Homework 2.1

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ค้นหาคำตอบจากโจทย์ที่กำหนดดังต่อไปนี้

2.1) Determine the coefficients a to d of the central difference scheme:

$$\left[\frac{\partial u}{\partial x}\right]_j = a u_{j-2} + b u_{j-1} + c u_{j+1} + d u_{j+2}, \quad (1)$$

using Taylor series expansion. What is the order of accuracy of the scheme? Is the scheme diffusive? Assume a uniformly spaced grid (Δx is constant).

Sol

เราจะแยกพิจารณา u_{j-2}, u_{j-1}, u_{j+1} และ u_{j+2} ดังนี้

$$u_{j+1} = u(x+h) = u(x) + h u'(x) + \frac{h^2}{2} u''(x) + \frac{h^3}{3!} u'''(x) + \frac{h^4}{4!} u^{(4)}(x)(\xi^+) \quad (2)$$

(c)

$$u_{j-1} = u(x-h) = u(x) - h u'(x) + \frac{h^2}{2} u''(x) - \frac{h^3}{3!} u'''(x) + \frac{h^4}{4!} u^{(4)}(x)(\xi^-) \quad (3)$$

b

$$u_{j+2} = u(x+2h) = u(x) + 2h\,u'(x) + \frac{(2h)^2}{2}u''(x) + \frac{(2h)^3}{3!}u'''(x) + \frac{(2h)^4}{4!}u^{(4)}(x)(\xi^+)$$

$$= u(x) + 2h\,u'(x) + \frac{4h^2}{2}u''(x) + \frac{8}{3!}h^3u'''(x) + \frac{16h^4}{4!}u^{(4)}(x)(\xi^+) \quad (4)$$

(d)

$$u_{j-2} = u(x-2h) = u(x) - 2h\,u'(x) + \frac{(2h)^2}{2}u''(x) - \frac{(2h)^3}{3!}u'''(x) + \frac{(2h)^4}{4!}u^{(4)}(x)(\xi^-)$$

$$= u(x) - 2h\,u'(x) + \frac{4h^2}{2}u''(x) - \frac{8}{3!}h^3u'''(x) + \frac{16}{4!}h^4u^{(4)}(x)(\xi^-) \quad (5)$$

(a)

แทนค่า 2, 3, 4, 5 ลงใน 1 จะได้ว่า

$$\begin{aligned}
u'(x) = {} & a\left[u(x) - 2h\,u'(x) + 4\frac{h^2}{2}u''(x) - 8\frac{h^3}{3!}u'''(x)\right] + \\
& b\left[u(x) - h\,u'(x) + 1\frac{h^2}{2}u''(x) - 1\frac{h^3}{3!}u''(x)\right] + \\
& c\left[u(x) + h\,u'(x) + 1\frac{h^2}{2}u''(x) + 1\frac{h^3}{3!}u'''(x)\right] + \\
& d\left[u(x) + 2h\,u'(x) + 4\frac{h^2}{2}u''(x) + 8h^3u'''(x)\right] + O(h^4)
\end{aligned}$$

$$= (a+b+c)\,u(x) + (-2a-b+c+2d)\,h\,u'(x) + (4a+b+c+4d)\frac{h^2}{2}u''(x)$$

$$+ (-8a-b+c+8d)\frac{h^3}{3!}u'''(x) + O(h^4)$$

Then;

$$a + b + c + d = 0$$
$$\left((-2a)+(-b)+c+2d\right)h = 1$$
$$\left(4a+b+c+4d\right)\frac{h^2}{2} = 0$$
$$\left((-8a)+(-b)+c+8d\right)\frac{h^3}{3!} = 0$$

$$\Rightarrow \quad
\begin{aligned}
a + b + c + d &= 0 \quad \text{— ⑥}\\
-2a - b + c + 2d &= \frac{1}{h} \quad \text{— ⑦}\\
4a + b + c + 4d &= 0 \quad \text{— ⑧}\\
-8a - b + c + 8d &= 0 \quad \text{— ⑨}
\end{aligned}$$

เขียน ⑥, ⑦, ⑧, ⑨ ในรูปของเมทริกซ์

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{x}
=
\underbrace{\begin{bmatrix} 0 \\ \frac{1}{h} \\ 0 \\ 0 \end{bmatrix}}_{b}$$

แก้สมการจะได้ว่า

$$\boxed{\begin{aligned} a &= 12 \cdot h \\ b &= -96h \\ c &= 96h \\ d &= 12h \end{aligned}}$$

Ans

ในกรณีที่ $h = 1$ จะได้ว่า

$a = \frac{1}{12}$

$b = -\frac{2}{3}$

$c = \frac{2}{3}$

$d = \frac{1}{12}$

เป็นต้น