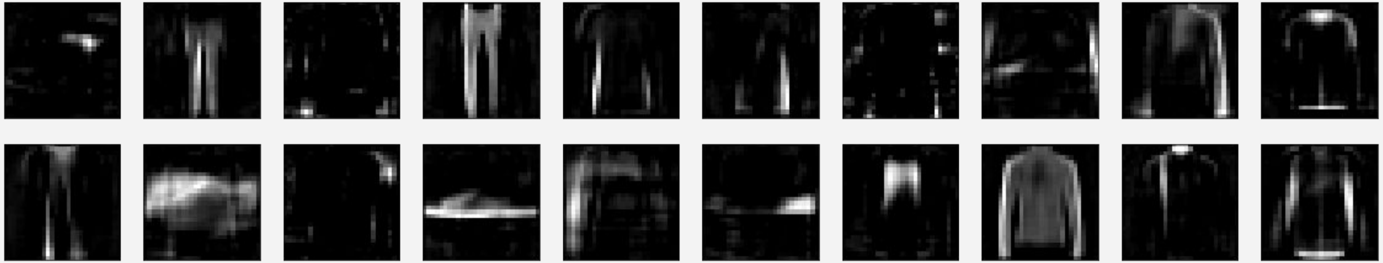
p=0.05, beta=0.001 -
Sparsity: 0.505

p=0.01, beta=0.005 -
Sparsity: 0.785

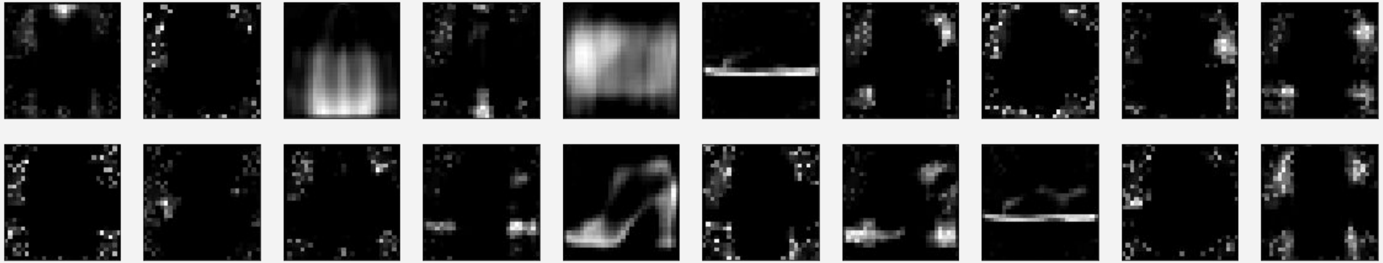
Results - Atoms



No Constraint

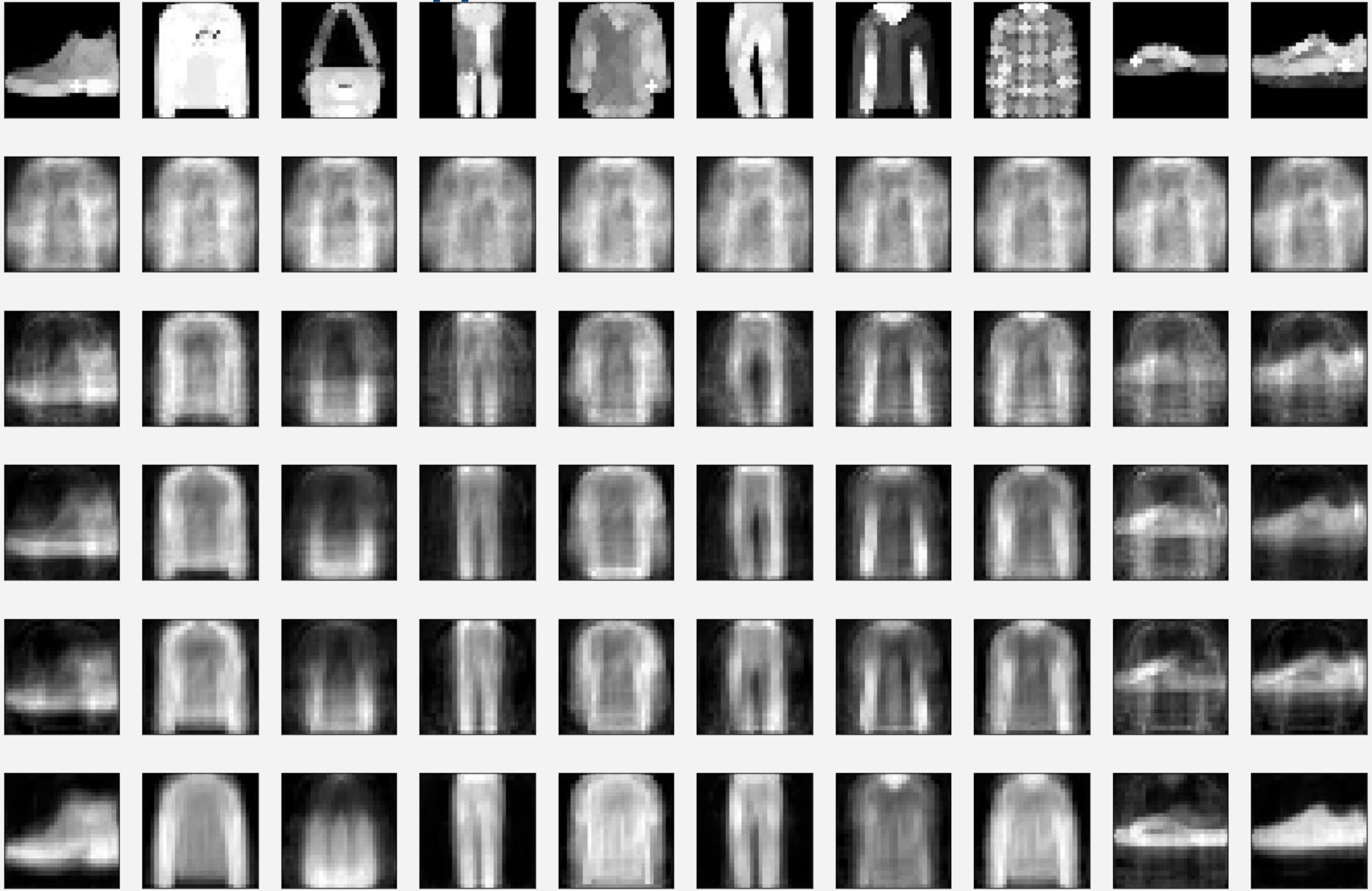


p=0.1, beta=0.01



p=0.01, beta=0.005

Results - Max-approximations to dilation



Original images

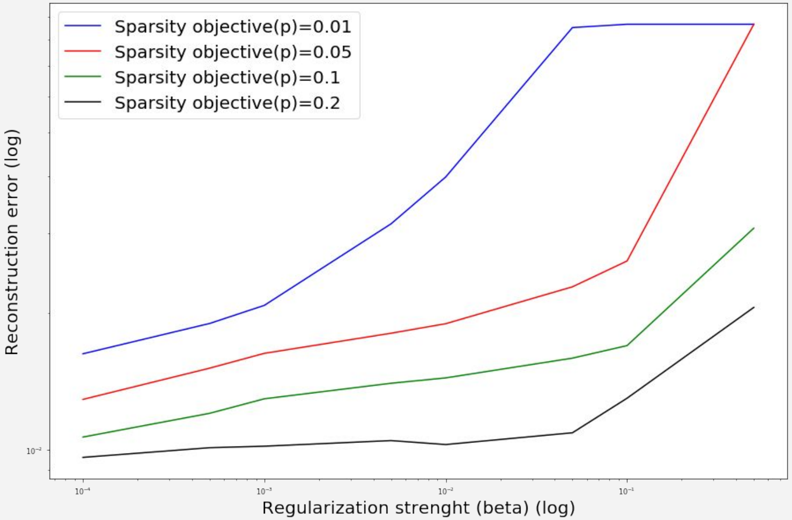
No Constraint -
Max-Approximation error: 18.04

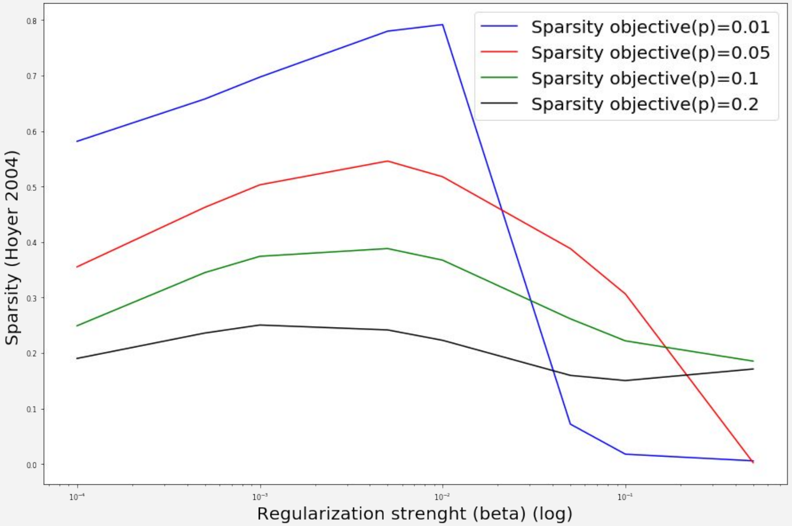
p=0.2, beta=0.001 -
Max-Approximation error: 1.179

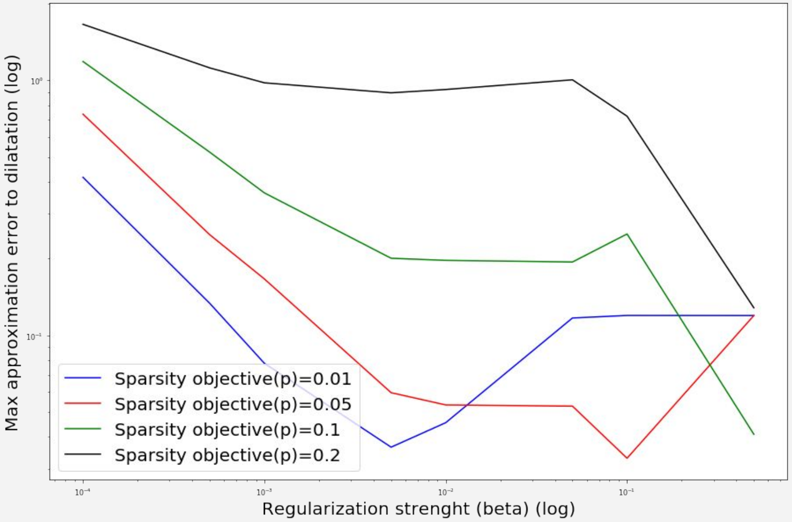
p=0.01, beta=0.01 -
Max-Approximation error: 0.264

p=0.05, beta=0.001 -
Max-Approximation error: 0.182

p=0.01, beta=0.005 -
Max-Approximation error: 0.0339

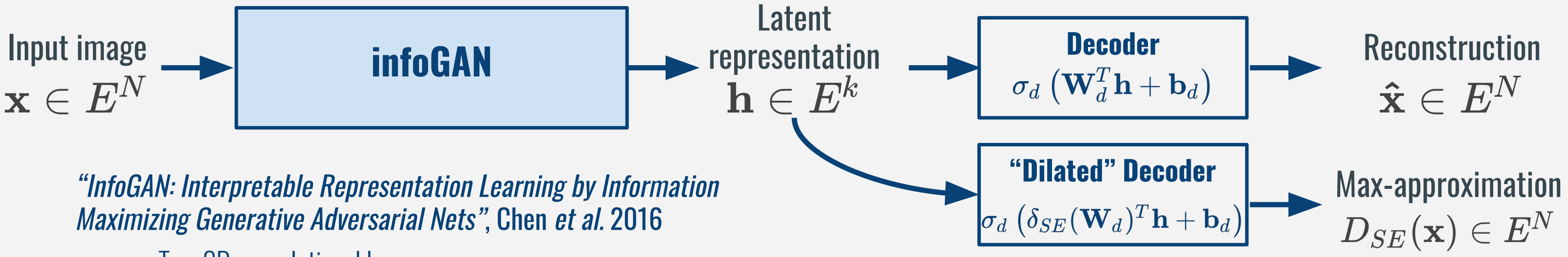

Sparsity objective(p)=0.01
Sparsity objective(p)=0.05
Sparsity objective(p)=0.1
Sparsity objective(p)=0.2
Reconstruction error (log)
10^{-2}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
Regularization strenght (beta) (log)


Sparsity objective(p)=0.01
Sparsity objective(p)=0.05
Sparsity objective(p)=0.1
Sparsity objective(p)=0.2
Sparsity (Hoyer 2004)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
Regularization strenght (beta) (log)


Max approximation error to dilatation (log)
10^{0}
10^{-1}
Sparsity objective(p)=0.01
Sparsity objective(p)=0.05
Sparsity objective(p)=0.1
Sparsity objective(p)=0.2
10^{-4}
10^{-3}
10^{-2}
10^{-1}
Regularization strenght (beta) (log)

04 - Using a Deeper Architecture

An Asymmetric Auto-Encoder



Input image $\mathbf{x} \in E^N$

infoGAN

Latent representation $\mathbf{h} \in E^k$

Decoder $\sigma_d\left(\mathbf{W}_d^T\mathbf{h} + \mathbf{b}_d\right)$

Reconstruction $\hat{\mathbf{x}} \in E^N$

"Dilated" Decoder $\sigma_d\left(\delta_{SE}(\mathbf{W}_d)^T\mathbf{h} + \mathbf{b}_d\right)$

Max-approximation $D_{SE}(\mathbf{x}) \in E^N$

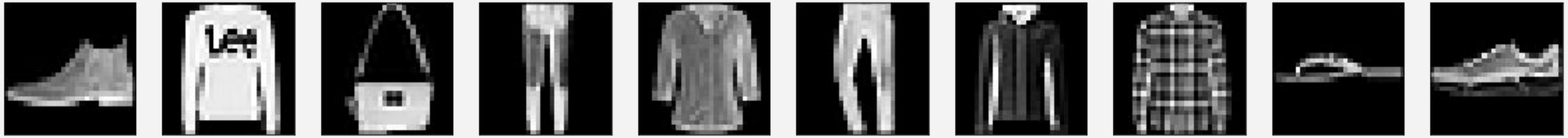
"InfoGAN: Interpretable Representation Learning by Information Maximizing Generative Adversarial Nets", Chen *et al.* 2016

- Two 2D convolutional layers
- Two fully connected layers

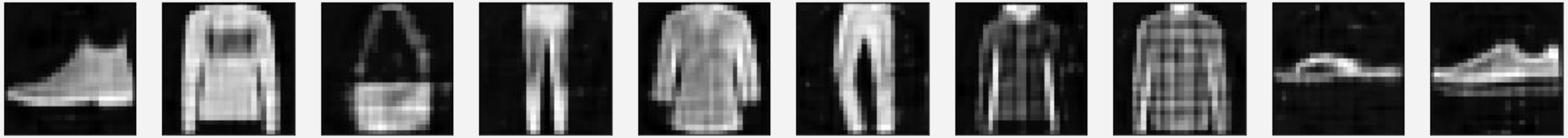
Motivations:

- Designed for a representation learning task on MNIST data set.
- Simple architecture.
- Use of convolutional layers, well adapted to computer vision tasks.
- Use of widely adopted state of the art techniques in deep learning: batch-normalization, leakyRELU, etc.

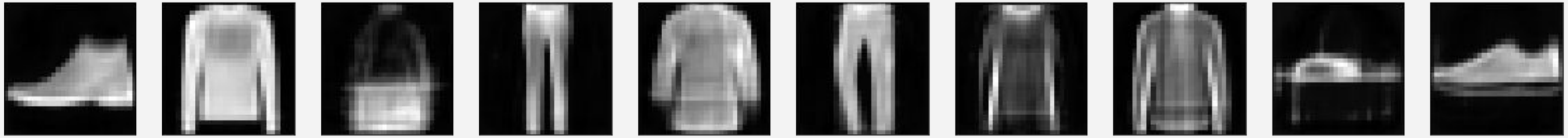
Results - Reconstructions



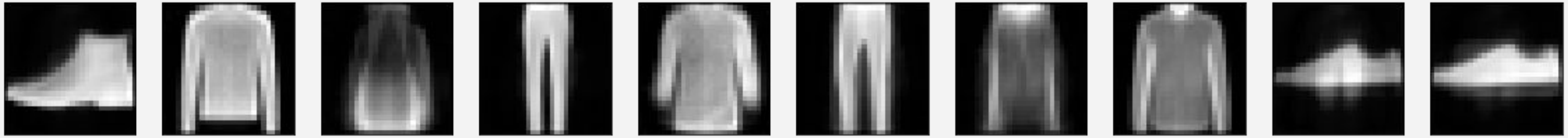
No Constraint - *Reconstruction error: 0.00646*



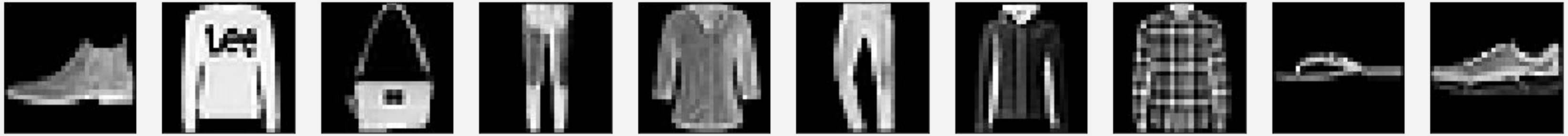
p=0.05, beta=0.005 - *Reconstruction error: 0.0125*



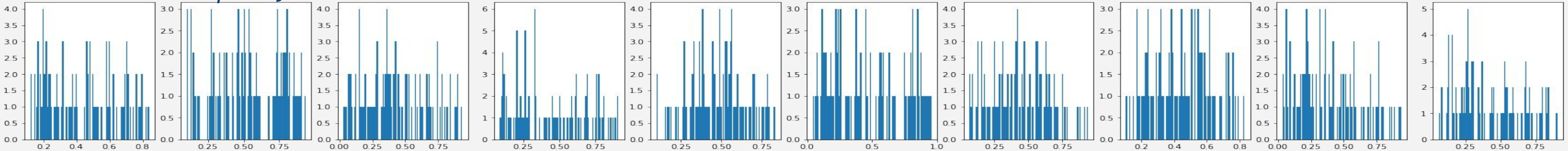
p=0.01, beta=0.01- *Reconstruction error: 0.0212*



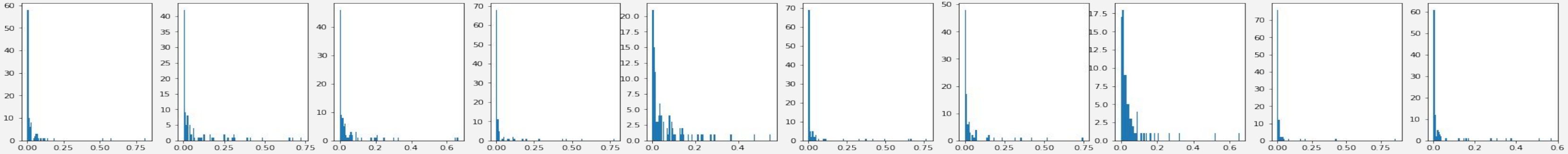
Results - Encodings



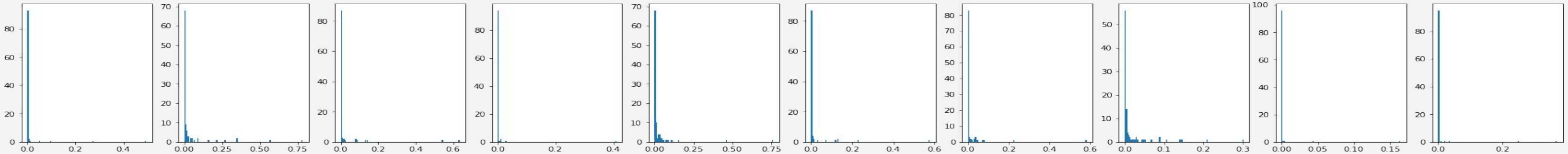
No Constraint - *Sparsity: 0.0956*



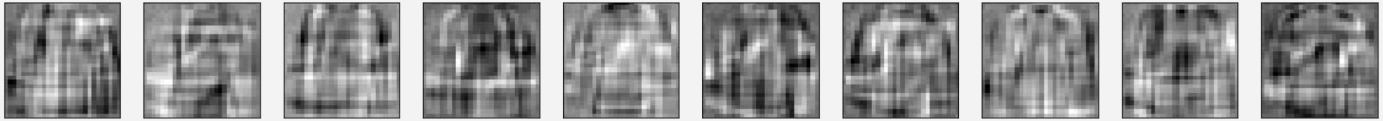
p=0.05, beta=0.005 -Sparsity*: 0.615*



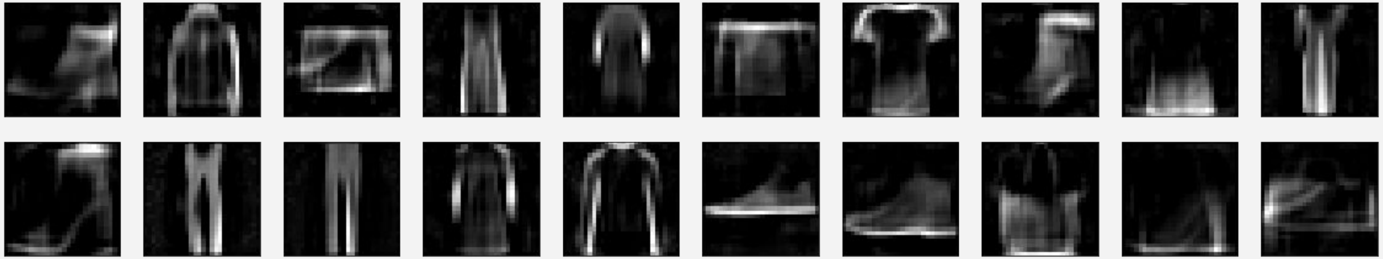
p=0.01, beta=0.01- *Sparsity 0.826*



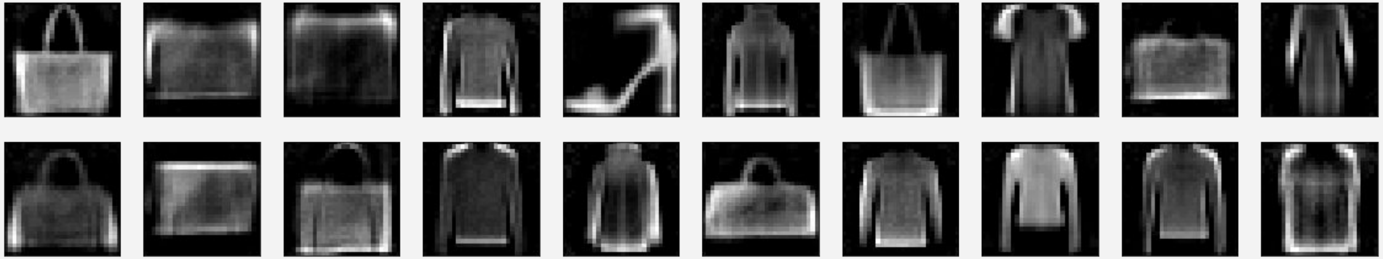
Results - Atoms



No Constraint

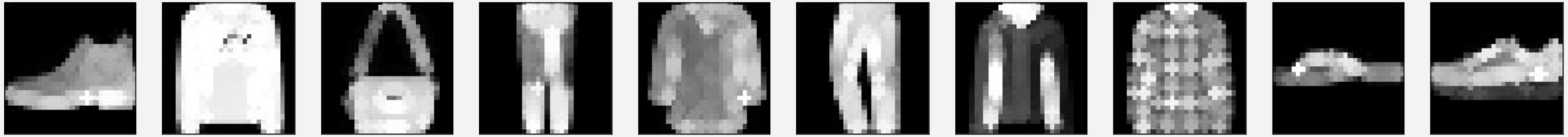


p=0.05, beta=0.005

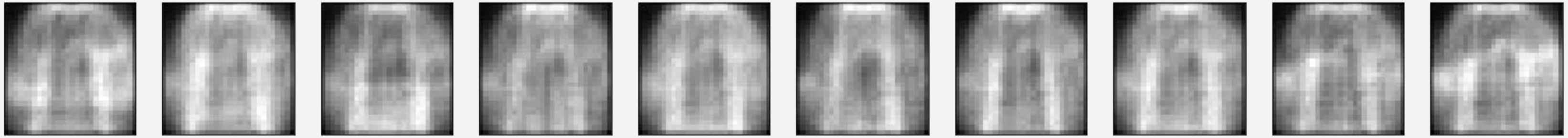


p=0.01, beta=0.01

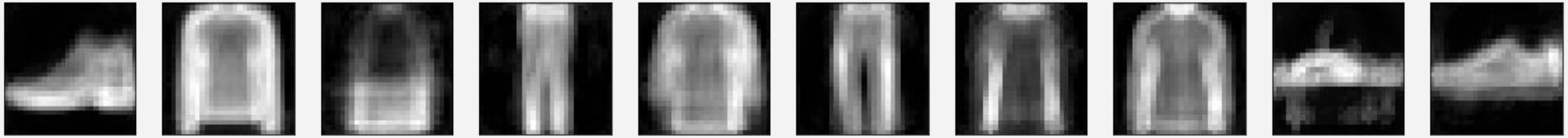
Results - Max-Approximations to dilation



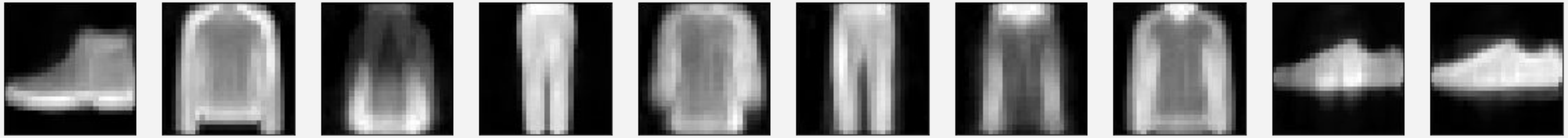
No Constraint - *Max-Approximation error: 14.31*



p=0.05, beta=0.005 - *Max-Approximation error: 0.123*



p=0.01, beta=0.01 - *Max-Approximation error: 0.0297*



06 - Conclusion and Future Works

Conclusion and possible improvements

- Applying Multi-scale morphological decomposition of the input.
- Enforcing sparsity of the atoms as well, and/or replacing the regularization function by the distance of the sparsity measure of Hoyer 2004 to a fix value (0.6).
- Learning a max-plus factorization by using morphological perceptron and max-plus convolution in the decoder.

05 - Multi-Scales Morphological Decompositions

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Results